RESEARCH STATEMENT

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My research is in the field of combinatorial and geometric representation theory. It mainly concerns Kashiwara's crystals, which are combinatorial objects related to Lie algebras (and more generally symmetrizable Kac-Moody algebras) and their representations. Studying Kashiwara's crystals directly from definitions can be cumbersome, but they can often be realized by other means. These realizations can be purely combinatorial, or can involve more sophisticated tools like quiver varieties. They can give insight into the representation theory, and can reveal connections with other areas.

My largest and most active current project concerns a realization of crystals in terms of certain Mirković-Vilonen (MV) polytopes. This realization was originally only defined in finite type. Recently, Pierre Baumann, Joel Kamnitzer and I [2] proposed a definition of MV polytopes in all symmetric affine types, and showed that these have numerous properties in common with the finite-type situation; in particular, they can be used to realize the correct affine crystals. This raises several interesting questions, mainly involving generalizing results concerning finite type MV polytopes to the affine situation. For instance finite-type MV polytopes can be interpreted in terms of at least two geometric models (MV cycles in the affine grassmannian and Lusztig/Nakajima quiver varieties), and can also be understood algebraically using Lusztig's PBW bases. In [2] we show how the affine polytopes relate to quiver varieties, but do not develop a connection with affine MV cycles or with affine PBW bases, both of which do exist. I plan to investigate those connections. I also plan to investigate MV polytopes for Kac-Moody algebras which are not of symmetric affine type. In the case of the non-symmetric affine types, there is now a clear conjecture as to what the correct polytopes should be, and some ideas that may lead to a proof.

What follows is not an exhaustive list of my research. For instance, I have been involved with a project studying the relationship between the braiding on the category of representations of a quantum group, and something called the crystal commutor on the corresponding category of crystals, and authored or coauthored five papers [13, 14, 21, 24, 25] motivated by these ideas. I am also currently working with Steven Sam (a graduate student at MIT) on a project producing combinatorial realizations of crystals from torus actions on quiver varieties, which we expect will give a conceptual explanation for, and also generalize, a family of crystals introduced a few years ago by Fayers [6]. I will not describe these projects here, but will instead focus on my most active current work.

Section 1 contains some background and history related to my work. Section 2 discusses the project on affine MV polytopes. Section 3 discusses a smaller current research project, which I include mainly because it leads to some very approachable open questions that could be explored by younger researchers (either undergraduates or beginning graduate students).

1. History and background

Simple finite-dimensional Lie algebras have been studied for at least a century and have many important applications. More recently, Kac and Moody introduced a generalization of these algebras, now known as Kac-Moody algebras (see [12]).

Associated to a Kac-Moody algebra $\mathbf{g}$ is its universal enveloping algebra, an associative algebra $U(\mathbf{g})$ with the same representation theory as $\mathbf{g}$ itself. In the 1980s, people began studying the quantized universal enveloping algebra $U_q(\mathbf{g})$, which is roughly a 1-parameter deformation of $U(\mathbf{g})$. This was first introduced in a mathematically precise way by Drinfel'd [5] and Jimbo [11], although many of the ideas date to earlier work in mathematical physics. The representation theory of $U_q(\mathbf{g})$ for generic q is very similar to that of $\mathbf{g}$ itself, but studying the q-deformed version often gives new insight.

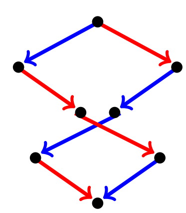
The algebra $U_q(\mathbf{g})$ has applications in diverse fields, ranging from topology (through knot invariants and 3-manifold invariants, see e.g. [26]) to statistical mechanics (where solutions to the quantum Yang-Baxter equation allow for explicit calculations of correlation functions, see e.g. [8, Chapter 9]). For the research discussed here, the importance of $U_q(\mathbf{g})$ is mostly that it led to Kashiwara's construction of crystals.

Much of my work concerns the affine Kac-Moody algebras. Many of these are two dimensional extensions of loop algebras for the finite type algebras,

$$\hat{\mathbf{g}} \cong \mathbf{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

The others are obtained from these by an operation of "twisting" by an automorphism of a finite type Dynkin diagram. Affine Kac-Moody algebras are particularly interesting for a number of reasons. For instance, they can be used to construct conformal field theories (see e.g. [7]), which are important in mathematical physics. They are also the most relevant cases for the statistical mechanics applications mentioned above.

