Non-commutative Galois theory and the classification of small-index subfactors.

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joint work with Bigelow, Morrison, Penneys, Snyder

November 30, 2011

Galois Theory

Theorem (Fundamental Theorem of Galois Theory)

Suppose L is a finite, Galois field extension of k. The intermediate subfields between L and k are in one-to-one correspondence with subgroups of the finite group $G = \mathrm{Aut}(L/k)$.

Corollary (Fundamental Corollary of Galois Theory)

A general degree-five polynomial is not solvable by radicals.

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Proof.

An equation over k is solvable by radicals if L, the field containing its solutions, can be reached by adjoining radicals one at a time:

$$k \subset K_1 \subset \cdots \subset K_n \subset L.$$

This is equivalent to the algebraic condition that $G = \mathrm{Aut}(L/k)$ is solvable, ie has a composition series in which all quotients G_i/G_{i-1} are cyclic groups of prime order:

$$1 \subset G_1 \subset \cdots \subset G_n \subset G.$$

But S_5 is not solvable. □

Subfactors of von Neumann algebras

Operator algebra is the study of operators on (usually infinite dimensional) vector spaces. Such vector spaces are unweildly to say the least. We impose closure/completeness conditions on the vector spaces (Hilbert spaces) and also on the kinds of operators we look at (bounded).

Definition

A von Neumann algebra is a unital subalgebra of bounded operators on a Hilbert space which is closed in a given topology.
A factor A is a highly non-commutative von Neumann algebra: $A \cap A' = \mathbb{C} \cdot 1$.
A subfactor is a pair of factors, one contained in the other: $1 \in A \subset B$.

Summary: a subfactor is a pair $A \subset B$, where A and B are (usually) infinite algebras, and both are 'as non-commutative as possible.'

Question

Emily, why would you study those things?

Answer

Factors have no 2-sided ideals. This means we can think of them as 'non-commutative fields.' A subfactor, therefore, is a non-commutative analog of a field extension.

Example

Given a factor R and a finite group G which acts outerly on R, their semidirect product is again a factor:

$$R \rtimes G = \{ (r, g) \mid (r, g) \cdot (s, h) = (r \cdot g(s), gh) \}.$$

Then $R \simeq R \rtimes 1 \subset R \rtimes G$ is a subfactor.

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Subfactor Invariants

In the galois theory analogy, the role of the automorphism group of a field extension is played by the 'standard invariant' of a subfactor. Subfactors have three important and related invariants:

Definition

The index of a subfactor $A \subset B$ is a real number $[B : A]$ between 1 and ∞.
The principal graphs of a subfactor are a pair of bipartite graphs. Vertices are irreducible bimodules over A and/or B, and edges describe behavior under $\otimes$.
The standard invariant of a subfactor is a graded algebra, with $+$, $\cdot$, and $\otimes$. (Actually, it's a tensor category).

Less precisely, the index measures relative size, and the principal graph and standard invariant describe how A sits inside B.

Definition

The <u>*index*</u> *of a subfactor* $A \subset B$ *is a real number* $[B : A]$ *between* 1 *and* ∞.

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The <u>*standard invariant*</u> *of a subfactor is a graded algebra, with* $+$, $\cdot$, *and* $\otimes$. *(Actually, it's a tensor category).*

From the standard invariant, you can compute the principal graphs by looking at idempotents.

From the principal graph, you can compute the index by looking at the graph norm (ie, the operator norm of the adjacency matrix).

Theorem (Jones)

The possible indices for a subfactor are

$$\{4\cos(\frac{\pi}{n})^2 | n \geq 3\} \cup [4, \infty].$$

Example: $R \rtimes H \subset R \rtimes G$

Again, let G be a finite group with subgroup H, and act outerly on a factor R. Consider $A = R \rtimes H \subset R \rtimes G = B$. Then $[B : A] = [G : H] = \frac{|G|}{|H|}$.

The dual principal graph of $A \subset B$ is the induction-restriction graph for irreducible representations of H and G.

Example ($S_3 \leq S_4$)

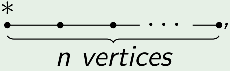
trivial standard sign

trivial standard V sign$\otimes$standard sign

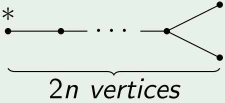
(The principal graph is an induction-restriction graph too, for H and various subgroups of H.)

