$\widehat{\mathfrak{sl}}_n$ crystals and cylindric partitions[1]

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[1] Slides and notes available at www-math.mit.edu/∼ptingley/

Outline

1. Motivation and background
   - Crystals, Characters and Combinatorics
   - $\widehat{\mathfrak{sl}}_n$ and its crystals
2. Partiton and cylindric partition models
   - The Misra-Miwa-Hayashi realization
   - Cylindric partitions and higher level representations
   - Two applications
   - Relationship with the Kyoto path model
3. Current work
   - Fayers' crystals
   - Future directions

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- Any representation of $\mathfrak{sl}_3$ decomposes (as a vector space) into the direct sum of the simultaneous eigenspaces for the diagonal matrices (weight spaces).

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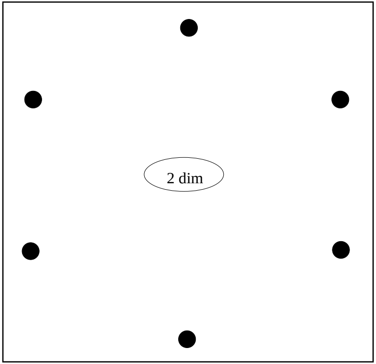

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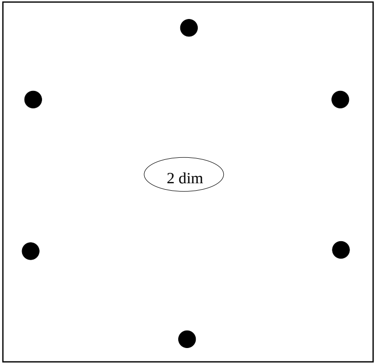

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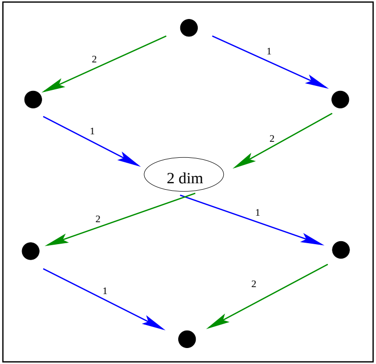

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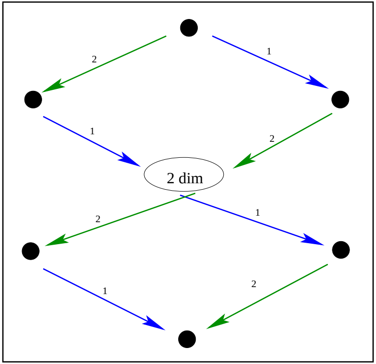

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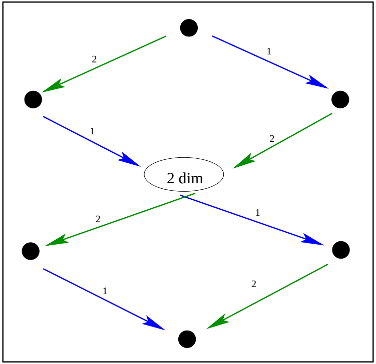

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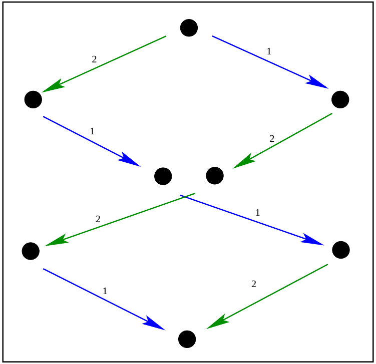

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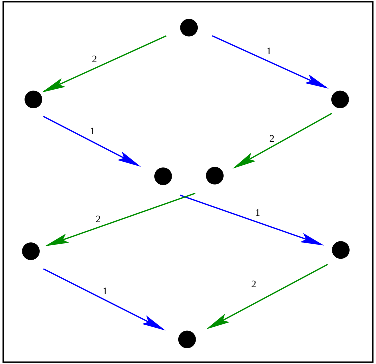

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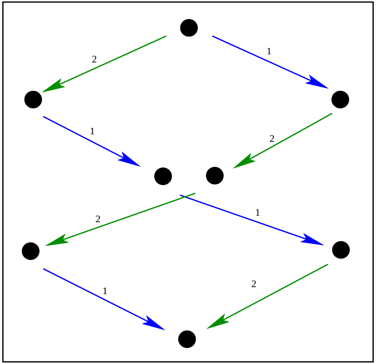

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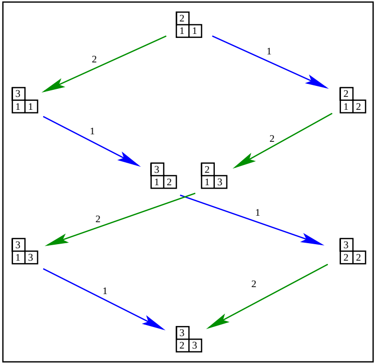

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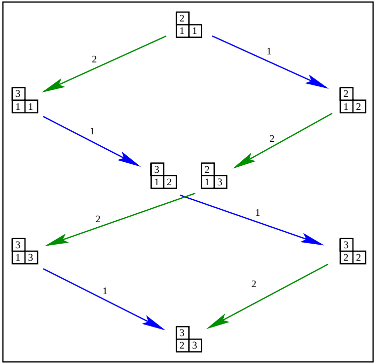

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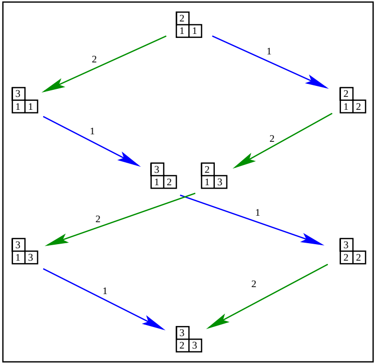

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- Here you see that the graded dimension of the representation is the generating function for semi-standard Young tableaux.

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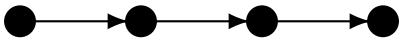

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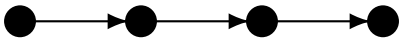

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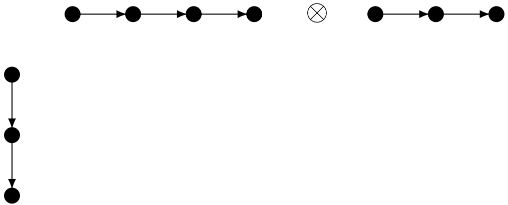

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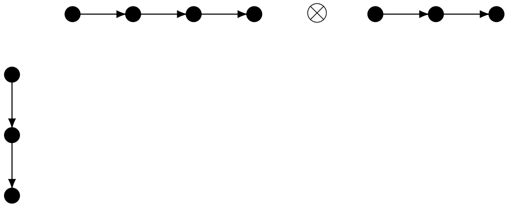

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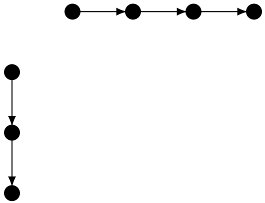

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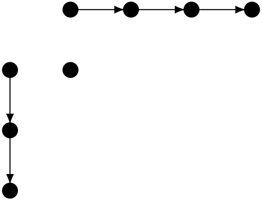

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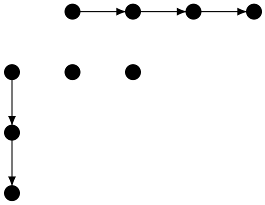

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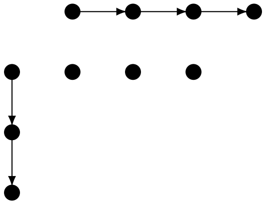

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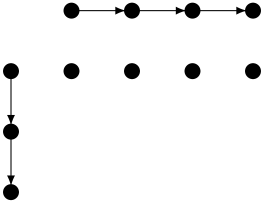

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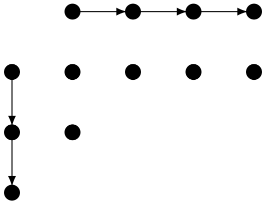

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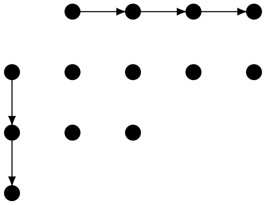

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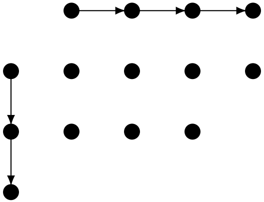

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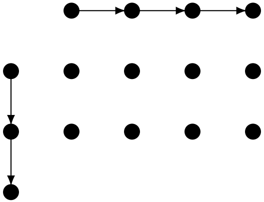

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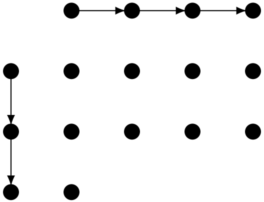

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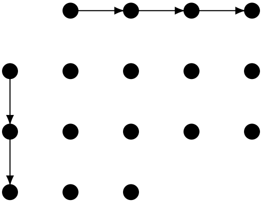

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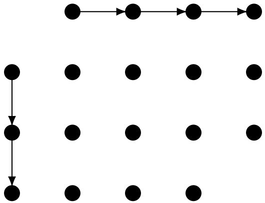

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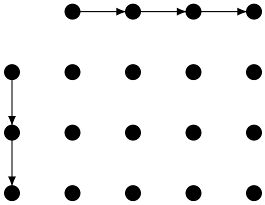

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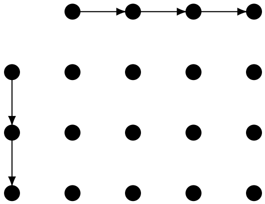

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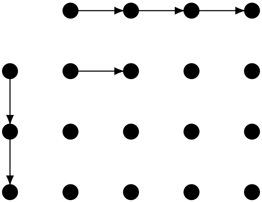

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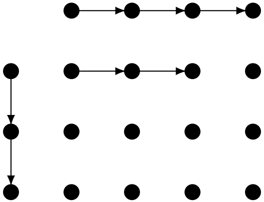

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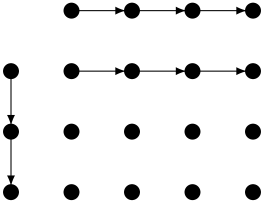

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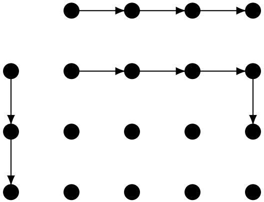

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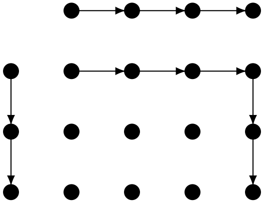