1.1. **Crystals.** Crystal bases were developed by Kashiwara in the early 1990s (see [15]). For any highest weight representation $V(\lambda)$ of $U_q(\mathbf{g})$, Kashiwara defines new operators $\tilde{E}_i, \tilde{F}_i$ modifying the actions of the usual Chevalley generators E_i and F_i. Every integrable highest weight representation $V(\lambda)$ of $U_q(\mathbf{g})$ has a basis B such that the span L of B over the local ring $\mathbb{C}[q]_0$ is preserved by $\tilde{E}_i$ and $\tilde{F}_i$, and such that the residues e_i, f_i of these operators at $q = 0$ act by partial permutations on the image $B(\lambda)$ of B in L/qL. The result is combinatorial data called the crystal: the discrete set $B(\lambda)$ along with the partial permutations e_i, f_i. This can be depicted as a colored directed graph, where the underlying set forms the vertex set for the graph, and one puts an i colored arrow from b to b' if and only if $f_i(b) = b'$. For instance, in the case of the adjoint representation for $\mathfrak{sl}_3$, using red arrows to denote f_1 and blue arrows to denote f_2, one obtains



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There is a unique (up to isomorphism) crystal $B(\lambda)$ for each irreducible highest weight representation $V(\lambda)$. These form a directly system, which has a limit $B(\infty)$.

Crystals can often be realized in a purely combinatorial way, and these realizations give information about the original representations. For instance, there is a simple combinatorial tensor product rule for crystals which can be used to find tensor product multiplicities for the corresponding representations.

1.2. **PBW bases and the MV polytope realization.** There are many realizations of Kashiwara's crystals. For instance, in the case of $\mathfrak{sl}_n$ there is a well-known construction where the underlying set of $B(\lambda)$ consists of semi-standard Young tableau of a fixed shape. Here I will describe a combinatorial realization based on certain Mirković-Vilonen (MV) polytopes, and how this is related to Lusztig's algebraic theory of PBW bases.

For now, assume $\mathbf{g}$ is of finite type. Choose any basis $\{v_1, \ldots, v_N\}$ for $\mathbf{g}^-$ (the part of $\mathbf{g}$ generated by only the Chevalley generators F_i). Then the set of monomials

$$\{v_1^{a_1} v_2^{a_2} \cdots v_N^{a_N} \mid a_1, \ldots, a_N \in \mathbb{Z}_{\geq 0}\}$$

is a basis for $U^-(\mathbf{g})$, known as the PBW basis. It requires some care to define such a construction for $U_q(\mathbf{g})$, as $\mathbf{g}^-$ is not naturally contained in $U_q^-(\mathbf{g})$, but in finite type it can be done. For each reduced expression $w_0 = s_{i_1} s_{i_2} \ldots s_{i_N}$ for the longest element of the Weyl group for $\mathbf{g}$, Lusztig defines "root vectors" $F_{\alpha_{i_1}}, F_{s_{i_1}\alpha_{i_2}}, F_{s_{i_1}s_{i_2}\alpha_{i_3}}, \ldots, F_{s_{i_1}\cdots s_{i_{N-1}}\alpha_{i_N}} \in U_q^-(\mathbf{g})$. This gives a collection of elements in $U_q^-(\mathbf{g})$, which in the limit at $q=1$ form a weight basis of $\mathbf{g}^- \subset U^-(\mathbf{g})$. Lusztig's PBW basis elements are the monomials

$$F^{(n_N)}_{s_{i_1}\cdots s_{i_{N-1}}\alpha_{i_N}} \cdots F^{(n_3)}_{s_{i_1}s_{i_2}\alpha_{i_3}} F^{(n_2)}_{s_{i_1}\alpha_{i_2}} F^{(n_1)}_{\alpha_{i_1}}$$

for $n_1, n_2, \ldots, n_N \in \mathbb{Z}_{\geq 0}$. Here the superscript (n) means the quantum divided power,

$$X^{(n)} = \frac{X^n}{[n][n-1]\cdots[2]} \text{ where } [k] = q^{k-1} + q^{k-3} + \cdots + q^{-k+1}.$$