Theorem (Jones, Ocneanu, Kawahigashi, Izumi, Bion-Nadal)

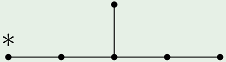
The principal graph of a subfactor of index less than 4 is one of

$A_n =$ 

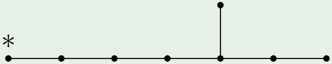
$, n \geq 2$ *index* $4\cos^2(\frac{\pi}{n+1})$

$D_{2n} =$ 

$, n \geq 2$ *index* $4\cos^2(\frac{\pi}{4n-2})$

$E_6 =$ 

index $4\cos^2(\frac{\pi}{12}) \approx 3.73$

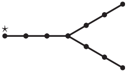
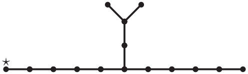
$E_8 =$ 

index $4\cos^2(\frac{\pi}{30}) \approx 3.96$

Theorem (Popa)

The principal graphs of a subfactor of index 4 are extended Dynkin diagrams.

Haagerup's list

- In 1993 Haagerup classified possible principal graphs for subfactors with index between 4 and $3 + \sqrt{3} \approx 4.73$:
  - , , , $\ldots$, (≈ 4.30, 4.37, $4.38, \ldots$)
  - , (≈ 4.56)
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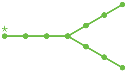
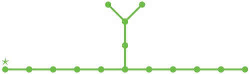
  - , , , $\ldots$, (≈ 4.30, 4.37, $4.38, \ldots$)
  - , (≈ 4.56)
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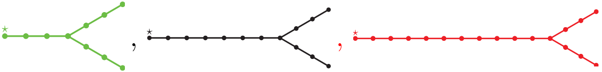
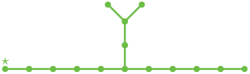
  -  , . . . , (≈ 4.30, 4.37, $4.38, \ldots$)
  -  , (≈ 4.56)
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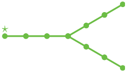
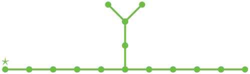
  - , , , . . ., ($\approx 4.30,\ 4.37,\ 4.38, \ldots$)
  - , (≈ 4.56)
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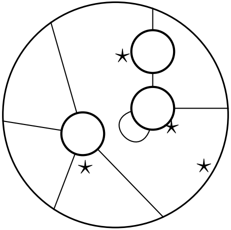
- Haagerup and Asaeda & Haagerup (1999) constructed two of these possibilities; the first exotic subfactors.
- Bisch (1998) and Asaeda & Yasuda (2007) ruled out infinite families.
- In 2009 we (Bigelow-Morrison-Peters-Snyder) constructed the last missing case. `arXiv:0909.4099`

Definition

A planar diagram has

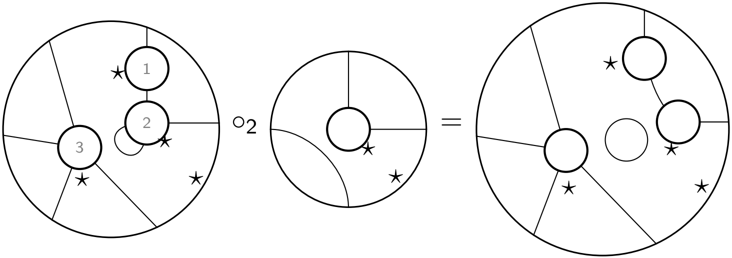
- *a finite number of inner boundary circles*
- *an outer boundary circle*
- *non-intersecting strings*
- *a marked point $\star$ on each boundary circle*



In normal algebra (the kind with sets and functions), we have one dimension of composition:

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

In planar algebras, we have two dimensions of composition:



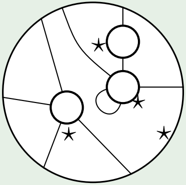
In ‘normal algebra,’ we have a set and some functions which give it structure. For example, a group is a set G with a multiplication law $\circ : G \times G \to G$.

A planar algebra also has sets, and maps giving them structure; there are a lot more of them.