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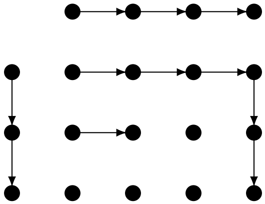

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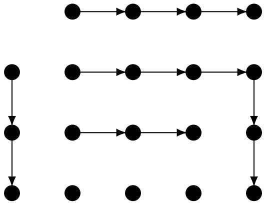

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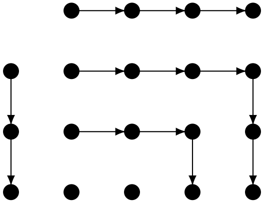

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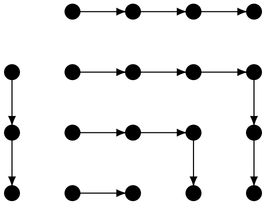

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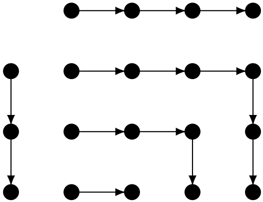

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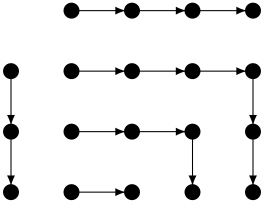

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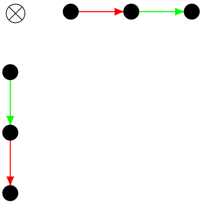

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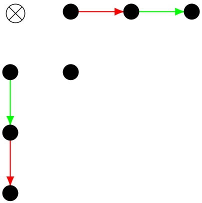

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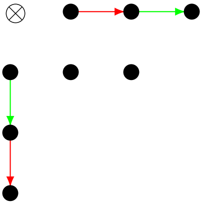

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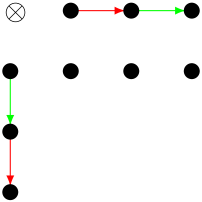

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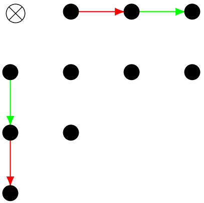

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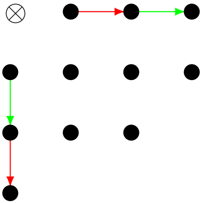

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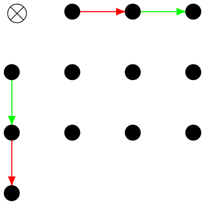

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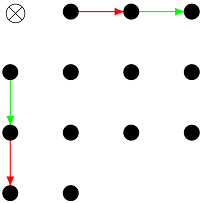

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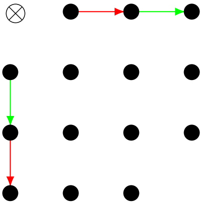

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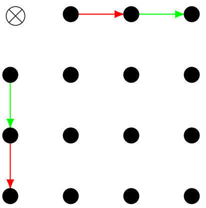

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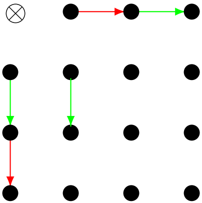

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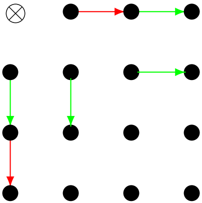

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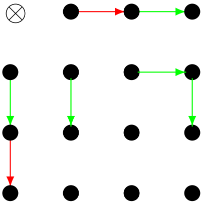

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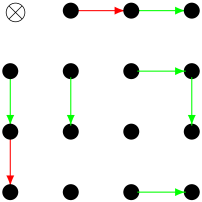

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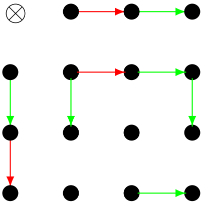

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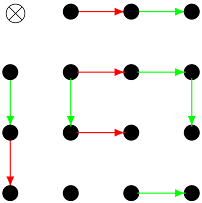

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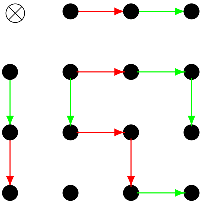

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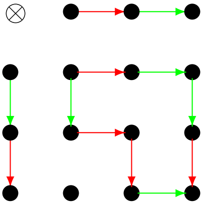

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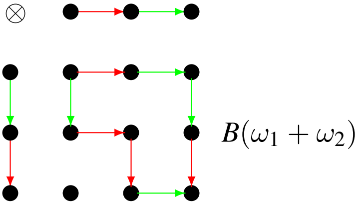

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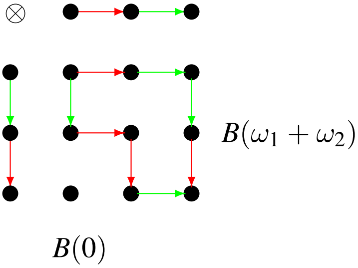

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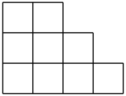

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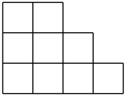
\#  in

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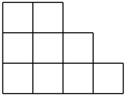
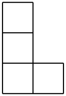
\#  in 

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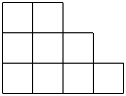
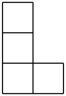
\#  in  $\otimes$

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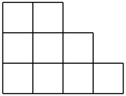
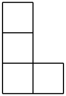
 in  $\otimes$ 

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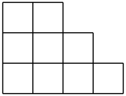
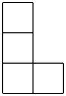
\#  in  $\otimes$ 

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- We would need to discuss the actually operators on tableaux to finish, but the point is it is combinatorial, and reasonably easy to compute.

The infinity crystal

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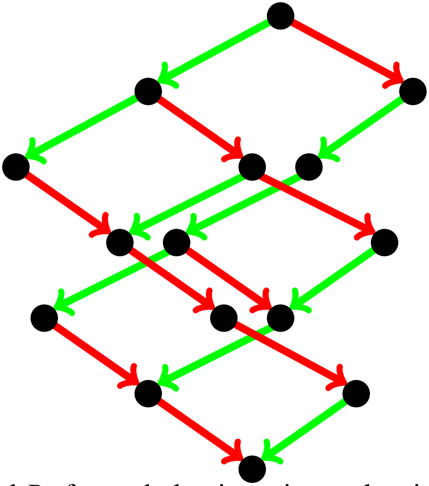

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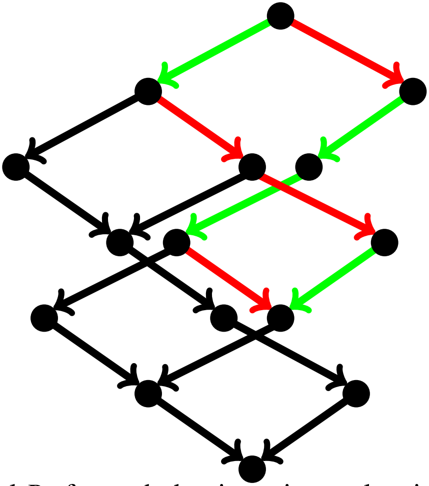

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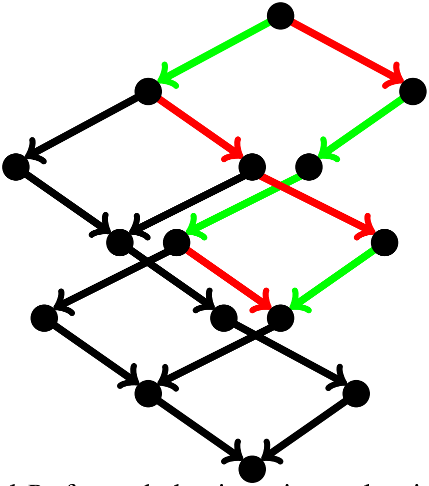

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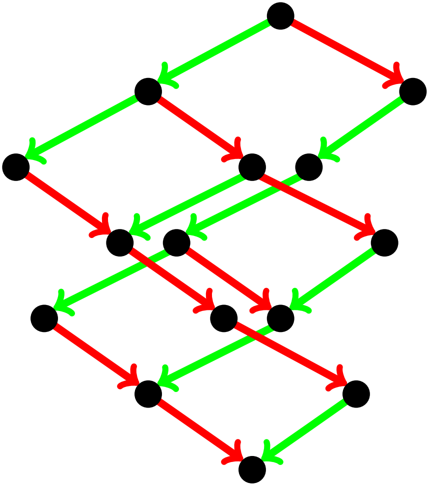

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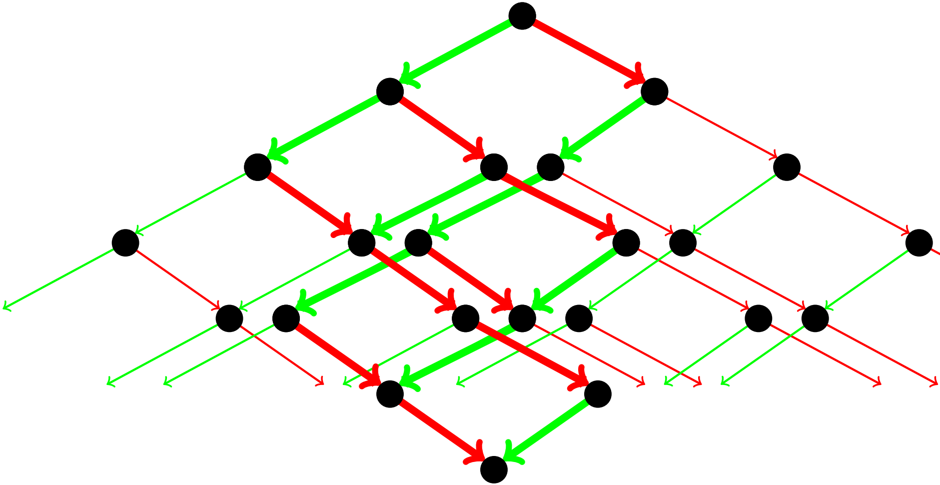

