# Classifying subfactors: beyond index 5

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"The little desert? Some subfactors with index in the interval $(5, 3+\sqrt{5})$", with Scott Morrison.

Suppose $N \subset M$ is a subfactor, ie a unital inclusion of type $II_1$ factors.

### Definition

*The* <u>*index*</u> *of* $N \subset M$ *is* $[M : N] := \dim_N L^2(M)$.

### Example

If $R$ is the hyperfinite $II_1$ factor, and $G$ is a finite group which acts outerly on $R$, then $R \subset R \rtimes G$ is a subfactor of index $|G|$.

If $H \leq G$, then $R \rtimes H \subset R \rtimes G$ is a subfactor of index $[G : H]$.

### Theorem (Jones)

*The possible indices for a subfactor are*

$$\{4\cos(\frac{\pi}{n})^2 | n \geq 3\} \cup [4, \infty].$$

Let $X = {}_N M_M$ and $\overline{X} = {}_M (M^{op})_N$, and $\otimes = \otimes_N$ or $\otimes_M$ as needed.

**Definition**

*The standard invariant of $N \subset M$ is the (planar) algebra of bimodules generated by $X$:*

$$\begin{array}{ccccccccc} X & , & X \otimes \overline{X} & , & X \otimes \overline{X} \otimes X & , & X \otimes \overline{X} \otimes X \otimes \overline{X} & , & \dots \\ \overline{X} & , & \overline{X} \otimes X & , & \overline{X} \otimes X \otimes \overline{X} & , & \overline{X} \otimes X \otimes \overline{X} \otimes X & , & \dots \end{array}$$

**Definition**

*The principal graph of $N \subset M$ has vertices for (isomorphism classes of) irreducible $N$-$N$ and $N$-$M$ bimodules, and an edge from ${}_N Y_N$ to ${}_N Z_M$ if $Z \subset Y \otimes X$ (iff $Y \subset Z \otimes \overline{X}$).*

*Ditto for the dual principal graph, with $M$-$M$ and $M$-$N$ bimodules.*

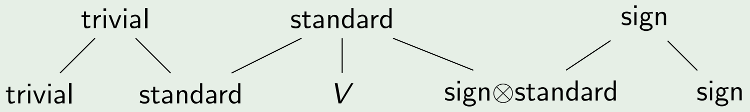
# Example: $R \rtimes H \subset R \rtimes G$

Again, let $G$ be a finite group with subgroup $H$, and act outerly on $R$. Consider $N = R \rtimes H \subset R \rtimes G = M$.

The irreducible $M$-$M$ bimodules are of the form $R \otimes V$ where $V$ is an irreducible $G$ representation. The irreducible $M$-$N$ bimodules are of the form $R \otimes W$ where $W$ is an $H$ irrep.

The dual principal graph of $N \subset M$ is the induction-restriction graph for irreps of $H$ and $G$.

**Example ($S_3 \leq S_4$)**




(The principal graph is an induction-restriction graph too, for $H$ and various subgroups of $H$.)

Where do subfactors come from?

- Groups
- Quantum Groups
- Rational Conformal Field Theories
- Out of thin air (from connections or planar algebras).

# Reconstruction Theorems

The standard invariant of a subfactor can be described by

- A planar algebra (Jones)
- A biunitary connection (Ocneanu)

Certain planar algebras or connections give subfactors:

- subfactor planar algebras, which have an inner product defined by $\langle x, y \rangle := \mathrm{tr}(y^* x)$
- flat connections

Both the planar algebra and the biunitary connection of a subfactor are finite if the principal graph is finite.

