Knots, the four-color Theorem, and von Neumann Algebras

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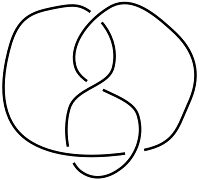
Knots and diagrams

Definition

A knot is the image of a smooth embedding $S^1 \to \mathbb{R}^3$.



Question: Are knots one-dimensional, or three?

Knots and diagrams

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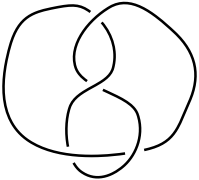
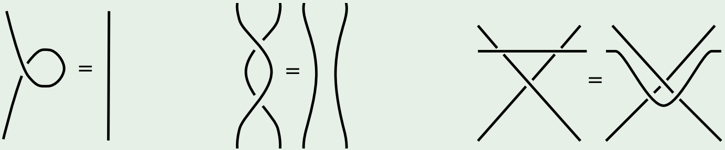


Question: Are knots one-dimensional, or three?
Answer: No.

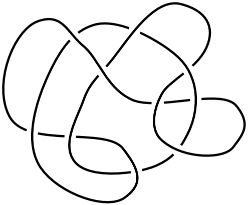
Different diagrams for the same knot

Theorem (Reidemeister)

If two diagrams represent the same knot, then you can move between them in a series of Reidemeister moves:



Seeing that two knots are the same



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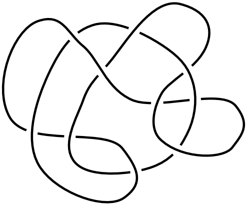

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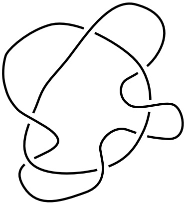

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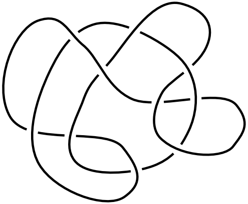

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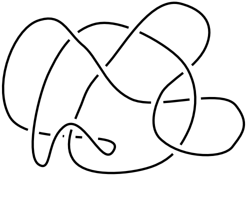

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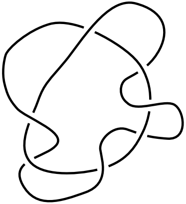

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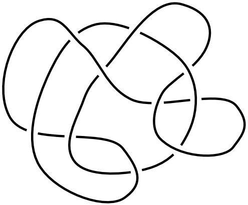
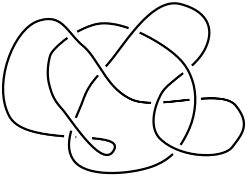
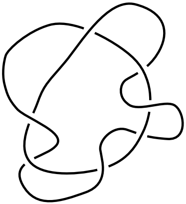
 ,
 ,


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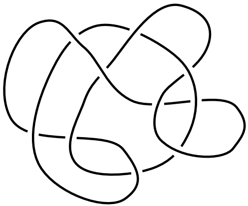
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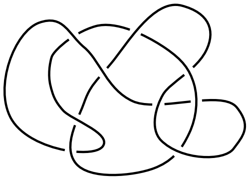
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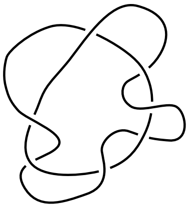

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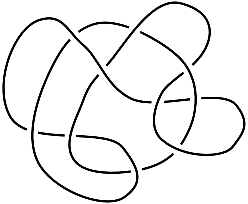
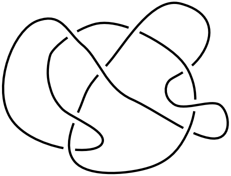
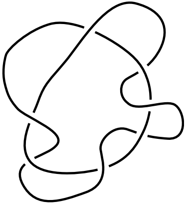
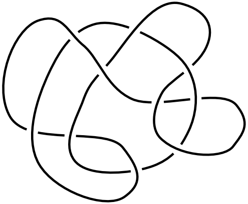
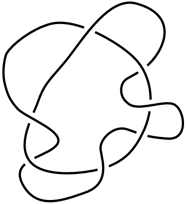
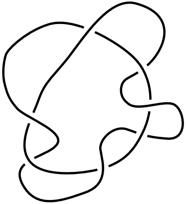
 ,  , 

Seeing that two knots are the same

 ,  , 

Seeing that two knots are different

Haken's Algorithm

In 1961, Haken publishes a 130-page description of an algorithm to determine whether a given knot is the unknot or not.