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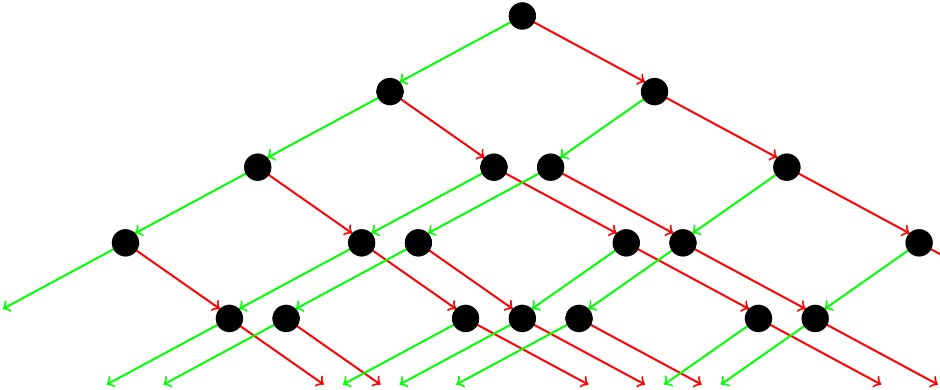

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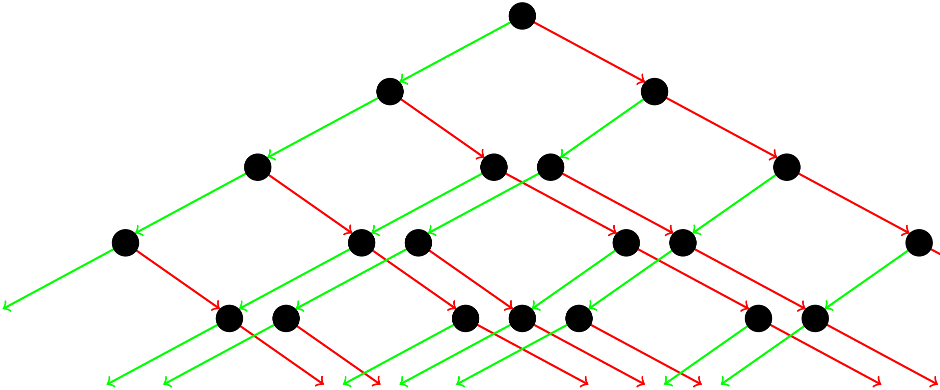

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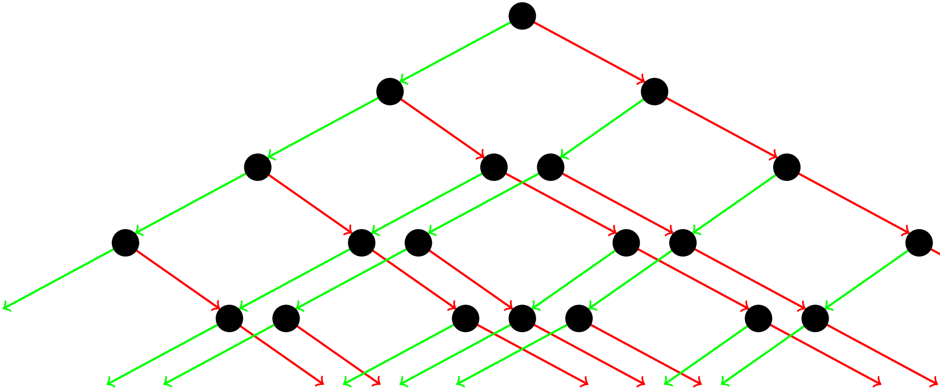

- In any realization, I want to understand these injections.
- They come from the fact that there is a canonical basis of $U_q^-(\mathbf{g})$ which descends to a basis of each $V(\lambda)$.

$\widehat{\mathfrak{sl}}_n$ and its crystals

•

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$\widehat{\mathfrak{sl}}_n$ and its crystals

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$\widehat{\mathfrak{sl}}_n$ and its crystals

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$$\mathfrak{sl}_n \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}C$$

$\widehat{\mathfrak{sl}}_n$ and its crystals

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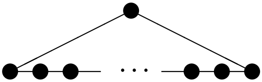

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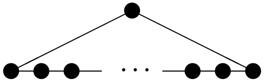

- $\widehat{\mathfrak{sl}}_n'$ is generated by $\{E_i, F_i\}_{0 \leq i \leq n-1}$ subject to the relations that for each pair $0 \leq i < j \leq n-1$, $\{E_i, F_i, E_j, F_j\}$ generate a copy of

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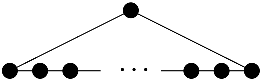

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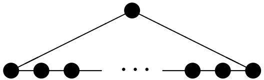

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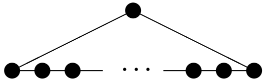

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- In fact, it is a theorem of Kashiwara that, to check a graph is a crystal, it suffices to look at rank 2 behavior.

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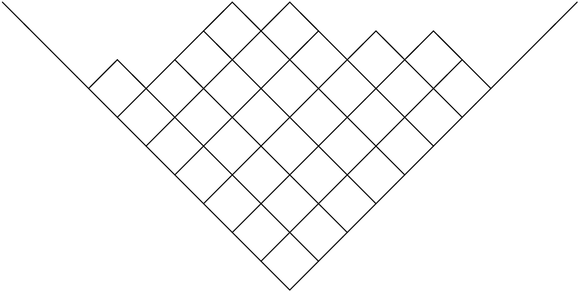

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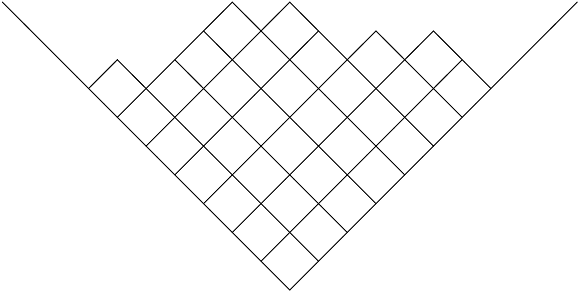

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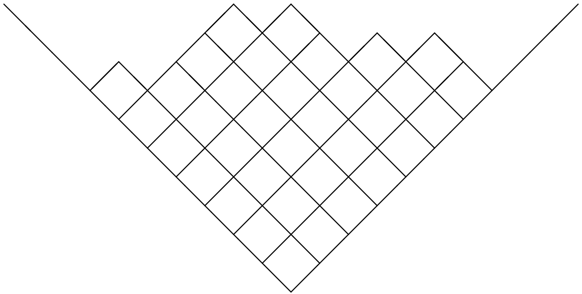

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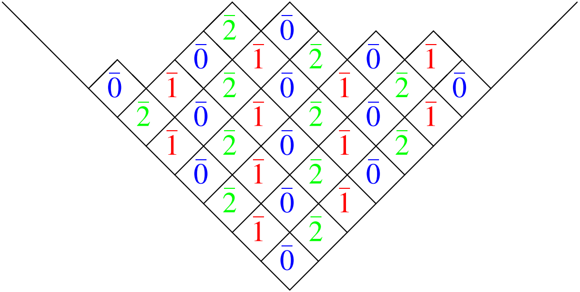

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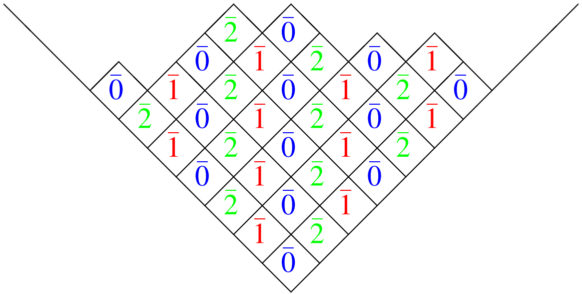

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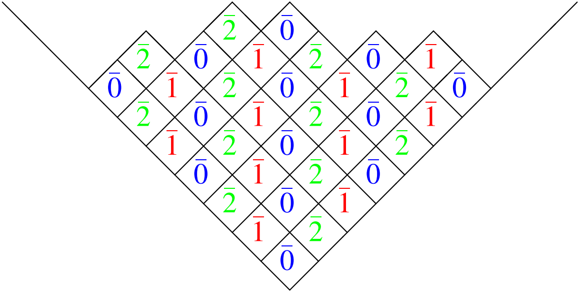

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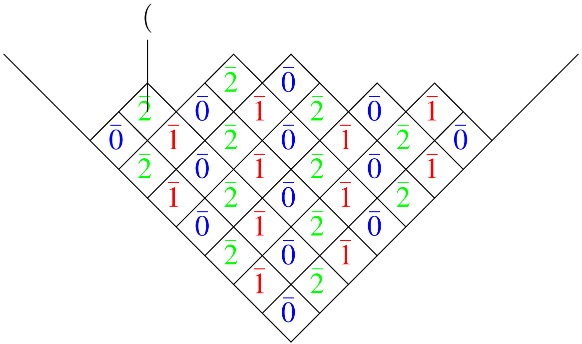

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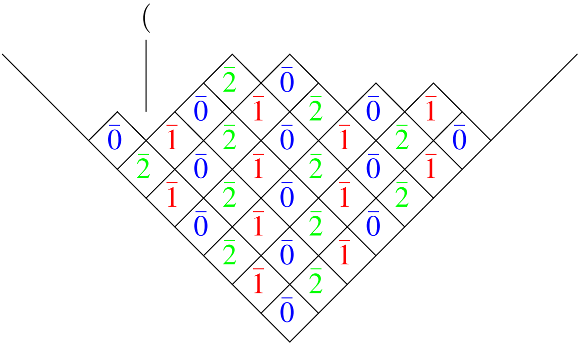

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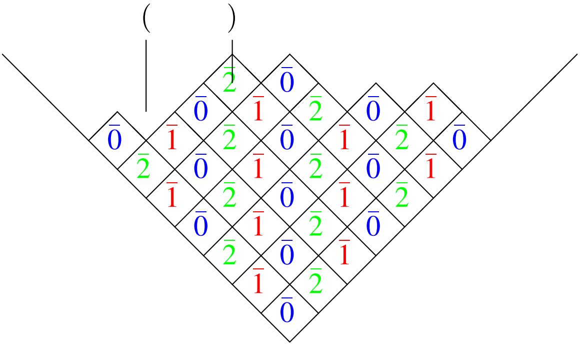

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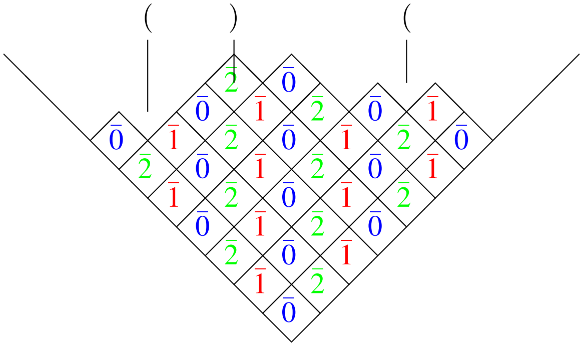