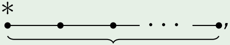
**Theorem (Jones-Penneys, Morrison-Walker)**

*If $\mathcal{P}$ is a subfactor planar algebra with principal graph $\Gamma$, a copy of $\mathcal{P}$ can be found in $GPA(\Gamma)$.*

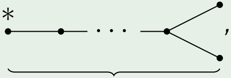
# Index less than 4

**Theorem (Jones, Ocneanu, Kawahigashi, Izumi, Bion-Nadal)**

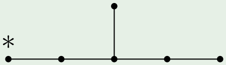
*The principal graph of a subfactor of index less than 4 is one of*

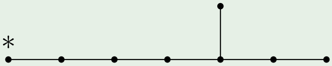
$A_n =$  , $n \geq 2$ — *index* $4\cos^2(\frac{\pi}{n+1})$

*n vertices*

$D_{2n} =$  , $n \geq 2$ — *index* $4\cos^2(\frac{\pi}{4n-2})$

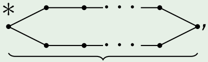
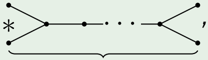
*2n vertices*

$E_6 =$  — *index* $4\cos^2(\frac{\pi}{12}) \approx 3.73$

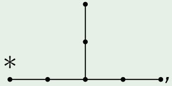
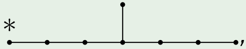
$E_8 =$  — *index* $4\cos^2(\frac{\pi}{30}) \approx 3.96$

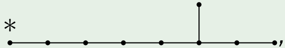
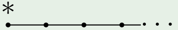
## Theorem (Popa and others)

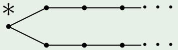
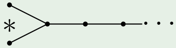
*The principal graphs of a subfactor of index 4 are extended Dynkin diagram:*

$A_n^{(1)} =$ , $n \geq 1$, $D_n^{(1)} =$ , $n \geq 3$,

$n+1$ *vertices* $\qquad$ $n+1$ *vertices*

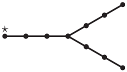
$E_6^{(1)} =$ , $\quad E_7^{(1)} =$ ,

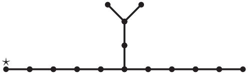
$E_8^{(1)} =$ , $\quad A_\infty =$  ,

$A_\infty^{(1)} =$  , $\quad D_\infty =$ 

*There are multiple subfactors for some of these principal graphs (eg, $n-1$ non-isomorphic hyperfinite subfactors for $D_n^{(1)}$).*

# Haagerup's list

- In 1993 Haagerup classified possible principal graphs for subfactors with index between 4 and $3 + \sqrt{3} \approx 4.73$:
  -  ,  ,  , . . . ,

    $(\approx 4.30,\ 4.37,\ 4.38, \ldots)$
  -  , $(\approx 4.56)$
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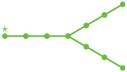
  -  ,  ,  , . . . ,
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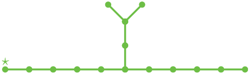
  -  , $(\approx 4.56)$
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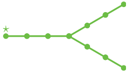
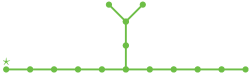
  - , , , $\ldots$, ($\approx 4.30$, $4.37$, $4.38, \ldots$)
  - , ($\approx 4.56$)
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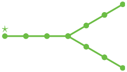
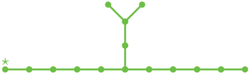
  - , , , ..., ($\approx 4.30$, $4.37$, $4.38, \ldots$)
  - , ($\approx 4.56$)
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- Haagerup and Asaeda & Haagerup (1999) constructed two of these possibilities.
- Bisch (1998) and Asaeda & Yasuda (2007) ruled out infinite families.
- In 2009 we (Bigelow-Morrison-Peters-Snyder) constructed the last missing case. `arXiv:0909.4099`

- Why did Haagerup stop at $3+\sqrt{3}$?
- Why try to extend it?

The classification is again in terms of principal graphs.