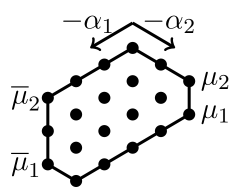
It turns out that Lusztig's PBW basis is a crystal basis $U_q^-(\mathbf{g})$, and hence parameterizes $B(\infty)$. This associates to each $b \in B(\infty)$ a collection of monomials, one for each reduced expression for w_0. This collection defines the MV polytope. To explain how, consider as an example $\mathbf{g} = \mathfrak{sl}_3$. In this case there are exactly two reduced expressions for w_0:

$$\mathbf{i}_1 := s_1 s_2 s_1 \quad \text{and} \quad \mathbf{i}_2 := s_2 s_1 s_2.$$

Using a superscript of $\mathbf{i}_1, \mathbf{i}_2$ to denote the expression for w_0 being used, one finds that, e.g.,

$$(F^{\mathbf{i}_1}_{\alpha_2})^{(1)}(F^{\mathbf{i}_1}_{\alpha_1+\alpha_2})^{(2)}(F^{\mathbf{i}_1}_{\alpha_1})^{(3)} = (F^{\mathbf{i}_2}_{\alpha_1})^{(4)}(F^{\mathbf{i}_2}_{\alpha_1+\alpha_2})^{(1)}(F^{\mathbf{i}_2}_{\alpha_2})^{(2)} \mod q,$$

and thus these are expressions for the same $b \in B(\infty)$. These two monomials define a polygon in weight space $\mathfrak{h}^*$, where the lengths of the edges along the left side record the exponents in the $\mathbf{i}_1$ monomial, and the edges on the right side record the exponents of the $\mathbf{i}_2$ monomial:



More generally, every expression for w_0 will give a path in weight space, and the union of these paths is the 1-skeleton of the MV polytope P_b.

It is natural to ask which polytopes are MV polytopes. That is, which polytopes occur as P_b for some $b \in B(\infty)$? In rank-2 cases, the answer is given in terms of certain tropical Plücker relations. For $\widehat{\mathfrak{sl}}_3$, these are equivalent to the conditions

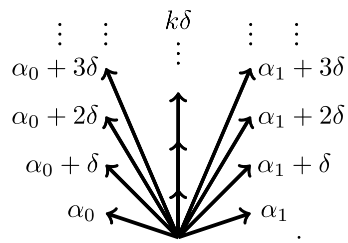
(i) $(\mu_2 - \overline{\mu}_1, \omega_2) \leq 0$,
(ii) $(\overline{\mu}_2 - \mu_1, \omega_1) \leq 0$, and
(iii) At least one of these is an equality,

where the vertices are as in the above figure and ω_1, ω_2 are the two fundamental weights. Remarkably, this is enough to understand all cases: in general, a polytope is an MV polytope for $\mathbf{g}$ if and only if all 2-faces are MV polytopes of the correct type. This gives a combinatorial characterization of the polytopes, independent of Lusztig's PBW bases. One can also give combinatorial definition of the crystal operators f_i, so this does lead to a purely combinatorial description of $B(\infty)$.

2. Mirković-Vilonen polytopes in affine type

In recent and ongoing work with Pierre Baumann and Joel Kamnitzer, we give a realization of $B(\infty)$ in symmetric affine types via an analogue of MV polytopes. For $\widehat{\mathfrak{sl}}_2$ this is done combinatorially [1]. For the other symmetric affine types, we make use of a geometric model for $B(\infty)$ in terms of Lusztig's quiver varieties, but in the end obtain a combinatorial definition (see [2]).

2.1. $\widehat{\mathfrak{sl}}_2$ **MV polytopes.** The $\widehat{\mathfrak{sl}}_2$ root system has two simple roots α_0 and α_1. The positive roots are $\alpha_0 + k\delta$ and $\alpha_1 + k\delta$ for all $k \geq 0$, and $j\delta$ for $j \geq 1$, where $\delta = \alpha_0 + \alpha_1$. These can be drawn as



There are infinitely many roots, but they can be arranged in only three lines. We define an $\widehat{\mathfrak{sl}}_2$ GGMS polytope to be a convex polytope in $\mathrm{span}_{\mathbb{R}}\{\alpha_0, \alpha_1\}$ such that all edges are integer multiples of the roots, as drawn above. We can now give the definition of $\widehat{\mathfrak{sl}}_2$ MV polytopes.