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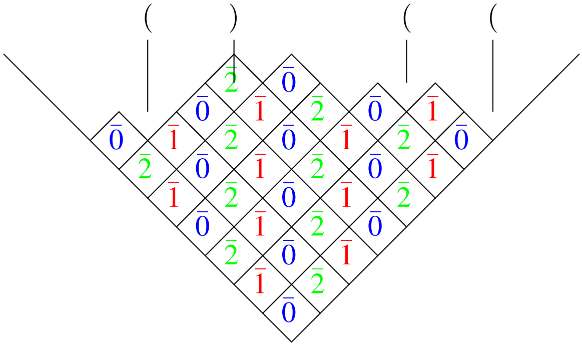

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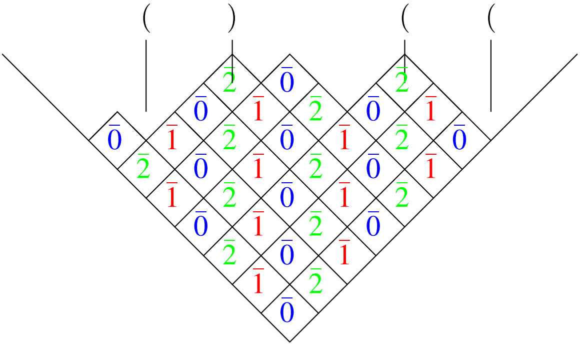

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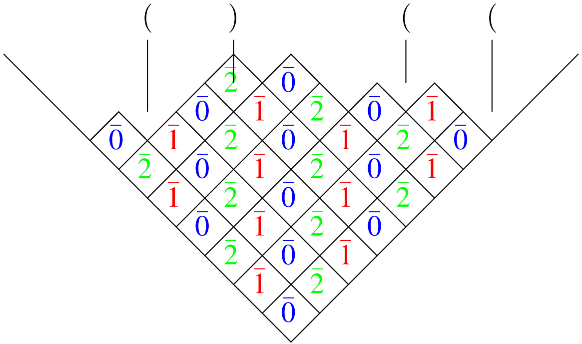

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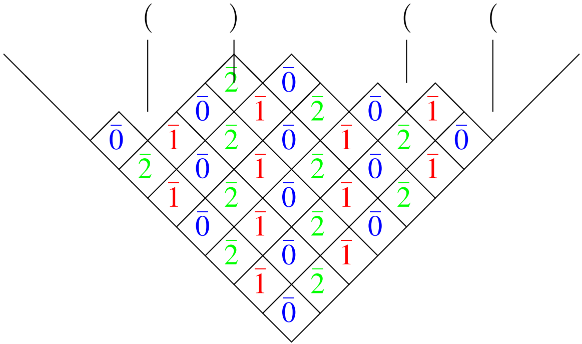

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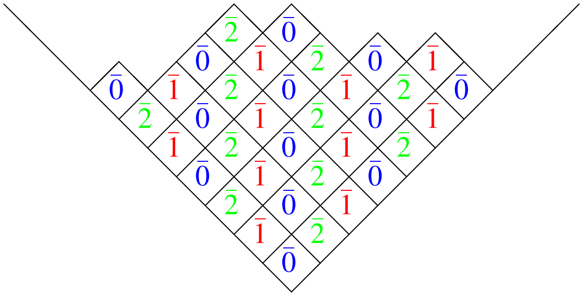

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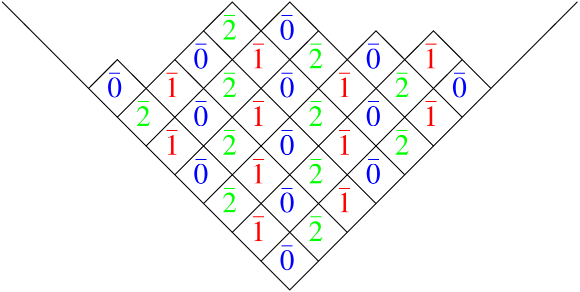

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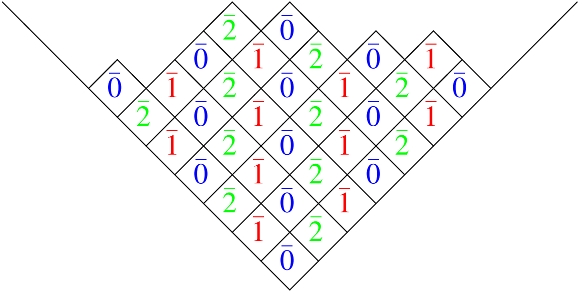

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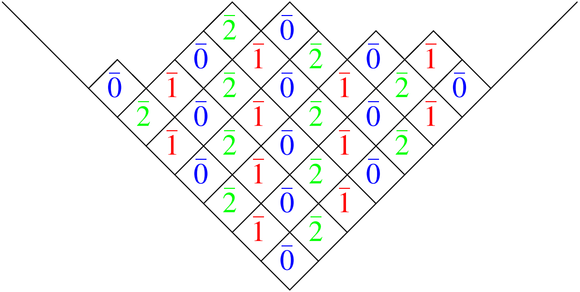

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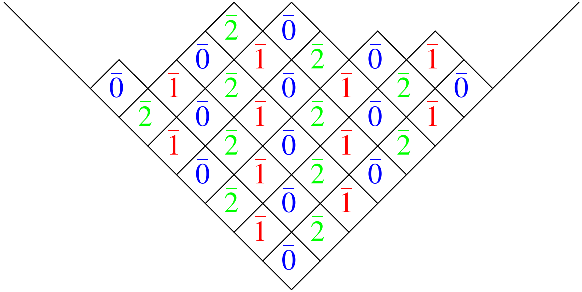

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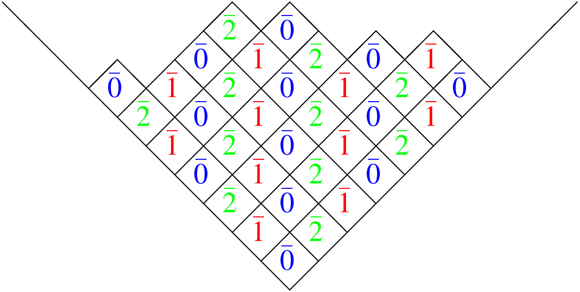

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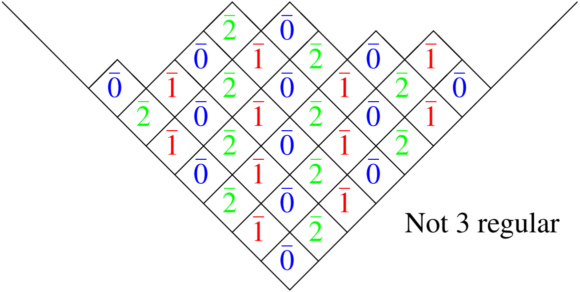

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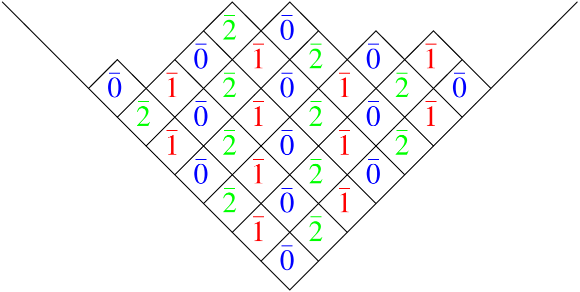

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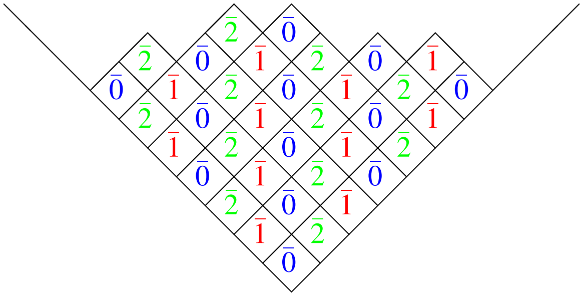

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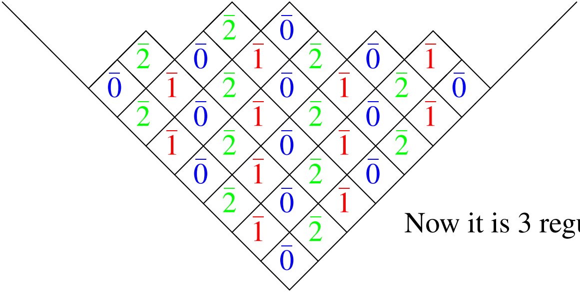

Now it is 3 regular

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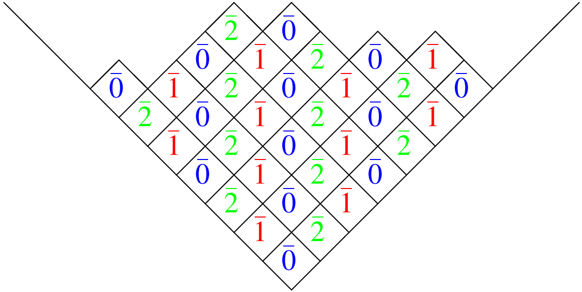

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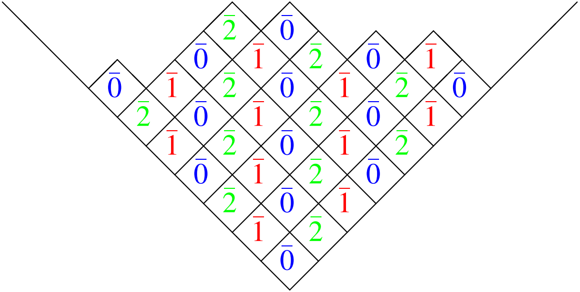

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- For instance, we now know that the q-character of V_{Λ_0} is equal to the generating function of 3-regular partitions counted by size.