This algorithm runs in exponential time and memory (exponential in the number of crossings) ... and is really hard to program.

Question

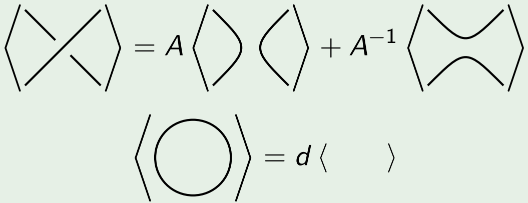
Are there better ways to tell if a knot is or isn't the unknot?

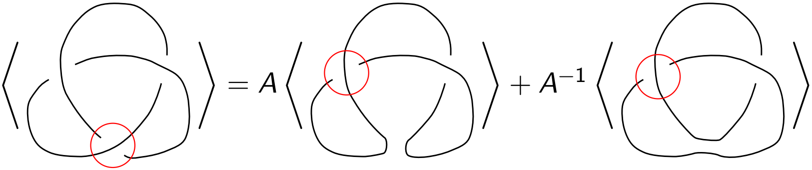
Knot invariants

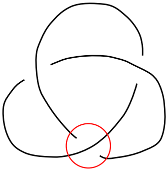
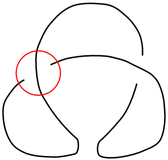
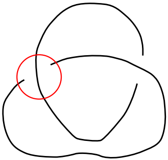
A knot invariant is a map from knot diagrams to something simpler: either $\mathbb{C}$, or polynomials, or 'simpler' diagrams. Crucially, the value of the invariant shouldn't change under Reidemeister moves.

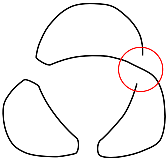
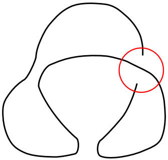
Definition

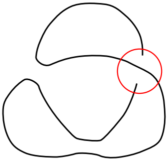
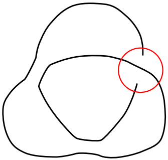
The Kauffman bracket of a knot is a map from knot diagrams to $\mathbb{C}[[A]]$. *Let* $d = -A^2 - A^{-2}$. *Then define*

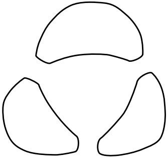
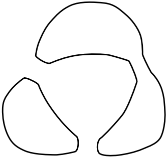
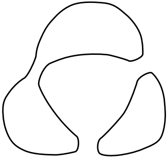


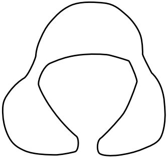
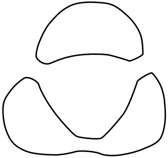
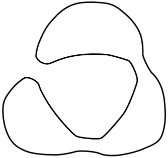

$= A$
$+ A^{-1}$

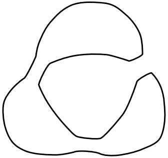
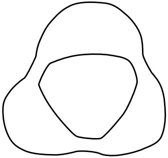
$\Big\langle$  $\Big\rangle = A \Big\langle$  $\Big\rangle + A^{-1} \Big\langle$  $\Big\rangle$

$= A^2 \Big\langle$  $\Big\rangle + \Big\langle$  $\Big\rangle$

$+ \Big\langle$  $\Big\rangle + A^{-2} \Big\langle$  $\Big\rangle$

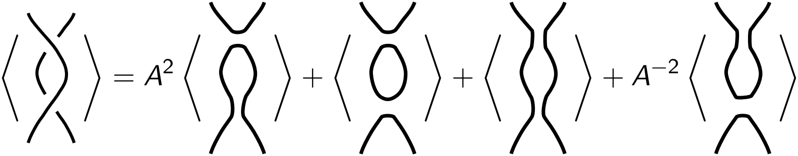
$= A^3 \Big\langle$  $\Big\rangle + A \Big\langle$  $\Big\rangle + A \Big\langle$  $\Big\rangle$