Definition 2.1. *An $\widehat{\mathfrak{sl}}_2$ MV polytope P is a triple $(P, \lambda, \overline{\lambda})$ of an $\widehat{\mathfrak{sl}}_2$ GGMS polytope and two partitions such that, for the vertices $\mu_k, \mu^k, \overline{\mu}_k, \overline{\mu}^k, \mu_\infty, \mu^\infty, \overline{\mu}_\infty, \overline{\mu}^\infty$ as in Figure 1,*

(i) *λ is a partition of $(\mu^\infty - \mu_\infty, \omega_1)$ and $\overline{\lambda}$ is a partition of $(\overline{\mu}^\infty - \overline{\mu}_\infty, \omega_1)$*
(ii) *For each $k \geq 1$, $(\overline{\mu}_k - \mu_{k-1}, \omega_1) \leq 0$ and $(\mu_k - \overline{\mu}_{k-1}, \omega_0) \leq 0$, with at least one of these being an equality.*
(iii) *For each $k \geq 1$, $(\overline{\mu}^k - \mu^{k-1}, \omega_0) \geq 0$ and $(\mu^k - \overline{\mu}^{k-1}, \omega_1) \geq 0$, with at least one of these being an equality.*
(iv) *Either $\lambda = \overline{\lambda}$, or λ is obtained from $\overline{\lambda}$ by adding or removing a single part of size $(\mu_\infty - \overline{\mu}_\infty, \alpha_1)/2$ (i.e. the width of the polytope).*
(v) $\lambda_1, \overline{\lambda}_1 \leq (\mu_\infty - \overline{\mu}_\infty, \alpha_1)/2$.

We think of λ and $\overline{\lambda}$ as being associated to the two vertical edges of the polytope.

Theorem 2.2. *(Baumann-Dunlap-Kamnitzer-Tingley [1]) the set of $\widehat{\mathfrak{sl}}_2$ MV polytopes, along with some explicitly defined combinatorial crystal operators e_0, e_1, f_0, f_1, is a realization of $B(\infty)$.*

The operators are not complicated. For example, f_0 increases the length of the top edge on the right side of a polytope P, and leaves the rest of the right side unchanged. There is a unique MV polytope with this new right side, and that is $f_0(P)$.

We prove Theorem 2.2 by a combinatorial argument using Kashiwara and Saito's characterization of $B(\infty)$ in terms of $*$-involution [17]. The $*$-involution on $B(\infty)$ becomes rotation by 180 degrees for polytopes.