Higher level crystals

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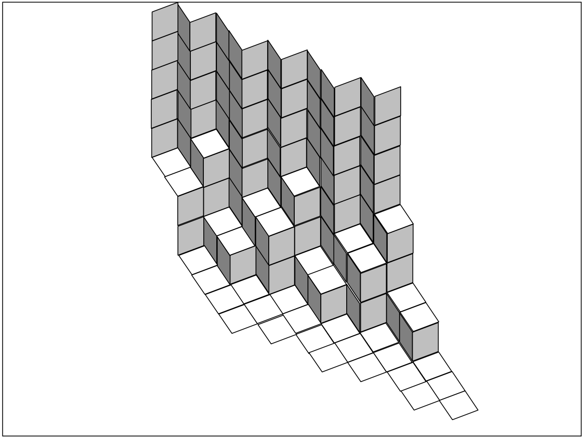

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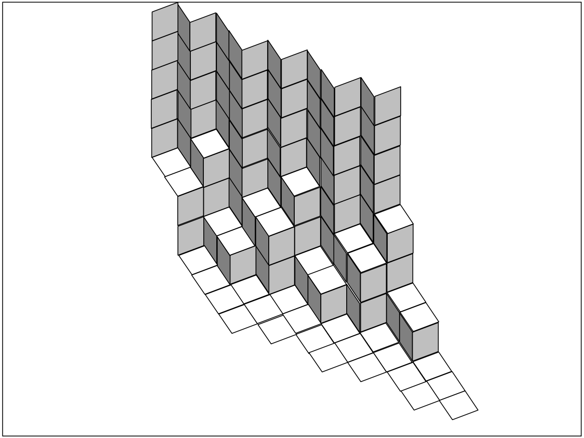

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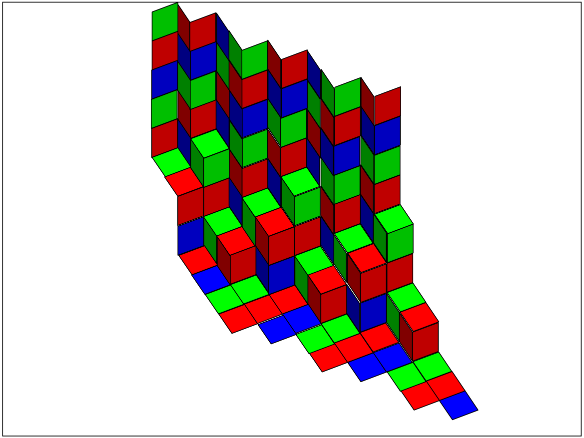

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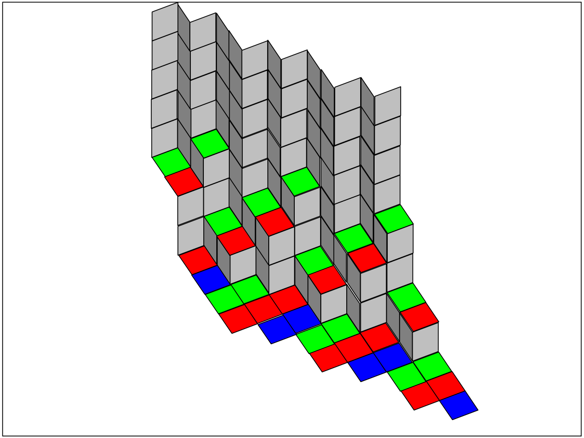

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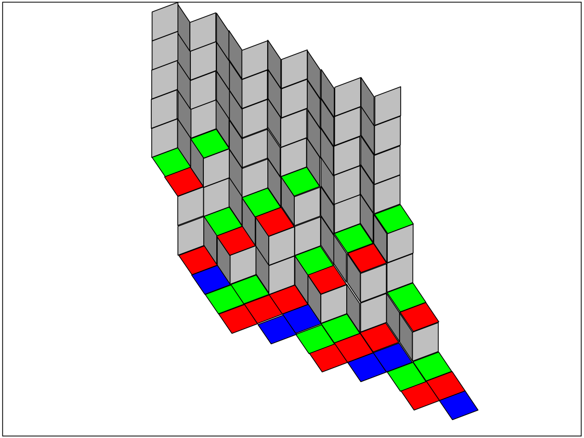

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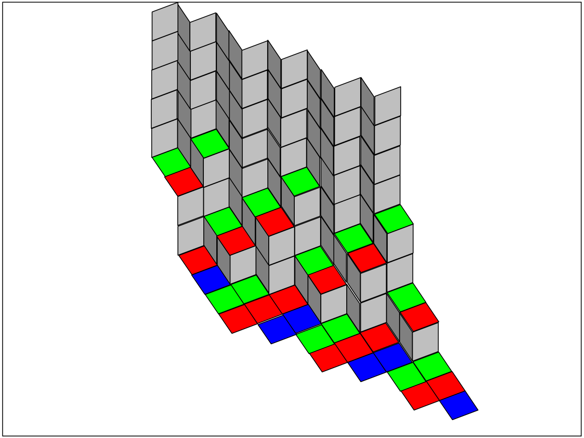

- People usually denote this by a tupple of partitions.

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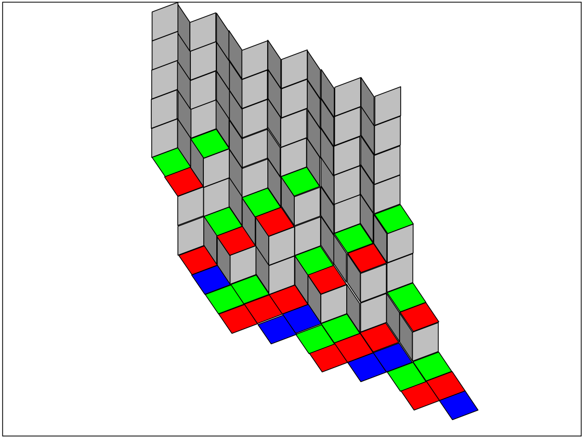
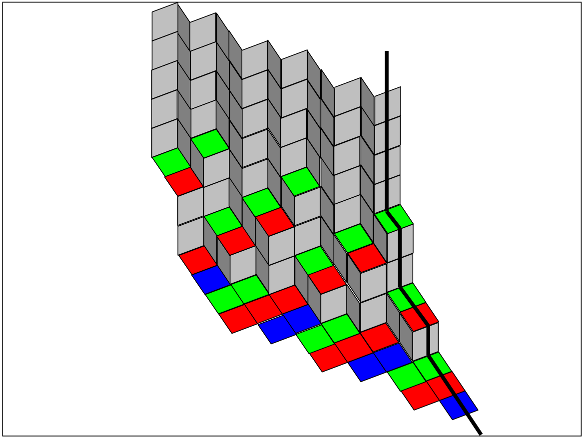
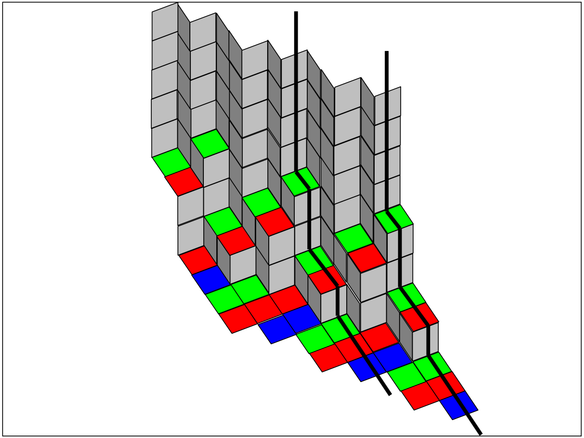


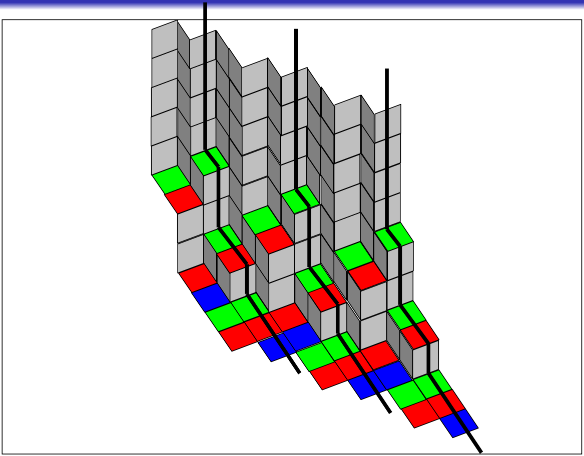
Higher level crystals



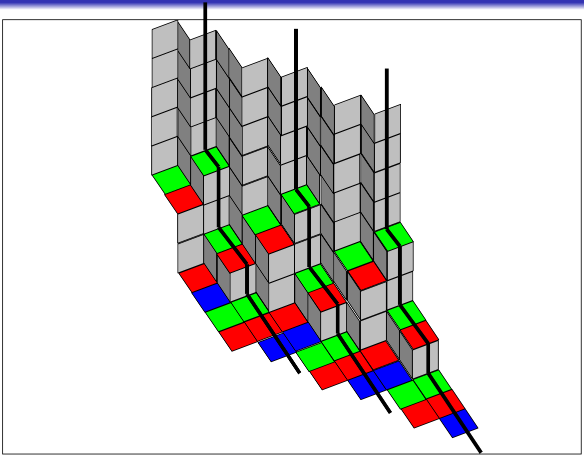
Higher level crystals

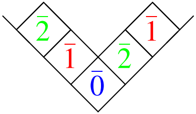


Higher level crystals



Higher level crystals





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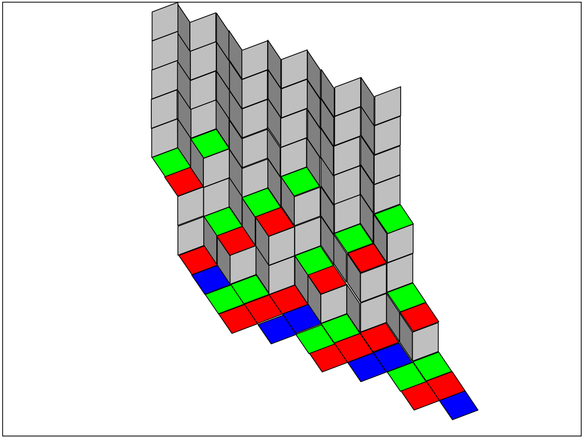

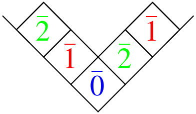

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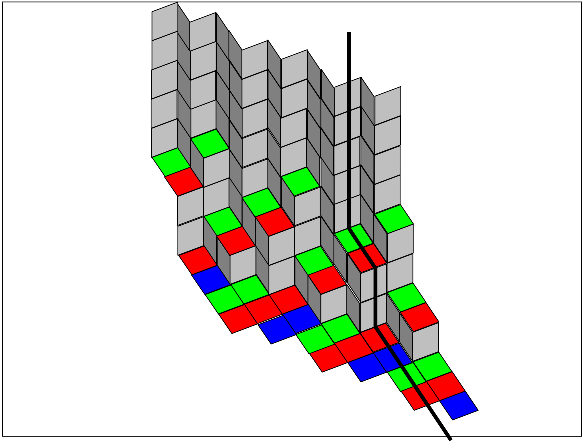