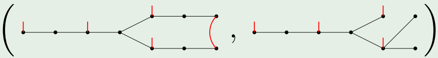
**Definition**

*The vertices of a principal graph pair are (isomorphism classes of) irreducible bimodules over $A$ and/or $B$. Let $X = {}_A B_B$.*

*In the standard invariant, there are four kinds of bimodules: $A-A$, $A-B$, $B-A$ and $B-B$. The principal graph has $A-A$ and $A-B$ bimodules, and ${}_A Y_A$ and ${}_A Z_B$ are connected by an edge if $Z \subset Y \otimes X$.*

*The dual principal graph has $B-A$ and $B-B$ projections, and ${}_B V_A$ and ${}_B W_B$ are connected by an edge if $W \subset V \otimes X$.*

## Example (The Haagerup subfactor's principal graph pair)



Which pairs can go together? The vertices of a principal graph are (isomorphism classes of) projections in $\mathrm{End}(X^{\otimes n})$

- The graphs must have the same graph norm;
- The graphs' depths can differ by at most 1;
- The pair must satisfy an associativity test:

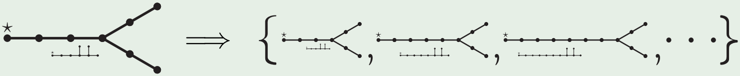
$$(X \otimes Y) \otimes X \cong X \otimes (Y \otimes X)$$

A computer can efficiently enumerate such pairs with index below some number $L$ up to a given rank or depth, obtaining a collection of allowed vines and weeds.

## Definition

*A vine represents an integer family of principal graphs, obtained by* *<u>translating</u>* *the vine.*

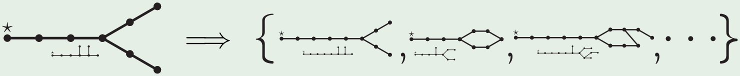
## Example



## Definition

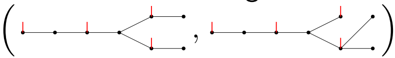
*A weed represents an infinite family, obtained by either translating or* *<u>extending</u>* *arbitrarily on the right.*

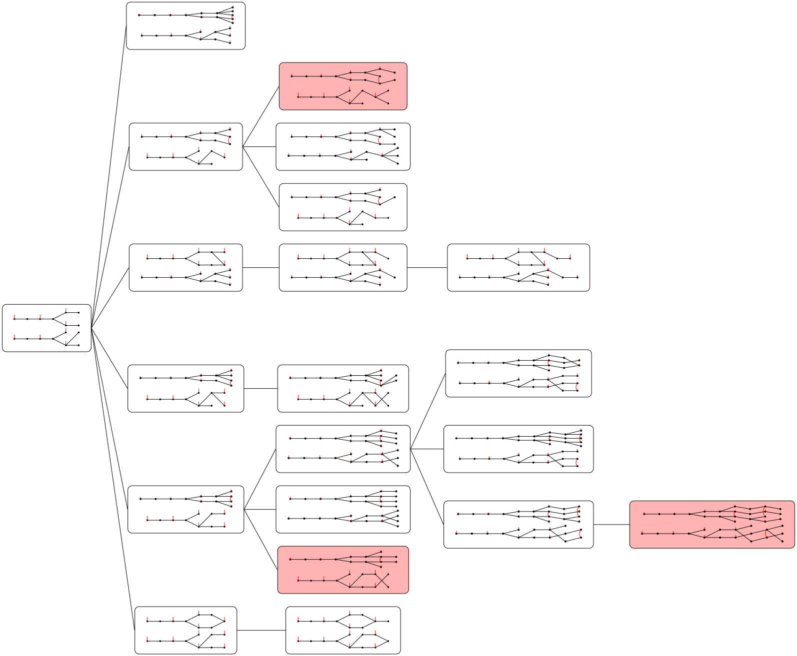
## Example



Each time we extend the depth, a weed turns into a set of vines and a (possibly empty) set of new, longer weeds. If all the weeds disappear, the enumeration is complete. This happens if the index is sufficiently small (e.g. Haagerup's theorem up to index $3+\sqrt{3}$), but generally we stop with some surviving weeds, and have to rule these out 'by hand'.