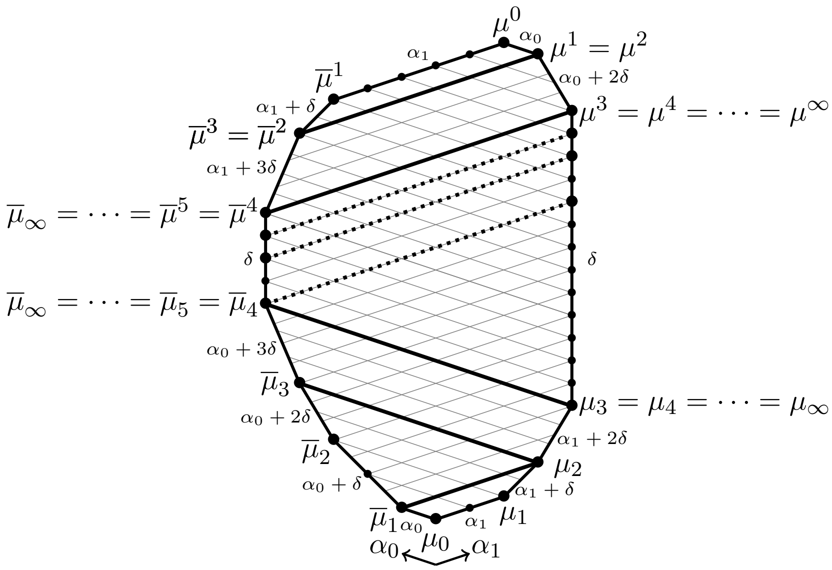


FIGURE 1. An $\widehat{\mathfrak{sl}}_2$ MV polytope. The small labels indicate the roots parallel to each edge. The bold diagonals point in directions α_0 or α_1. These are the diagonals where the inequalities from Definition 2.1 hold with equality. All the quadrilaterals obtained by cutting the polytope along these diagonals are themselves MV polytopes. In this example $\lambda = (9, 2, 1, 1)$ and $\overline{\lambda} = (2, 1, 1)$. We denote these by putting extra vertices on the edges parallel to δ. The conditions on λ and $\overline{\lambda}$ imply that one can draw in extra diagonals (shown with dotted lines) joining these vertices which are also parallel to α_1 (or in some examples to α_0).

2.2. **Other symmetric affine MV polytopes.** As in the $\widehat{\mathfrak{sl}}_2$ case, we define a GGMS polytope to be a convex polytope all of whose edges are integer multiples of positive roots in the affine root system. Once again, we need to attach extra decoration to each edge of the polytope parallel to δ. This time, we need to associate an n (= rank $\mathbf{g}$) tuple of partitions to each such edge, which is indexed by n chamber weights of the underlying finite type root system. A 2-face can have at most 2 edges parallel to δ; on each such a face, the label sets on those two edges differ in exactly one position. We insist that the partitions corresponding to the other $n-1$ labels agree on the two sides of each such a face.

We define an affine MV polytope to be a decorated GGMS polytope such that all 2-faces are MV polytopes of the correct type. For faces not parallel to the imaginary roots, this means of type $\mathfrak{sl}_3$ or $\mathfrak{sl}_2 \times \mathfrak{sl}_2$. For the faces parallel to the imaginary roots, this means that, after removing all the partitions that must necessarily agree on the two sides, one is left with a polytope as in Figure 1.

Theorem 2.3. *The set of affine MV polytopes, along with some combinatorially defined operators, realizes $B(\infty)$.*

This is made precise and proven in [2]. The combinatorial operators are the natural generalizations from the rank 2 situation.

Let me briefly sketch the idea for our proof. There is an algebra called the preprojective algebra associated to $\mathbf{g}$. This is related to the path algebra of the Dynkin diagram for $\mathbf{g}$. For each $\mathbf{v} \in \mathbb{N}^n$, let $\Lambda(\mathbf{v})$ be the variety of nilpotent actions of Λ on a fixed vector space of graded dimension $\mathbf{v}$, where the grading comes from the underlying Dynkin diagram (this is usually called Lusztig's nilpotent variety). The crystal $B(\infty)$ can be realized where the underlying set is the union over all $\mathbf{v}$ of the set of irreducible component of $\Lambda(v)$, so each $b \in B(\infty)$ corresponds to a component Z_b (see [17]).

Fix $\pi \in Z_b$ generic. Define a polytope by taking the convex hull of the dimension vectors of all subrepresentations of (π, V). It turns out that this is the GGMS polytope of the MV polytope P_b. That is, it is the MV polytope, but without the decoration on the edges parallel to δ (in finite type, it is exactly the MV polytope).

To see the decoration, we study finer structure on the category for representations of Λ. Roughly, we consider Harder-Nahrasimen filtrations with respect to certain stability conditions, and our decoration comes from analyzing certain subcategories of semi-stable representations.

2.3. **Future plans.** Let me now discuss some questions I plan to investigate related to this work.

Question 2.1. *How do affine MV polytopes relate to affine PBW bases?*

Fix $b \in B(\infty)$. In finite type, Section 1.2 describes how the MV polytopes for b records the PBW monomials approximating b for all reduced expressions for w_0. In [3], Beck describes a PBW-type basis in affine types, and Ito [10] generalizes this to give a PBW-type basis for each biconvex order on root directions (which, it turns out, is the natural generalization of a reduced expression for w_0 in affine type). The data involved in recording a PBW "monomial" is an integer for each positive real root, along with an n-tuple of partitions. It seems natural to guess that our MV polytopes in symmetric affine types are recording this PBW-type data for all biconvex orders. I have started working with Dinakar Muthiah (a graduate student at Brown) on the $\widehat{\mathfrak{sl}}_2$ case of this question, and it seem quite tractable.