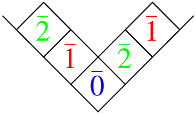

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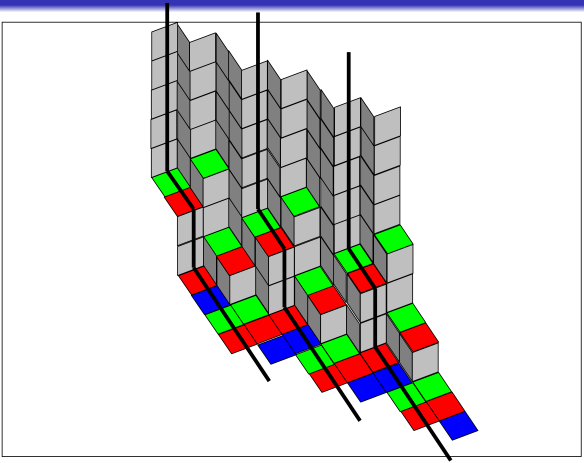

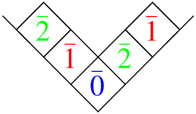

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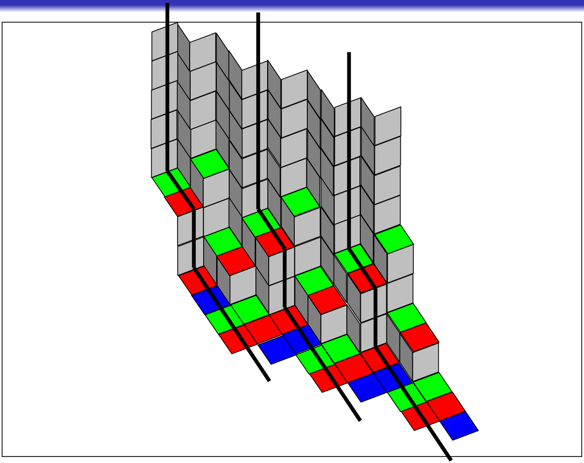
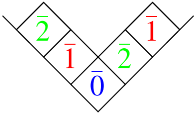
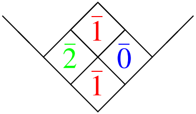
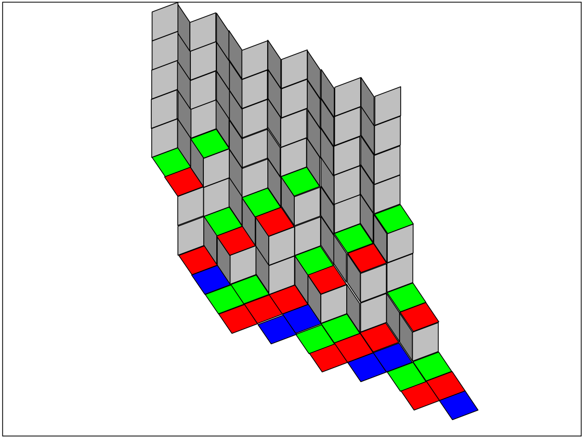


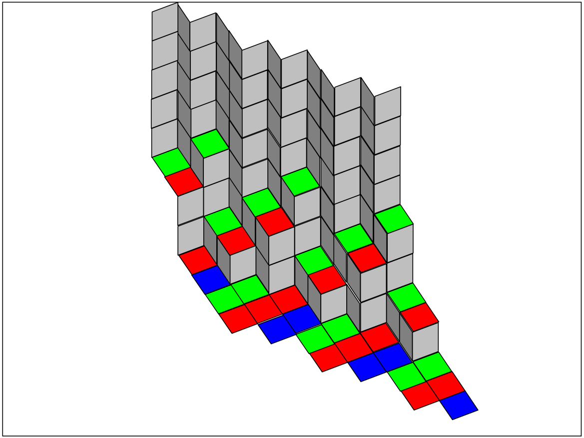




Higher level crystals

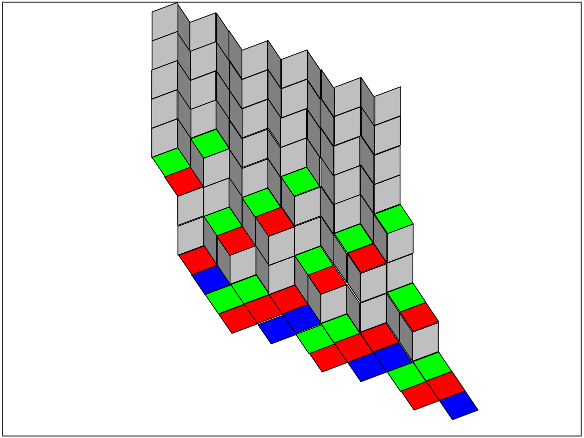


Higher level crystals



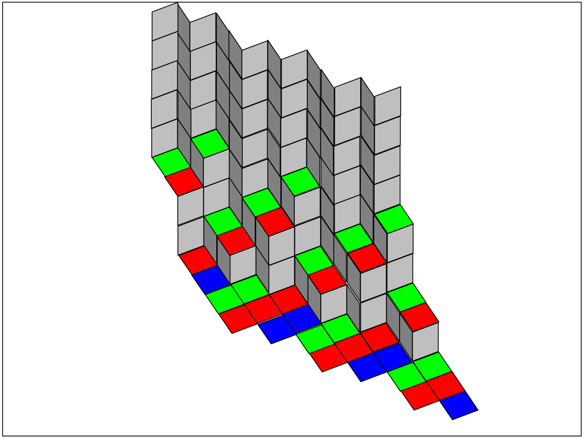
- There are natural crystal operations such that each connected component is a copy of $B(\Lambda)$.

Higher level crystals

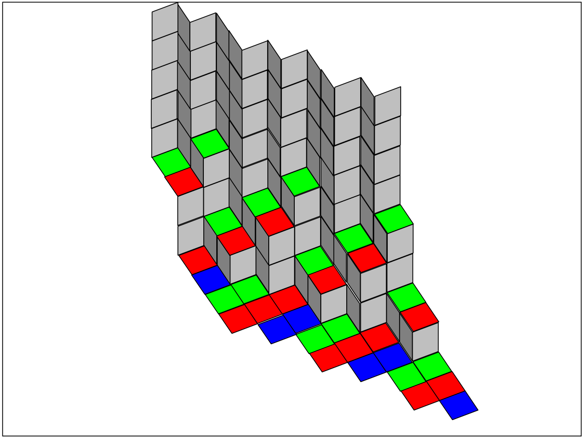


- There are natural crystal operations such that each connected component is a copy of $B(\Lambda)$.
- A cylindric partition is in the 'highest copy' if and only if it does not have three differently colored piles of the same height.

Higher level crystals

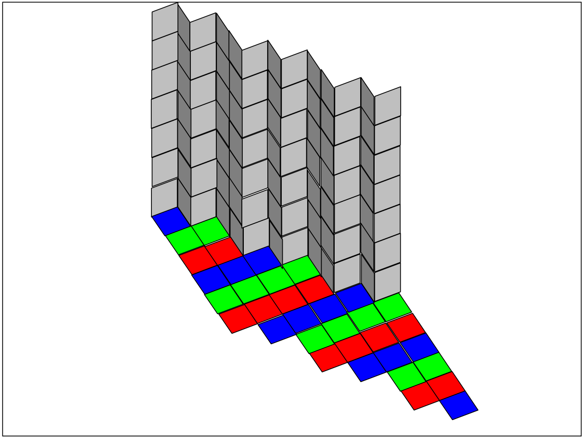


Higher level crystals



- The embeddings $B_\Lambda \hookrightarrow B_{\Lambda'}$ are given by “shifting".

Higher level crystals



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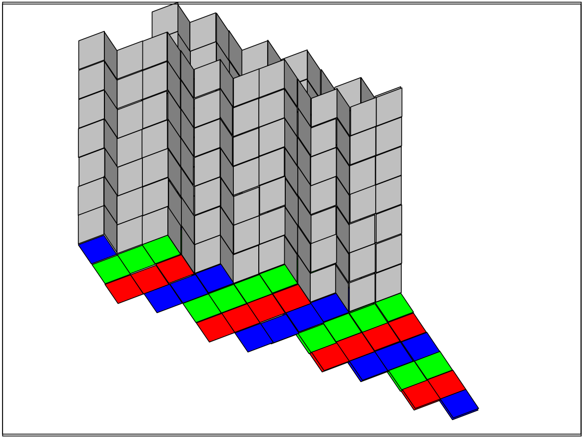
Higher level crystals



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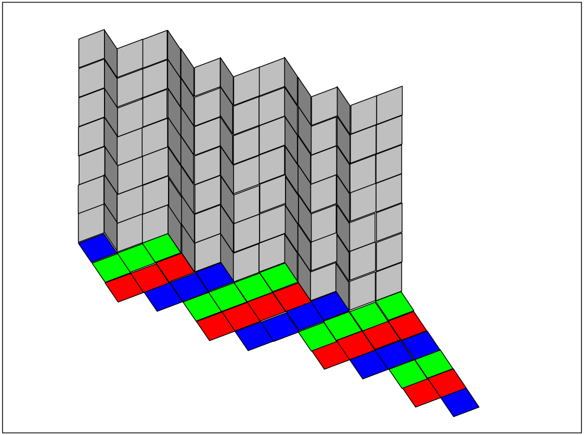

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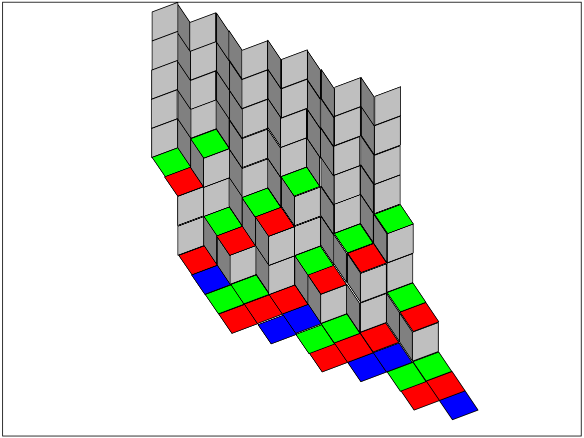

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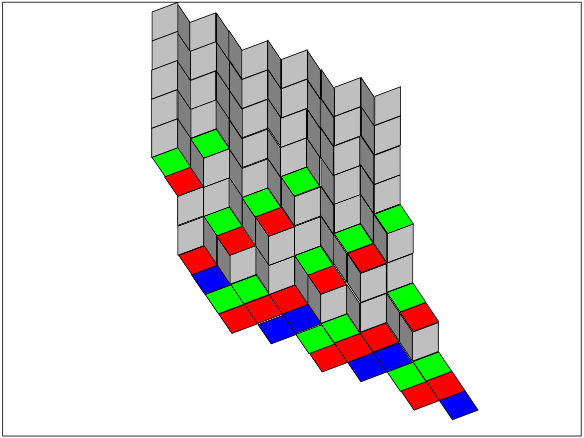