Definition

A planar algebra is

- *a family of vector spaces V_k, $k = 0, 1, 2, \ldots$, and*
- *an interpretation of any planar diagram as a multi-linear map*

among V_i:  $: V_2 \times V_5 \times V_4 \to V_7$

Definition

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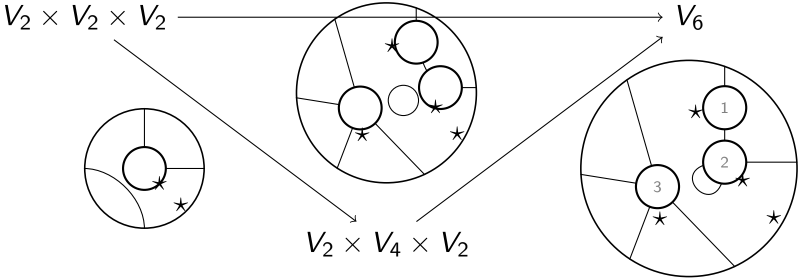
such that composition of multilinear maps, and composition of diagrams, agree.



Definition

A Temperley-Lieb diagram is a way of connecting up $2n$ points on the boundary of a circle, so that the connecting strings don't cross.

For example, TL_3: 

Example

The Temperley-Lieb planar algebra:

- The vector space TL_n has a basis consisting of all Temperley-Lieb diagrams on $2n$ points.
- A planar diagram acts on Temperley-Lieb diagrams by placing the TL diagrams in the input disks, joining strings, and replacing closed loops of string by $\cdot\delta$.

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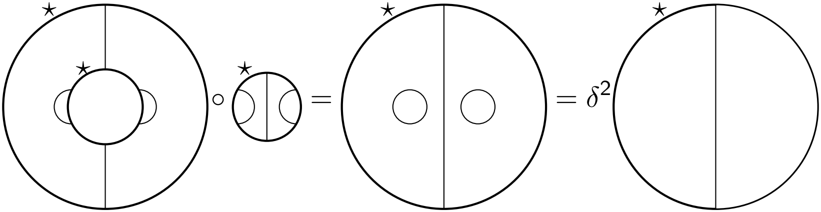


The standard invariant again

Let $X = {}_A B_B$ and $\overline{X} = {}_B(B^{op})_A$, and $\otimes = \otimes_A$ or $\otimes_B$ as needed.

Definition

The standard invariant of $A \subset B$ is the planar algebra of endomorphisms of powers of X:

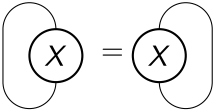
$$\mathrm{End}(1) \subset \mathrm{End}(X) \subset \mathrm{End}(X \otimes \bar{X}) \subset \mathrm{End}(X \otimes \bar{X} \otimes X) \subset \cdots$$

$$\mathrm{End}(1) \subset \mathrm{End}(\bar{X}) \subset \mathrm{End}(\bar{X} \otimes X) \subset \mathrm{End}(\bar{X} \otimes X \otimes \bar{X}) \subset \cdots$$

The planar algebra structure comes from labelling strings with X's, and inserting elements of $\mathrm{End}(X^{\otimes n})$ into disks with $2n$ strings.

subfactor planar algebras

The standard invariant $\mathcal{P}$ of a subfactor has some extra structure:

- $\mathcal{P}_0$ is one-dimensional
- All $\mathcal{P}_k$ are finite-dimensional
- Sphericality: 

- Inner product: each $\mathcal{P}_k$ has an adjoint $*$ such that the bilinear form $\langle x, y \rangle := yx^*$ is positive definite

Call a planar algebra with these properties a <u>subfactor planar algebra</u>.

Example

Temperley-Lieb is a subfactor planar algebra.

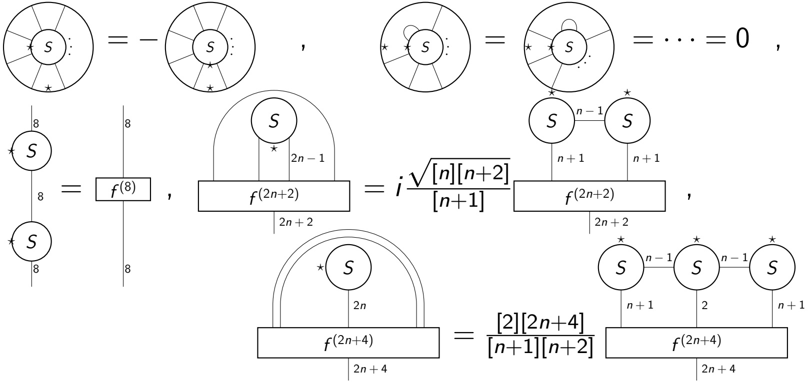
Theorem (Jones, Popa)

Subfactors give subfactor planar algebras, and subfactor planar algebras give subfactors.