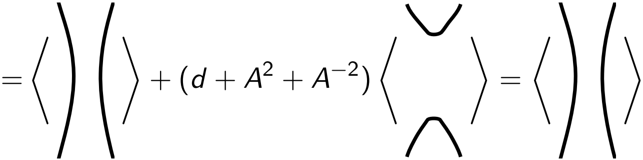
$+ A^{-1} \Big\langle$  $\Big\rangle + A \Big\langle$  $\Big\rangle + A^{-1} \Big\langle$  $\Big\rangle$

$+ A^{-1} \Big\langle$  $\Big\rangle + A^{-3} \Big\langle$  $\Big\rangle$

$$= A^3 d^3 + A d^2 + \cdots = -A^9 + A + A^{-3} + A^{-7}$$

The Kauffman bracket is invariant under Reidemeister 2:





Exercise

The Kauffman bracket is also invariant under Reidemeister 3, but it is not invariant under Reidemeister 1.

A modification of the Kauffman bracket which is invariant under Reidemeister 1 is known as the *Jones Polynomial*.

Question

Does there exist a non-trivial knot having the same Jones polynomial as the unknot?

The n-color theorems

We say a graph can be *n-colored* if you can color its faces using n different colors such that adjacent regions are different colors. Most coloring theorems are about planar graphs.

The two-color theorem

Any planar graph where every vertex has even degree can be two-colored.

A three-color theorem (Grötszch 1959)

Planar graphs with no degree-three vertices can be three-colored.

The five-color theorem (Heawood 1890, based on Kempe 1879)

Any planar graph can be five-colored.

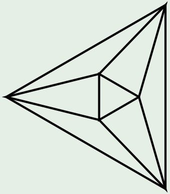
The four-color theorem (Appel-Haken 1976)

Any planar graph can be four-colored.

Definition/Theorem

The Euler characteristic of a graph is $V - E + F$. For planar graphs, $V - E + F = 2$.

Example



V=6

E=12

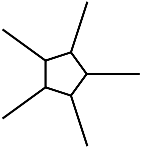
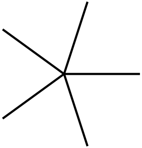
F=8

V-E+F=2

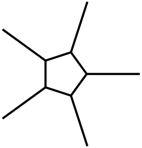
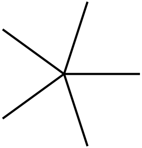
Corollary

Every planar graph has a face which is either a bigon, triangle, quadrilateral or pentagon.

Let's fail to prove the four-color theorem:

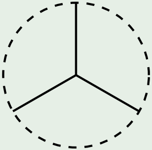
We first reduce it to a problem about trivalent graphs. If I can color  then I can color . So, replacing every degree-n vertex with a small n-gonal face doesn't change colorability.

Let's prove the five-color theorem:

We first reduce it to a problem about trivalent graphs. If I can color  then I can color . So, replacing every degree-n vertex with a small n-gonal face doesn't change colorability.

Any planar graph with boundary is a functional from a sequence of colors, to a number: how many ways are there to color in this graph so that the boundary colors are the given sequence?

Example



$$\{1, 2, 3\} \to 1$$
$$\{1, 2, 2\} \to 0$$
$$\{i, j, k\} \to \begin{cases} 1 & \text{if } i, j, k \text{ distinct;} \\ 0 & \text{else.} \end{cases}$$

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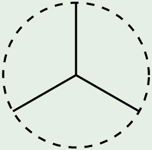

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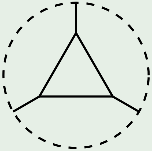
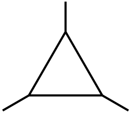
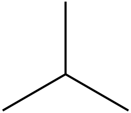
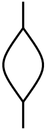
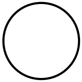
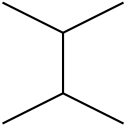
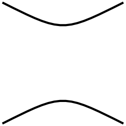
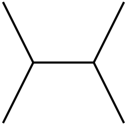
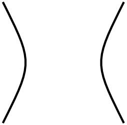


$$\{1,2,3\} \to n-3$$
$$\{1,2,2\} \to 0$$
$$\{i,j,k\} \to \begin{cases} n-3 & \text{if } i,j,k \text{ distinct;} \\ 0 & \text{else.} \end{cases}$$

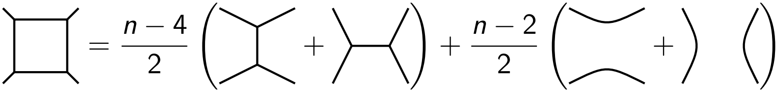
So,  $= (n-3)$  .