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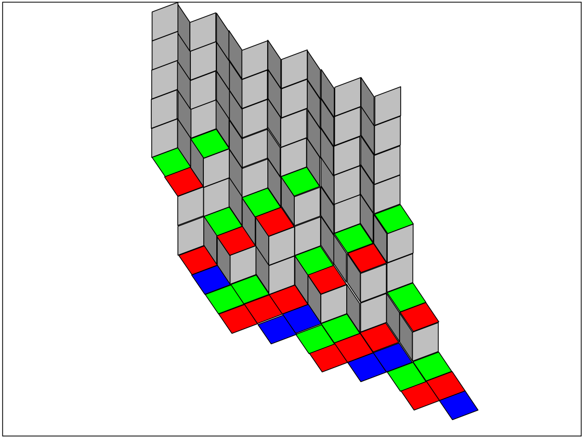

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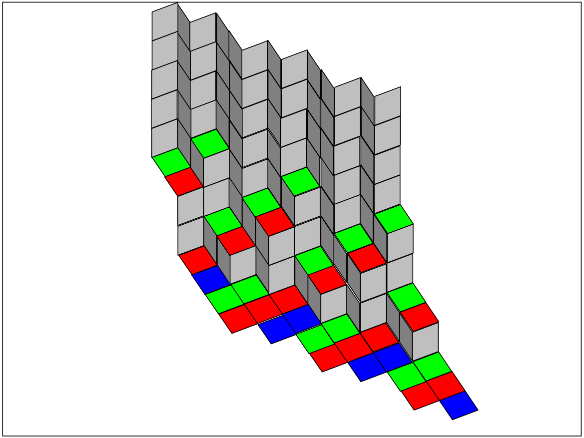

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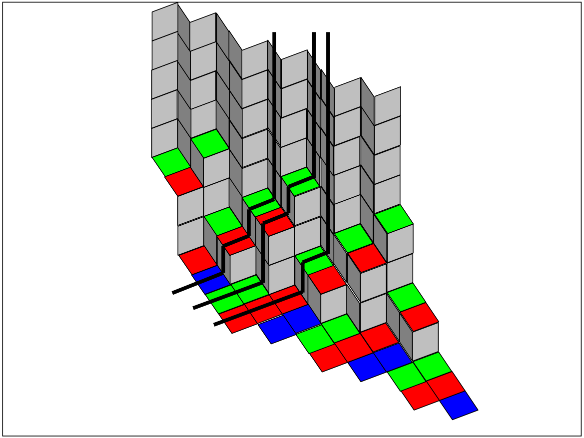

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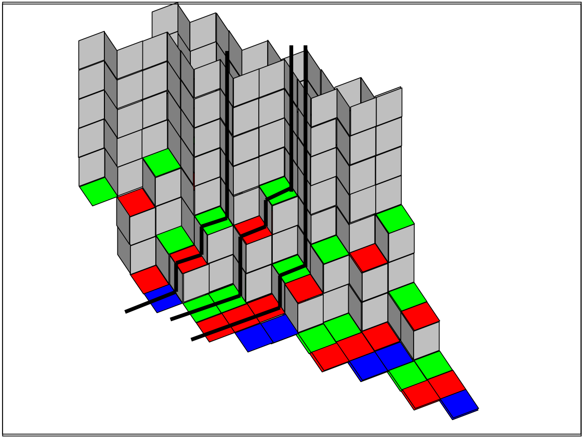

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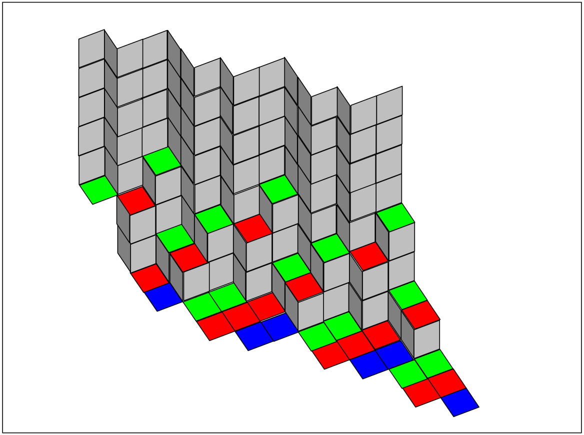

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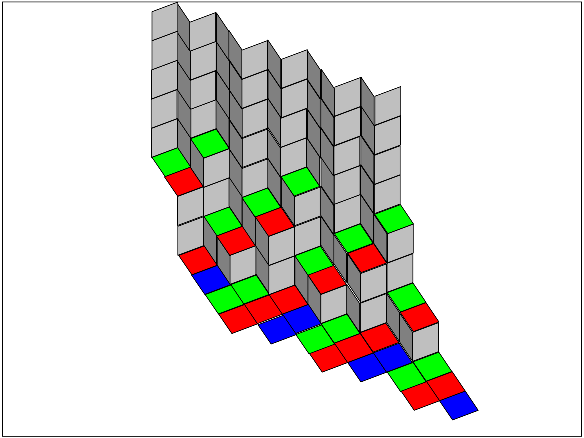
Higher level crystals

Higher level crystals

- The imbedding into B_∞ just records the vertical piles, not the arrangement into an ℓ-tuple of partitions.

Higher level crystals



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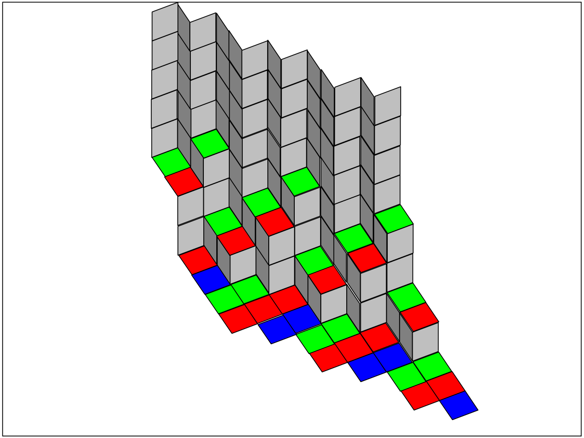
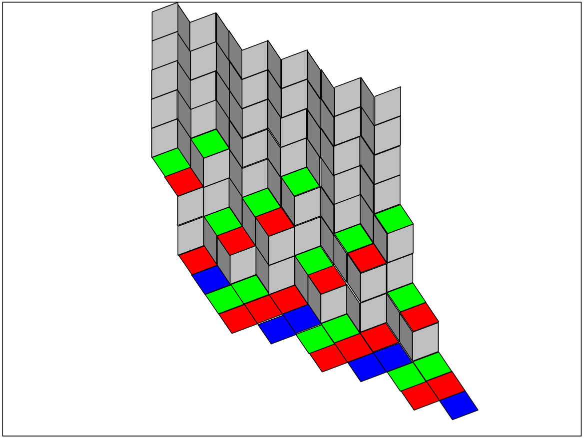


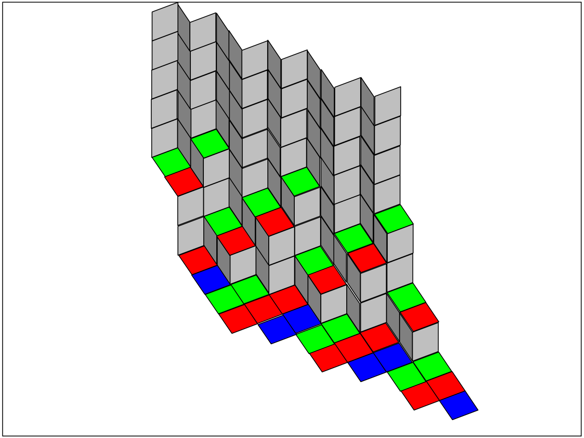
$\bar{2}$
$\bar{1}$
$\bar{0}$

Higher level crystals



$$\begin{array}{|c|} \hline \bar{2} \\ \hline \bar{1} \\ \hline \bar{0} \\ \hline \end{array} \quad \begin{array}{|c|} \hline \bar{1} \\ \hline \bar{0} \\ \hline \end{array}$$

Higher level crystals



$$\begin{array}{|c|}\hline \bar{2} \\ \hline \bar{1} \\ \hline \bar{0} \\ \hline \end{array} \quad \begin{array}{|c|}\hline \bar{1} \\ \hline \bar{0} \\ \hline \end{array} \quad \begin{array}{|c|}\hline \bar{2} \\ \hline \bar{1} \\ \hline \end{array}$$

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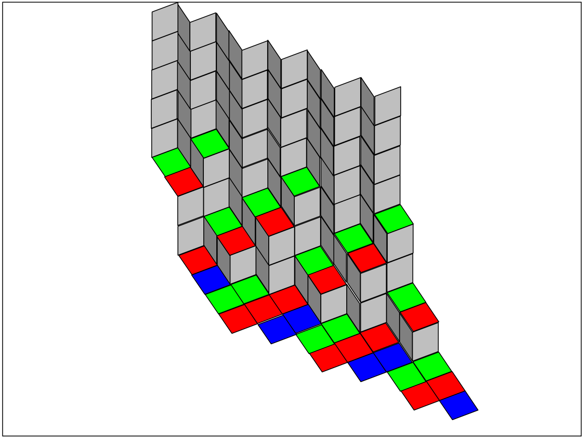
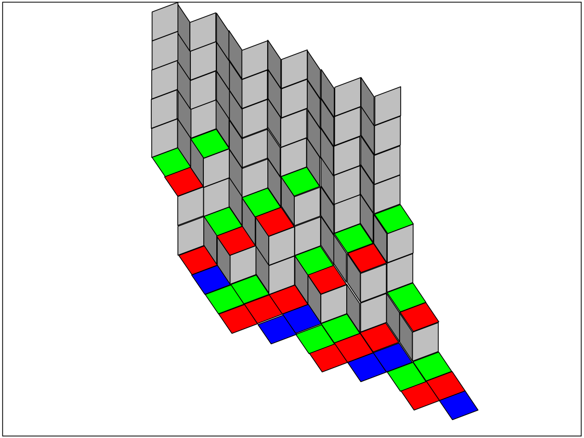