For example, here's what we get when we run this procedure with index limit 5, starting from the bigraph pair

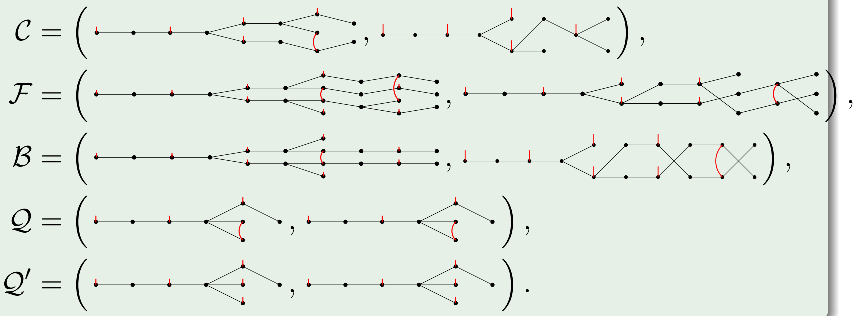
:



# The classification up to index 5

**Theorem (Morrison-Snyder, part I, `arXiv:1007.1730`)**

*Every (finite depth) $II_1$ subfactor with index less than 5 sits inside one of 54 families of vines, or 5 families of weeds:*



**Theorem (Morrison-Snyder, part I, `arXiv:1007.1730`)**

*Every (finite depth) $II_1$ subfactor with index less than 5 sits inside one of 54 families of vines, or 5 families of weeds.*

This is proved by exhaustive computer calculations, and

**Theorem (Morrison-Snyder, part I, `arXiv:1007.1730`)**

*There are no subfactors with index in $(4, 5)$ with supertransitivity one.*

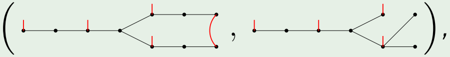
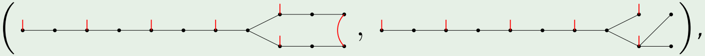
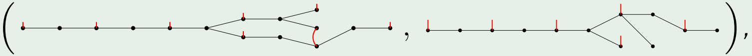
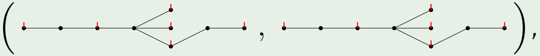
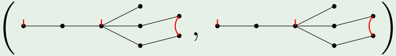
This is proved by careful attention to dimensions (and the difficulty of having an intermediate subfactor at small index).

**Definition**

*The supertransitivity of a graph of an irreducible subfactor is the number of edges between its initial point and the first branch point.*

## Theorem

*There are exactly ten non-trivial subfactors with index between* 4 *and* 5*:*

- 
- 
- 
- *The 3311 GHJ subfactor (MR999799), with index* $3+\sqrt{3}$ 
- *Izumi's self-dual 2221 subfactor (MR1832764), with index* $\frac{5+\sqrt{21}}{2}$ 

*along with the non-isomorphic duals of the first four, and the non-isomorphic complex conjugate of the last.*

How do you kill vines?

- non-associativity (The computer doesn't check that $X \otimes (Y \otimes X) \simeq (X \otimes Y) \otimes X$, only that $\#X \otimes (Y \otimes X) = \#(X \otimes Y) \otimes X$).
- number theory:

**Theorem (Coste-Gannon, '94)**

*The dimension of an object in a fusion category is a cyclotomic integer.*

**Theorem (Calegari-Morrison-Snyder, '10)**

*Only a finite number of graphs in any vine have cyclotomic index.*

How do you kill vines?

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**Theorem (Calegari-Morrison-Snyder, '10)**

*Only a finite number of graphs in any vine have cyclotomic index.*

How do you killl weeds?

- No longer have enough information to use non-associativity or number theory.
- Show there's no biunitary connection
- Show there's no planar algebra

## Theorem

*There are exactly ten non-trivial subfactors with index between* 4 *and* 5.