This is part of why 'non-commutative galois theory' is a good way to think about subfactors.

This also gives us a new way to construct subfactors, by giving a generators-and-relations presentation of the associated planar algebra.

[Bigelow, Morrison, Peters, Snyder] The extended Haagerup planar algebra is the positive definite planar algebra generated by a single $S \in V_{16}$, subject to the relations



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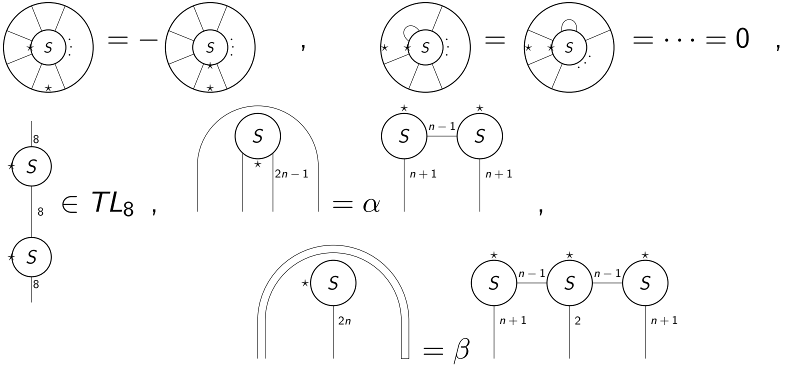


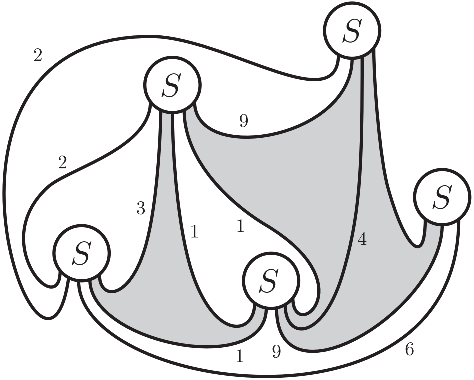
The extended Haagerup planar algebra is a subfactor planar algebra!

Proving that the extended Haagerup generators and relations give a subfactor planar algebra: getting the size right is the hard part. Let $\mathcal{P}$ be the extended Haagerup planar algebra. How do we know $\mathcal{P} \neq \{0\}$? How do we know $\dim(\mathcal{P}_0) = 1$?

Showing that $\mathcal{P} \neq \{0\}$ is technical and boring: It involves finding a copy of $\mathcal{P}$ inside a bigger planar algebra which we understand better.

$\dim(\mathcal{P}_0) = 1$ means we can evaluate any closed diagram as a multiple of the empty diagram. The evaluation algorithm treats each copy of S as a 'jellyfish' and uses the one-strand and two-strand substitute braiding relations to let each S 'swim' to the top of the diagram.

Begin with arbitrary planar network of Ss.



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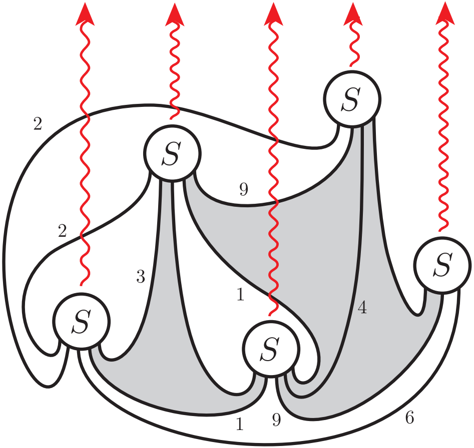

Now float each generator to the surface, using the relation.