Similarly,  $= (n-2)$ and  $= (n-1)$.

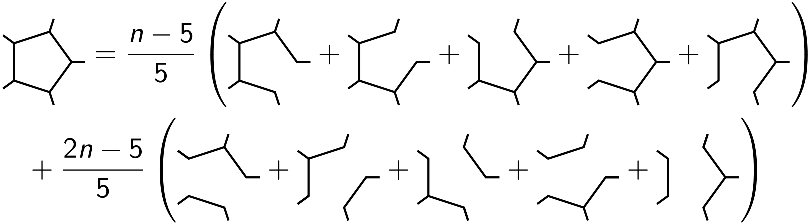
We also have a less obvious relation:

 $+$  $=$  $+$  .

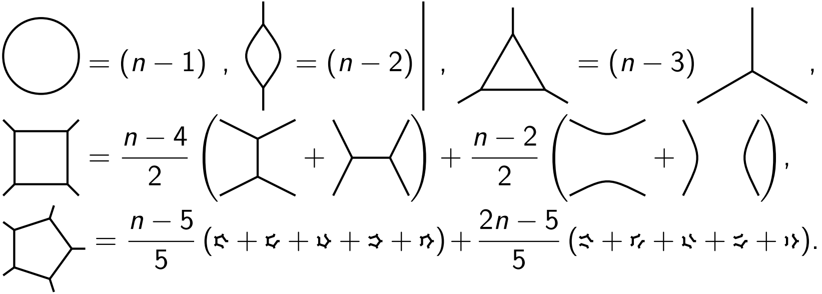
This last relation can be used to prove two more relations:



and



Proving the 5+-color theorem



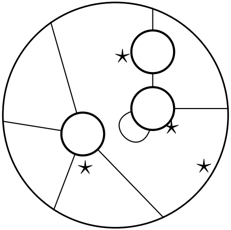
All these face-removing relations are positive for $n \geq 5$.
Any planar graph contains at least one circle, bigon, triangle, quadrilateral or pentagon (via Euler characteristic). So apply one of these positive relations and repeat until you have nothing left but a positive multiple of the empty diagram.

Planar algebras

Definition

A planar diagram has

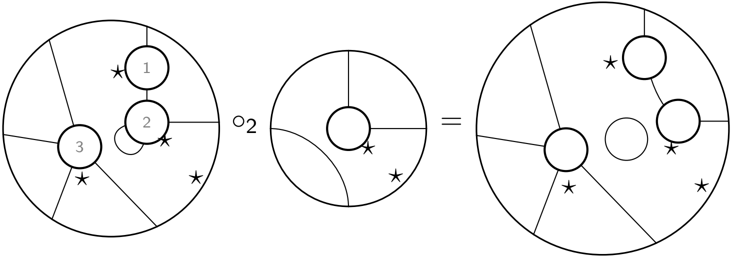
- *a finite number of inner boundary circles*
- *an outer boundary circle*
- *non-intersecting strings*
- *a marked point $\star$ on each boundary circle*



In normal algebra (the kind with sets and functions), we have one dimension of composition:

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

In planar algebras, we have two dimensions of composition

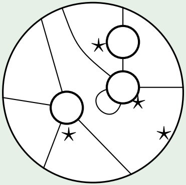


In abstract algebra, we often have a set whose structure is given by some functions. For example, a group is a set G with a multiplication law $\circ : G \times G \to G$.
A planar algebra also has sets, and maps giving them structure; there are a lot more of them.

Definition

A planar algebra is

- *a family of vector spaces V_k, $k = 0, 1, 2, \ldots$, and*
- *an interpretation of any planar diagram as a multi-linear map*

among V_i:  *:* $V_2 \times V_5 \times V_4 \to V_7$

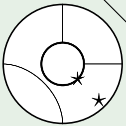
Definition

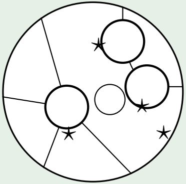
A planar algebra is

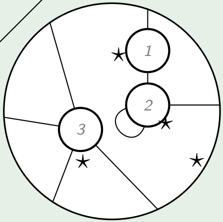
- *a family of vector spaces* V_k, $k = 0, 1, 2, \ldots$, *and*
- *Planar diagrams giving multi-linear map among* V_i,

such that composition of multilinear maps, and composition of diagrams, agree:

$$\begin{tikzcd} V_4 \times V_2 \times V_2 \arrow[rr] \arrow[dr] & & V_6 \\ & V_4 \times V_4 \times V_2 \arrow[ur] & \end{tikzcd}$$







First examples

Definition

A Temperley-Lieb diagram is a way of connecting up $2n$ points on the boundary of a circle, so that the connecting strings don't cross.