Question 2.2. *Can MV polytopes be defined outside of symmetric affine type?*

In the case of $A_2^{(2)}$ (the only rank-two affine root system other than $\widehat{\mathfrak{sl}}_2$), there is already a combinatorial solution to this question, which will appear in [1], but it is not likely to generalize. However, for the non-symmetric affine types, there is a clear conjecture: the MV polytopes should be the unique decorated polytopes such that all 2-faces are rank-2 MV polytopes of the right (finite or affine) type. The biggest hurdle appears to be showing that such polytopes exist. A good answer to Question 2.1 may lead to a construction of such polytopes in untwisted (but not necessarily symmetric) types, where PBW bases are known to exist.

Beyond affine type the situation looks quite difficult, but some of our methods do work, and even partial results may be interesting.

Question 2.3. *Can affine MV polytopes be defined using cycles in the double affine grassmannian?*

In finite type, MV polytopes were originally studied in the context of Mirković-Vilonen (MV) cycles. Analogues of MV cycles in the affine case have been constructed by Braverman, Finkelberg and Gaitsgory [4]. One would like to extract polytopes from the cycles, as in the finite type case. We conjecture that, in the symmetric affine cases, such a construction would lead to our decorated polytopes from Section 2.2.

3. Kirillov-Reshetikhin crystals, Demazure crystals, and Macdonald polynomials

I will finish by discussing another project I have worked on in the last year. This is still related to crystals, although is largely independent from the project of MV polytopes. One reason I feel this is worth discussing is that it leads to several questions that can be approached using the explicit combinatorial models for the relevant crystals, in particular using Kashiwara-Nakashima tableaux

[16] and the work of Schilling [22]. Some of these could be appropriate for undergraduate research projects.

In recent work [23] Anne Schilling and I established a precise relationship between certain energy functions on Kirillov-Reshetikhin crystals and the affine grading on certain Demazure crystals. Sanderson [20] and Ion [9] have developed an expression for type $A_n^{(1)}$, $D_n^{(1)}$ and $E_n^{(1)}$ Macdonald polynomials, specialized at $t = 0$, as specializations of Demazure characters. Together, this gives an expression for these specialized Macdonald polynomials as characters of KR crystals, where the powers of q in the Macdonald polynomial corresponds to the energy function.

Even more recently, Cristian Lenart [18] showed that, in type $C_n^{(1)}$, the specialization of the symmetric Macdonald polynomial at $t = 0$ can also be expressed as the character of a tensor product of type $C_n^{(1)}$ KR crystals, where the power of q records a combinatorial energy function. The tensor products of KR crystals B that show up in this construction are not "perfect" (a technical condition), and in particular $B \otimes B(\Lambda_0)$ is not irreducible. Thus, it is impossible to identify $B \otimes b_{\Lambda_0}$ with a Demazure subcrystal, as we do in the perfect cases. Examples suggest that it is instead the union of a Demazure subcrystal in each connected component of $B \otimes B(\Lambda_0)$. So, I ask:

Question 3.1. *If B is the tensor product of KR crystals used by Lenart to express the type $C_n^{(1)}$ Macdonald polynomials at $t = 0$, is $B \otimes b_{\Lambda_0}$ a union of Demazure crystals?*

If Question 3.1 has a positive answer, the next questions are

Question 3.2. *Which Demazure crystals show up? Are they (as in types $A_n^{(1)}$ and $D_n^{(1)}$) all Demazure crystals corresponding to translations in the affine Weyl group? Do (specializations of) the non-symmetric Macdonald polynomials appear?*

Finally, ignoring the connection with Macdonald polynomials, it is natural to ask

Question 3.3. *How general is the phenomenon that, given a tensor product B of KR crystals, $B \otimes b_\Lambda \subset B \otimes B(\Lambda)$ is a union of Demazure subcrystals of the various components of $B \otimes B(\Lambda)$?*

Recent work of Naoi [19] shows that $B \otimes b_\Lambda$ is a union of Demazure crystals in many cases where B is a tensor product of perfect KR crystals (of varying levels), but the non-perfect cases are open.

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