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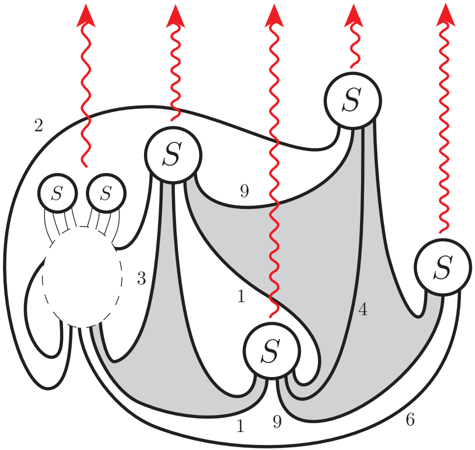

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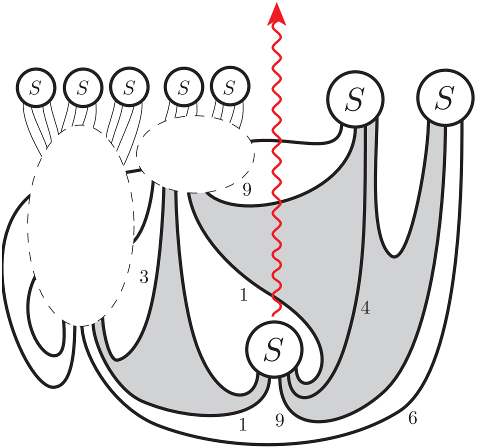

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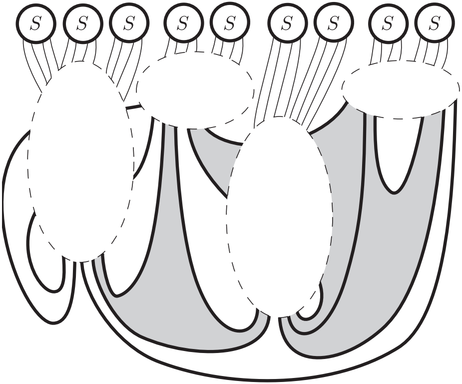
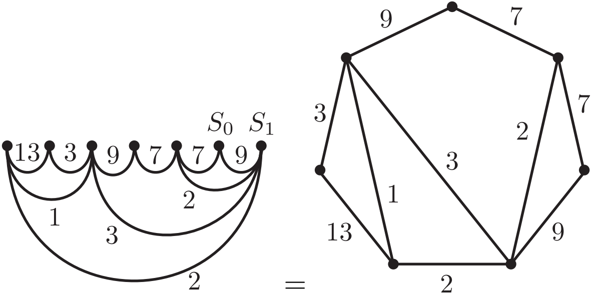
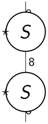


Now float each generator to the surface, using the relation.

The diagram now looks like a polygon with some diagonals, labelled by the numbers of strands connecting generators.



- Each such polygon has a corner, and the generator there is connected to one of its neighbours by at least 8 edges.
- Use 

$\in TL_8$ to reduce the number of generators, and recursively evaluate the entire diagram.

The classification is again in terms of principal graphs.

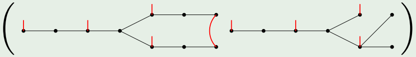
Definition

The vertices of a principal graph pair are (isomorphism classes of) minimal projections in the standard invariant: Recall that X is the bimodule ${}_A B_B$; the vertices are $q \in \mathrm{End}(X^{\otimes n})$ such that $q^2 = q, q^* = q$.

In the standard invariant, there are four kinds of bimodules: $A - A$, $A - B$, $B - A$ and $B - B$. The principal graph has $A - A$ and $A - B$ projections, and two projections q and q' are connected by an edge if $q < q' \otimes p_X$.

The dual principal graph has $B - A$ and $B - B$ projections, and two projections q and q' are connected by an edge if $q < q' \otimes p_X$.

Example (The Haagerup subfactor's principal graph pair)



Which pairs can go together? The vertices of a principal graph are (isomorphism classes of) projections in $\mathrm{End}(X^{\otimes n})$

- The graphs must have the same graph norm;
- The graphs depths can differ by at most 1;
- The pair must satisfy an associativity test:

$$(p_X \otimes q) \otimes p_X \cong p_X \otimes (q \otimes p_X)$$

A computer can efficiently enumerate such pairs with index below some number L up to a given rank or depth, obtaining a collection of allowed vines and weeds.