Proven in "Subfactors of index less than 5:"

- Morrison-Snyder, part 1: the principal graph odometer, `arXiv:1007.1730`
- Morrison-Penneys-Peters-Snyder, part 2: triple points, `arXiv:1007.2240`
- Izumi-Jones-Morrison-Snyder, part 3: quadruple points, `arXiv:1109.3190`
- Penneys-Tener, part 4: vines, `arXiv:1010.3797`

and

- Han, A construction of the "2221" planar algebra, `arXiv:1102.2052`

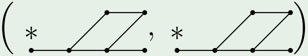
## Theorem (Izumi)

*The only subfactors with index exactly 5 are group-subgroup subfactors:*

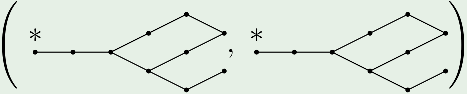
- $1 \subset \mathbb{Z}_5$;
- $\mathbb{Z}_2 \subset D_{10}$;
- $\mathbb{F}_5^\times \subset \mathbb{F}_5 \rtimes \mathbb{F}_5^\times$;
- $A_4 \subset A_5$;
- $S_4 \subset S_5$.

### Theorem

*There are two known subfactors coming from quantum groups ($SU(2)$ and $SU(3)$) with index between 5 and $3+\sqrt{5}$. They both have index $\approx 5.05$, and their principal graphs are*

$A =$ 

*and*

$B =$ 

### Theorem (Morrison-Peters)

*There are unique subfactors with principal graphs $A$ and $B$.*

**Theorem (Morrison-Peters)**

*The only 1-supertransitive subfactor with index between 5 and $3+\sqrt{5}$ has principal graph A.*

**Proof.**

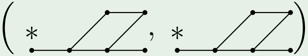
Careful attention to the dimensions appearing in potential principal graphs gives this result.

Suppose first our graph is finite-depth. There are at least two vertices at depth two. Neither can have dimension one, or there would be an intermediate subfactor. They cannot both have dimension bigger than two, because the allowed dimensions bigger than two would make the index too big. Thus at least one has dimension between 1 and 2.

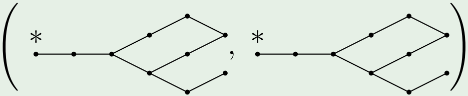
Considered from the point of view of this vertex, then, we are looking at a subfactor of index less than four. We understand these ... □

## Conjecture

*There are only two subfactors with index between 5 and* $3+\sqrt{5}$, *namely the quantum group subfactors with principal graphs*

$A =$ 

*and*

$B =$ 

With help from a computer, we can show

## Theorem (Trilobata)

*There are only two subfactors with index between 5 and* $3+\sqrt{5}$ *and rank* $\leq 38$, *namely the quantum group subfactors with principal graphs* $A$ *and* $B$.

The terrain changes:

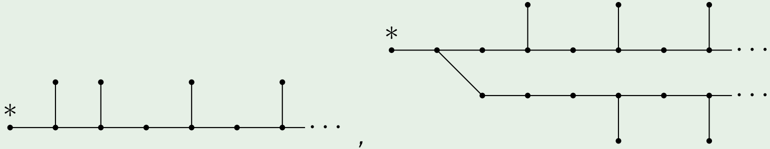
**Theorem (Bisch-Nicoara-Popa)**

*At index 6, there is an infinite one-parameter family of subfactors having isomorphic standard invariants.*

and

**Theorem (Bisch-Jones)**

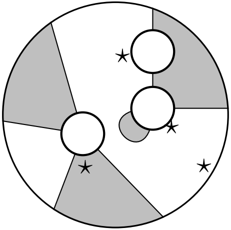
$A_2 * A_3$ *is an infinite depth subfactor at index* $2\tau^2 = 3 + \sqrt{5} \sim 5.23607$.