For example, TL_3: 

Example

The Temperley-Lieb planar algebra TL:

- The vector space TL_n has a basis consisting of all Temperley-Lieb diagrams on $2n$ points.
- A planar diagram acts on Temperley-Lieb diagrams by placing the TL diagrams in the input disks, joining strings, and replacing closed loops of string by $\cdot\delta$.

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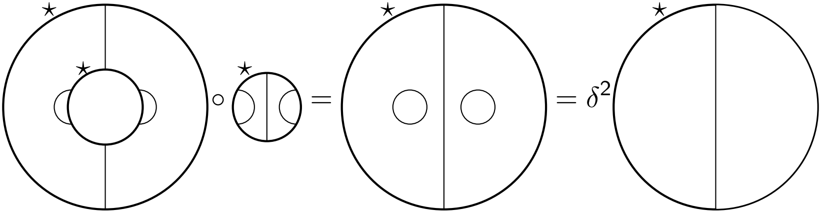
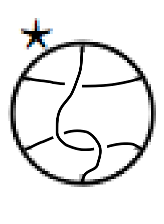
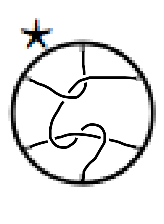
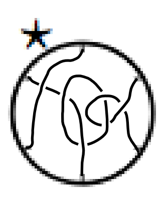


Definition

A tangle is a bunch of knotted strings whose endpoints are glued down around a circle.

 ,  ,  $, \in T_6$

Example

The planar algebra of knot tangles T:

- The vector space T_{2k} has a basis of tangles with $2k$ endpoints.
- a planar diagram acts on tangles by inserting them into the picture; the result is a new tangle.

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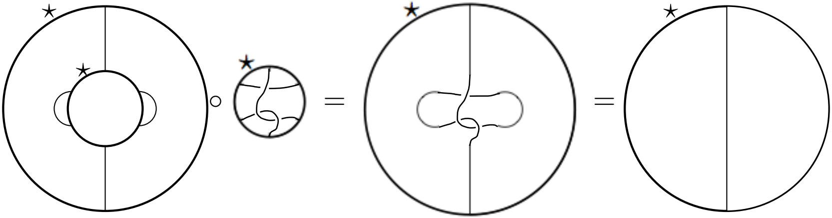

The planar algebra of knot tangles is generated, as a planar algebra, by a single crossing  subject to the Reidemeister relations.

(This is a planar algebras restatement of "knots are mostly planar, except where they cross; two diagrams are the same if we can get between them with the Reidemeister moves.")

The Kauffman bracket is a homomorphism of planar algebras between T and TL.

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Are there non-Reidemeister Reidemeister moves?

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Question

Are there non-Reidemeister Reidemeister moves? Can we define the planar algebra of knot tangles using the same generator and a different set of relations? Can we define the planar algebra of knot tangles using a different generator (and different relations)?

Operator Algebras

Linear algebra is the study of operators on finite dimensional vector spaces: matrices.

Operator algebra is the study of operators on infinite dimensional vector spaces. Such vector spaces are unweildly to say the least. We impose closure/completeness conditions on the vector spaces (*Hilbert spaces*) and also on the kinds of operators we look at (bounded).

A *von Neumann algebra* is a subalgebra of bounded operators on a Hilbert space which is closed in a given topology.

A *factor* is a highly non-commutative von Neumann algebra. The only n-by-n matrices which commute with all other n-by-n matrices are multiple of the identity. Similarly, the only operators in a factor which commute with all the other are multiples of the identity.

A *subfactor* is a pair of factors, one contained in the other.

Summary: a *s*ubfactor is a pair $A \subset B$, where A and B are (usually) infinite algebras, and both are 'as non-commutative as possible.'