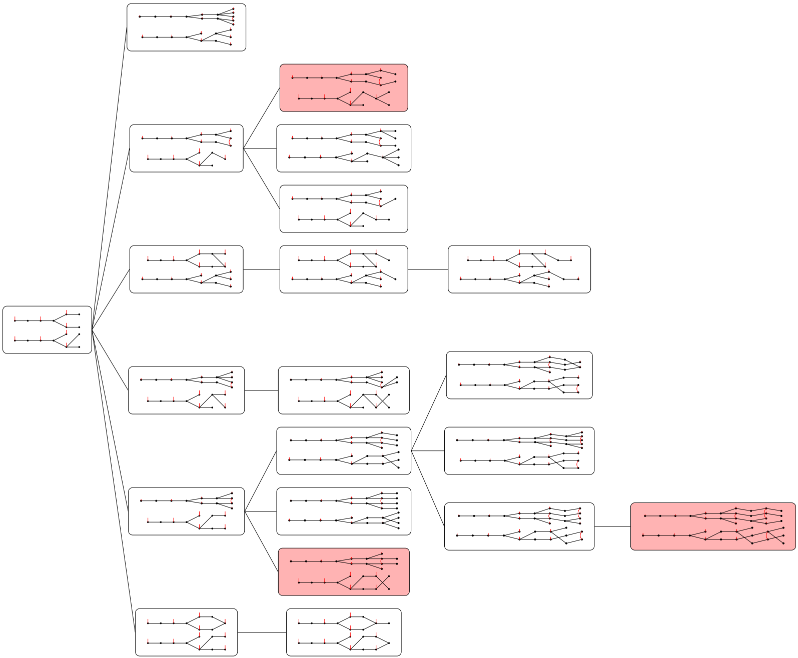
Definition

A vine represents an integer family of principal graphs, obtained by <u>translating</u> the vine.

Definition

A weed represents an infinite family, obtained by either translating or <u>extending</u> arbitrarily on the right.

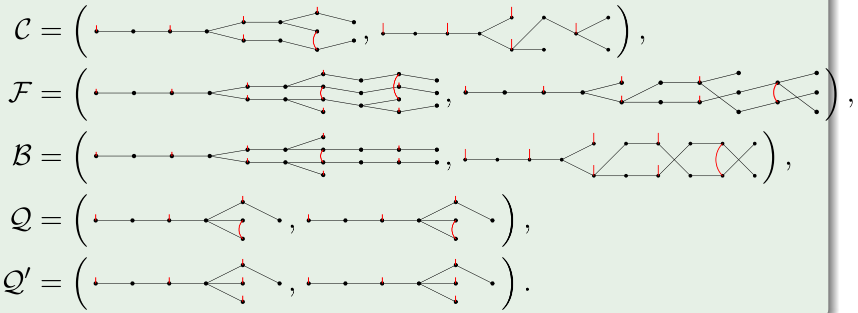
We can hope that as we keep extending the depth, a weed will turn into a set of vines. If all the weeds disappear, the enumeration is complete. This happens in favorable cases (e.g. Haagerup's theorem up to index $3 + \sqrt{3}$), but generally we stop with some surviving weeds, and have to rule these out 'by hand'.



The classification up to index 5

Theorem (Morrison-Snyder, part I, `arXiv:1007.1730`)

Every (finite depth) II_1 subfactor with index less than 5 sits inside one of 54 families of vines (see below), or 5 families of weeds:



Theorem (Morrison-Penneys-P-Snyder, part II, `arXiv:1007.2240`)

Using planar algebra techniques, there are no subfactors in the families $\mathcal{C}$, $\mathcal{F}$ or $\mathcal{B}$.

Theorem (Izumi-Jones-Morrison-Snyder, part III, `arXiv:1109.3190`)

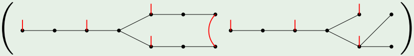
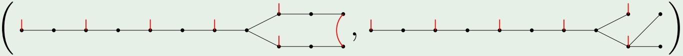
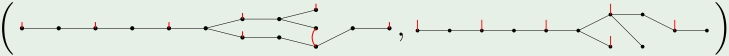
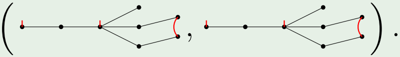
There are no subfactors in the families $\mathcal{Q}$ or $\mathcal{Q}'$.