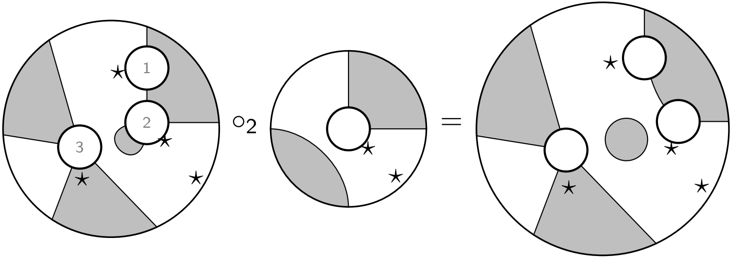
## Definition

*A shaded planar diagram has*

- *a finite number of inner boundary circles*
- *an outer boundary circle*
- *non-intersecting strings*
- *a marked point $\star$ on each boundary circle*



We can compose planar diagrams, by insertion of one into another (if the number of strings matches up):



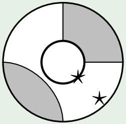

## Definition

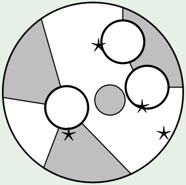
*The* <u>*shaded planar operad*</u> *consists of all planar diagrams (up to isomorphism) with the operation of composition.*

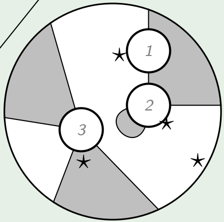
## Definition

*A planar algebra is a family of vector spaces* $V_{k,\pm}$, $k = 0, 1, 2, \ldots$ *which are acted on by the shaded planar operad.*

$$\begin{tikzcd}[column sep=small]
V_{2,-} \times V_{1,+} \times V_{1,+} \arrow[rr] \arrow[dr] & & V_{3,+} \\
& V_{2,-} \times V_{2,+} \times V_{1,+} \arrow[ur] &
\end{tikzcd}$$







## Example (The graph planar algebra $\mathcal{G}(\Gamma)$)

The underlying vector spaces $\mathcal{G}(\Gamma)_{n,\pm}$ are (formal sums of) loops of length $n$ on $\Gamma$, with the base point at either an even or odd depth vertex depending on $\pm$.

To define the action of a planar tangle $T$, we specify its values $T(\gamma_i)$, where the $\gamma_i$ are loops corresponding to the input vector spaces for $T$. This element $T(\gamma_i) \in \mathcal{G}_n$ is a sum of loops corresponding to the outside boundary of $T$:

$$T(\gamma_i) = \sum_{b \in \mathcal{L}} c(T, b) \partial_{\text{outer}}(b), \tag{0.1}$$

where the label set $\mathcal{L}$ consists of all ways to compatibly color the strands of $T$ with edges of $\Gamma$ and the regions of $T$ with vertices of $\Gamma$, such that around each inner or outer boundary of $T$ the colors agree with the loops $\gamma_i$. $\partial_{\text{outer}}(b)$ is the loop given by reading this labelling around the outer boundary. The coefficients $c(T, b)$ are ...

We care about graph planar algebras because

**Theorem (Jones-Penneys, Morrison-Walker)**

*If $\mathcal{P}$ is a subfactor planar algebra with principal graph $\Gamma$, a copy of $\mathcal{P}$ can be found in $GPA(\Gamma)$.*

Together with

**Theorem (Popa)**

*For finite-depth subfactors, the standard invariant is a complete invariant.*

We can prove

**Theorem (Morrison-Peters)**

*There are unique subfactors with principal graphs $A$ and $B$.*

**Proof.**

First we find biunitary connections for these graphs. There are (up to gauge equivalence) two for $A$ and one for $B$. So uniqueness of $B$ is established.

For any connection on a graph, the flat elements of the graph planar algebra form a subfactor planar algebra. However, it might not have the original graph as its principal graph.

The flat planar subalgebra for one of the connections on $A$ is too small to have principal graph $A$. The other connection then must (and does!) have the associated flat planar subalgebra have principal graph $A$.

As there is a unique (up to gauge equivalence) subfactor planar algebra of $GPA(A)$, there is a unique subfactor with principal graph $A$.

$\square$

The End!