Subfactors are big, and slippery. Just like with knots, invariants (if you can calculate them) are very useful.

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Question

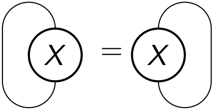
Emily, why would you study those things?

Answer

Factors have no ideals – making them 'non-commutative fields.' A subfactor, therefore, is a non-commutative analog of a field extension. The standard invariant of a subfactor is an analog to the Galois group of a field extension.

subfactor planar algebras

The standard invariant of a subfactor is a planar algebra $\mathcal{P}$ with some extra structure:

- $\mathcal{P}_0$ is one-dimensional
- All $\mathcal{P}_k$ are finite-dimensional
- Sphericality: 

- Inner product: each $\mathcal{P}_k$ has an adjoint $*$ such that the bilinear form $\langle x, y \rangle := yx^*$ is positive definite

Call a planar algebra with these properties a *subfactor planar algebra*.

Theorem (Jones, Popa)

Subfactors give subfactor planar algebras, and subfactor planar algebras give subfactors.

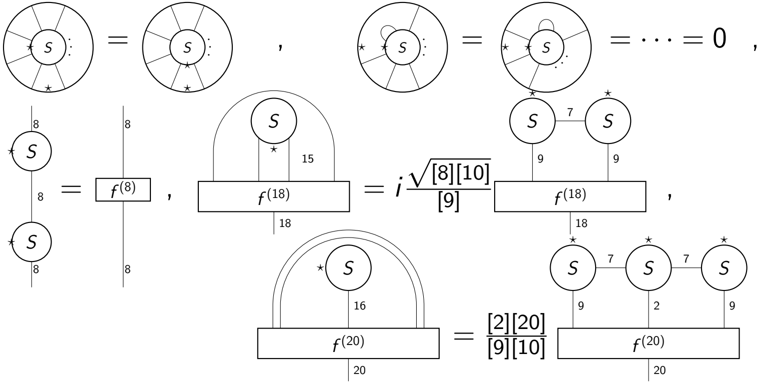
Example

Temperley-Lieb is a subfactor planar algebra if $\delta > 2$:

- P_0 is one dimensional
- $\dim(P_n) = c_n = \frac{1}{n+1}\begin{pmatrix} 2n \\ n \end{pmatrix}$
- circles are circles
- Positive definiteness is the difficulty, and the only place where $\delta > 2$ comes in.

The Extended Haagerup planar algebra

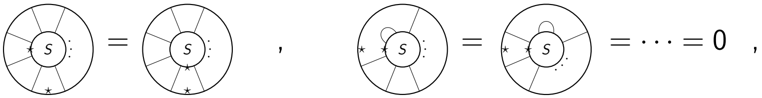
[Bigelow, Morrison, Peters, Snyder] The *extended Haagerup planar algebra* is the positive definite planar algebra generated by a single $S \in V_{16}$, subject to the relations

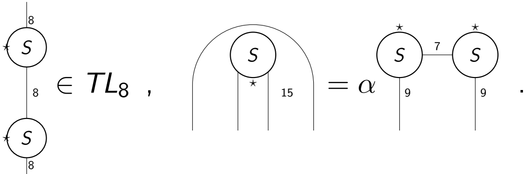


The extended Haagerup planar algebra is a subfactor planar algebra

The Extended Haagerup planar algebra redux

[Bigelow, Morrison, Peters, Snyder] The *extended Haagerup planar algebra* is the positive definite planar algebra generated by a single $S \in V_{16}$, subject to the relations





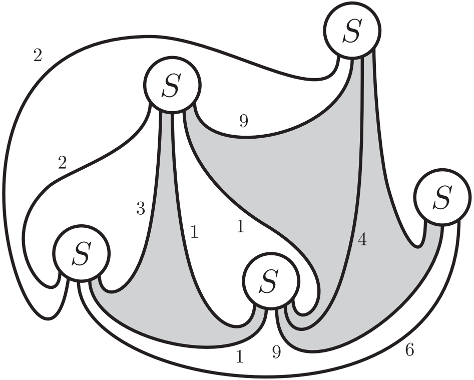
The extended Haagerup planar algebra is a subfactor planar algebra

Proving that the extended Haagerup generators and relations give a subfactor planar algebra: getting the size right is the hard part. Let V be the extended Haagerup planar algebra. How do we know $V \neq \{0\}$? How do we know $\dim(V_0) = 1$?