Theorem (Calegari-Morrison-Snyder, `arXiv:1004.0665`)

In any family of vines, there are at most finitely many subfactors, and there is an effective bound.

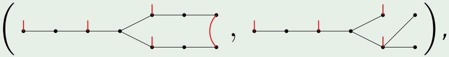
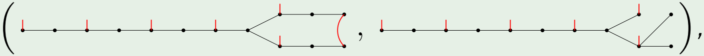
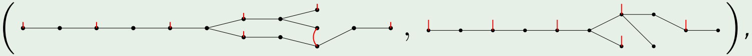
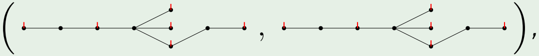
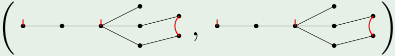
Corollary (Penneys-Tener, part IV, `arXiv:1010.3797`)

There are only four possible principal graphs of subfactors coming from the 54 families

- 
- 
- 
- 

Theorem

There are exactly ten subfactors other than Temperley-Lieb with index between 4 *and* 5*.*

- 
- 
- 
- *The 3311 GHJ subfactor (MR999799), with index* $3+\sqrt{3}$ 
- *Izumi's self-dual 2221 subfactor (MR1832764), with index* $\frac{5+\sqrt{21}}{2}$ 

along with the non-isomorphic duals of the first four, and the non-isomorphic complex conjugate of the last.

Theorem (Izumi)

The only subfactors with index exactly 5 *are group-subgroup subfactors:*

- $1 \subset \mathbb{Z}_5$;
- $\mathbb{Z}_2 \subset D_{10}$;
- $\mathbb{F}_5^\times \subset \mathbb{F}_5 \rtimes \mathbb{F}_5^\times$;
- $A_4 \subset A_5$;
- $S_4 \subset S_5$.

Index beyond 5

Somewhere between index 5 and index 6, things get wild:

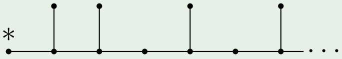
Theorem (Bisch-Nicoara-Popa)

At index 6, there is an infinite one-parameter family of subfactors (of the same factor) having isomorphic standard invariants.

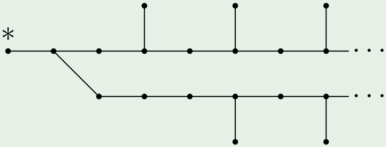
and

Theorem (Bisch-Jones)

$A_2 * A_3$ *is an infinite depth subfactor at index* $2\tau^2 \sim 5.23607$.

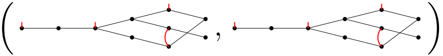


,

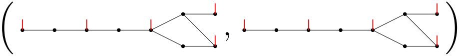
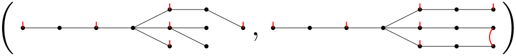
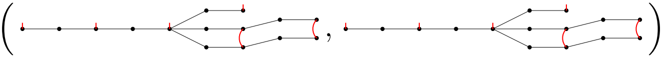
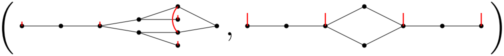


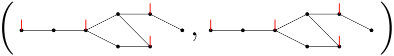
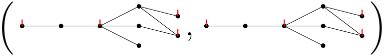
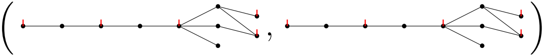
Classification above index 5 looks hard, but we can still fish for examples!

Here are some graphs that we find. (A few are previously known)

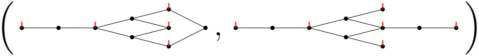
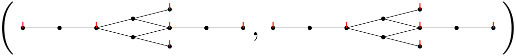
- 
  (from $SU_q(3)$ at a root of unity, index ~ 5.04892)

At index $2\tau^2 \sim 5.23607$

- 
- 
- 
- 

- 
  (“Haagerup +1” at index $\frac{7+\sqrt{13}}{2} \sim 5.30278$)
-  at
  $\frac{1}{2}\left(4+\sqrt{5}+\sqrt{15+6\sqrt{5}}\right) \sim 5.78339$
-  at
  $3+2\sqrt{2} \sim 5.82843$

And at index 6

- 
- 

and several more!

The End!