Showing that $V \neq \{0\}$ is technical and boring: It involves finding a copy of V inside a bigger planar algebra which we understand better.

$\dim(V_0) = 1$ means we can evaluate any closed diagram as a multiple of the empty diagram. The evaluation algorithm treats each copy of S as a 'jellyfish' and using the one-strand and two-strand substitute braiding relations to let each S 'swim' to the top of the diagram.

Begin with arbitrary planar network of Ss.



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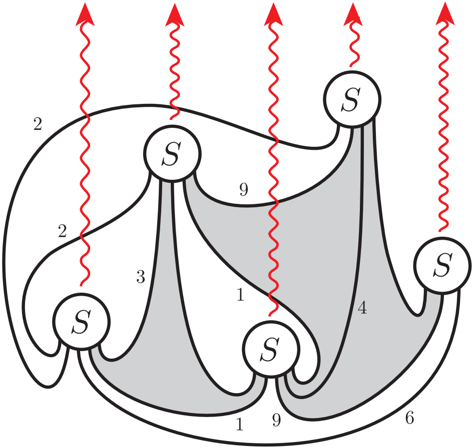

Now float each generator to the surface, using the relation.

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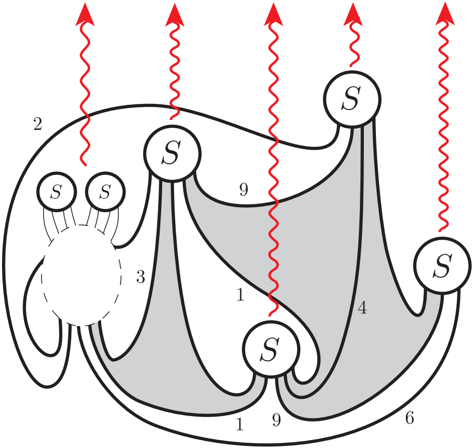

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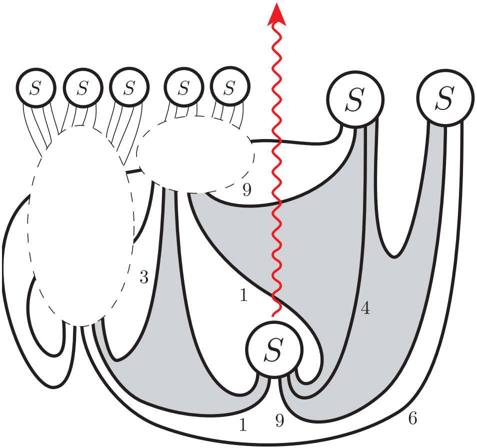

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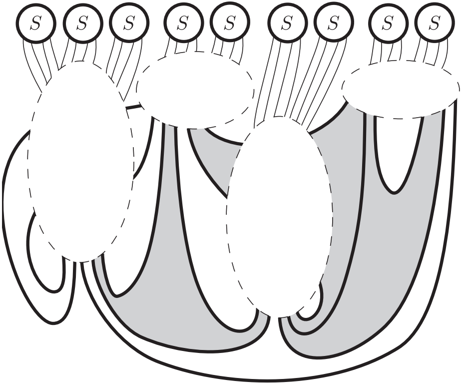
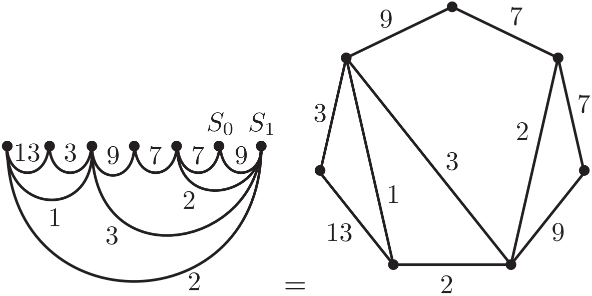


Now float each generator to the surface, using the relation.

The diagram now looks like a polygon with some diagonals, labelled by the numbers of strands connecting generators.



- Each such polygon has a corner, and the generator there is connected to one of its neighbours by at least 8 edges.
- Use $S^2 \in TL$ to reduce the number of generators, and recursively evaluate the entire diagram.

The End!