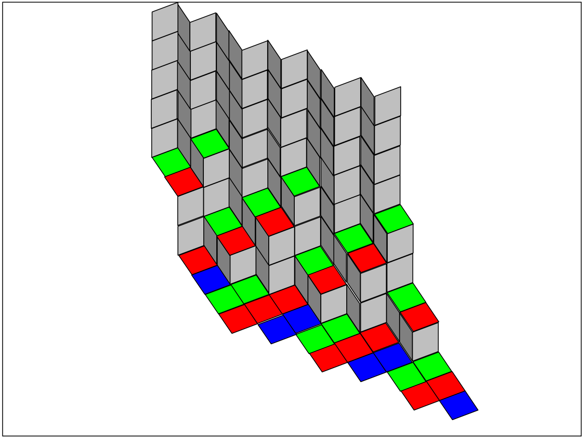
$\bar{2}$			
$\bar{1}$	$\bar{1}$	$\bar{2}$	
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{1}$

Higher level crystals



$\bar{2}$				
$\bar{1}$	$\bar{1}$	$\bar{2}$		
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{1}$	$\bar{2}$

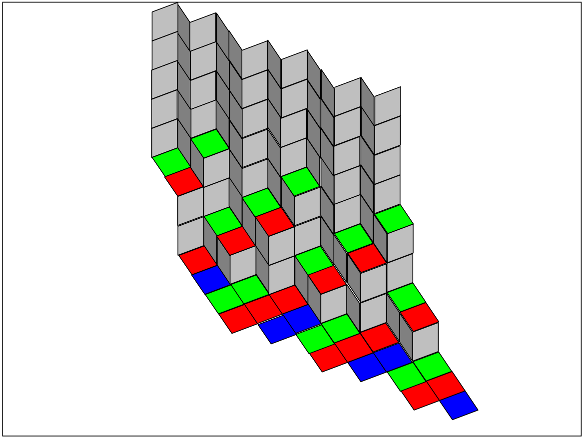
Higher level crystals



$$\begin{array}{ccccc} \boxed{\bar{2}} & & & & \\ \boxed{\bar{1}} & \boxed{\bar{1}} & \boxed{\bar{2}} & & \\ \boxed{\bar{0}} & \boxed{\bar{0}} & \boxed{\bar{1}} & \boxed{\bar{1}} & \boxed{\bar{2}} \end{array}$$

A “multi-segment"

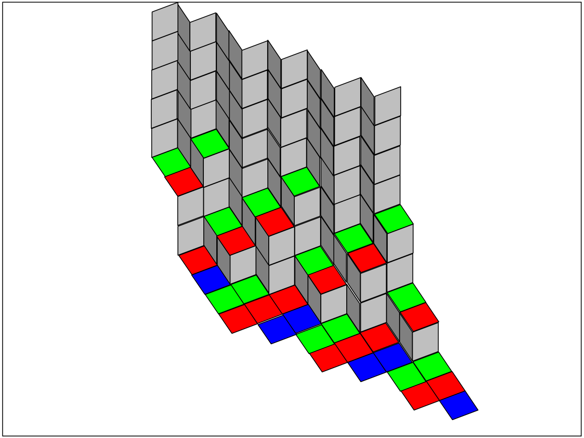
Higher level crystals



$\bar{2}$				
$\bar{1}$	$\bar{1}$	$\bar{2}$		
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{1}$	$\bar{2}$

- The B_∞ crystal structure reads boxes in order of height.

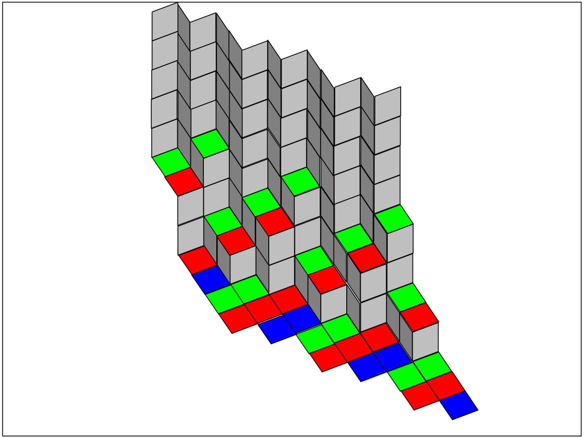
Higher level crystals



$\bar{2}$				
$\bar{1}$	$\bar{1}$	$\bar{2}$		
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{1}$	$\bar{2}$

- The B_∞ crystal structure reads boxes in order of height. For $f_{\bar{2}}$:

Higher level crystals



$$\begin{array}{c}\boxed{\bar{2}}\\\boxed{\bar{1}}\\\boxed{\bar{0}}\end{array}\quad\begin{array}{c}\boxed{\bar{1}}\\\boxed{\bar{0}}\end{array}\quad\begin{array}{c}\boxed{\bar{2}}\\\boxed{\bar{1}}\end{array}\quad\boxed{\bar{1}}\quad\boxed{\bar{2}}$$

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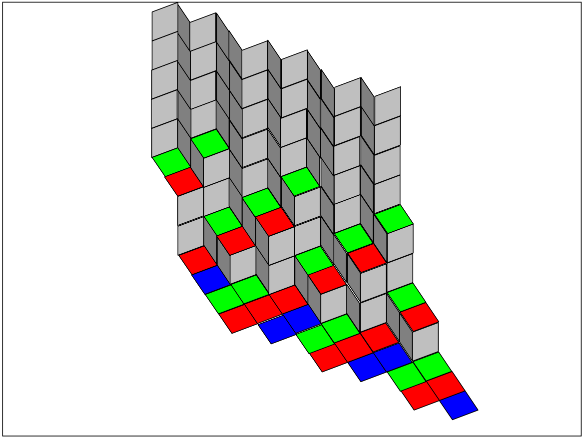
Higher level crystals



)	(			
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$\bar{1}$	$\bar{1}$	$\bar{2}$		
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{1}$	$\bar{2}$

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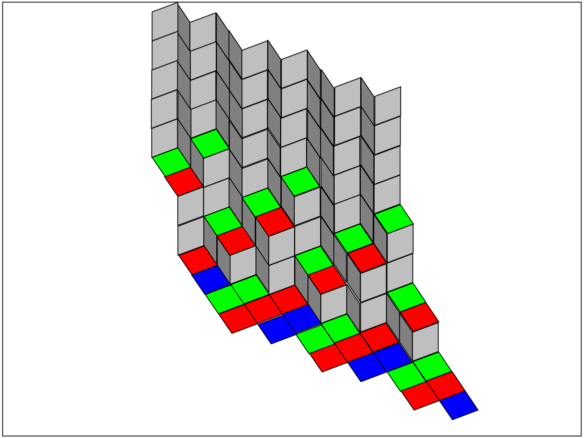

$$\begin{array}{ccccc}) & (&) & & \\ \boxed{\bar 2} & & & & \\ \boxed{\bar 1} & \boxed{\bar 1} & \boxed{\bar 2} & & \\ \boxed{\bar 0} & \boxed{\bar 0} & \boxed{\bar 1} & \boxed{\bar 1} & \boxed{\bar 2} \end{array}$$

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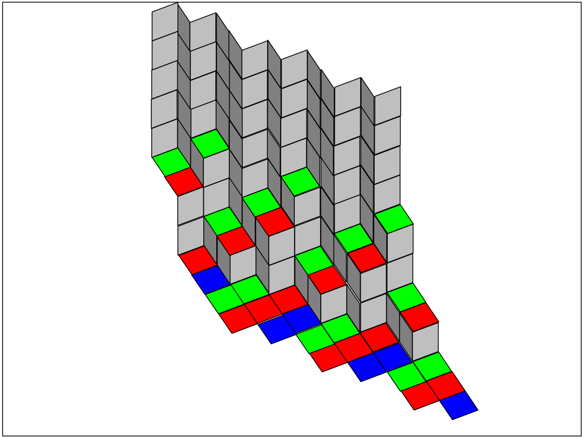

)	(	)	(	
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$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{1}$	$\bar{2}$

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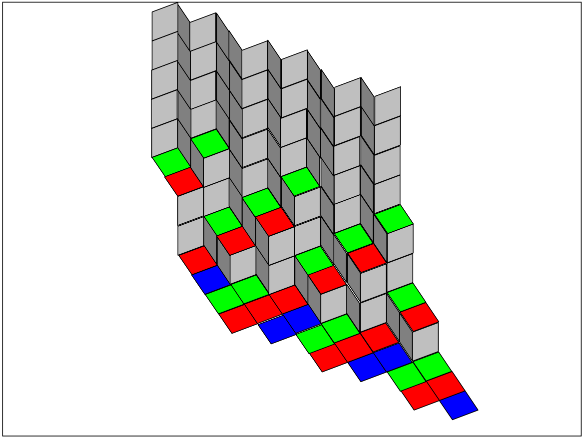

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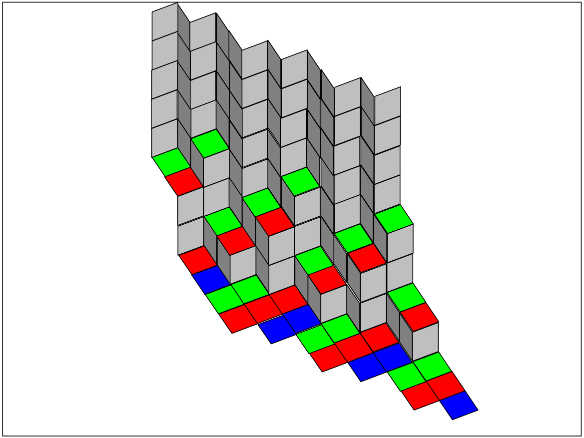

$$) \quad (\quad) \quad (\quad) \quad ($$

$$\begin{array}{ccccc} \boxed{\bar{2}} & & & & \\ \boxed{\bar{1}} & \boxed{\bar{1}} & \boxed{\bar{2}} & & \\ \boxed{\bar{0}} & \boxed{\bar{0}} & \boxed{\bar{1}} & \boxed{\bar{1}} & \boxed{\bar{2}} \end{array}$$

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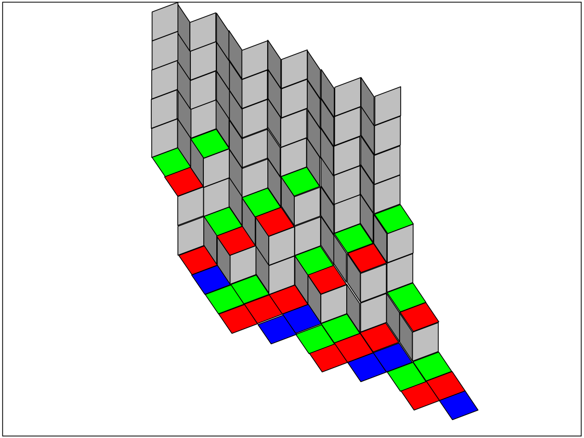

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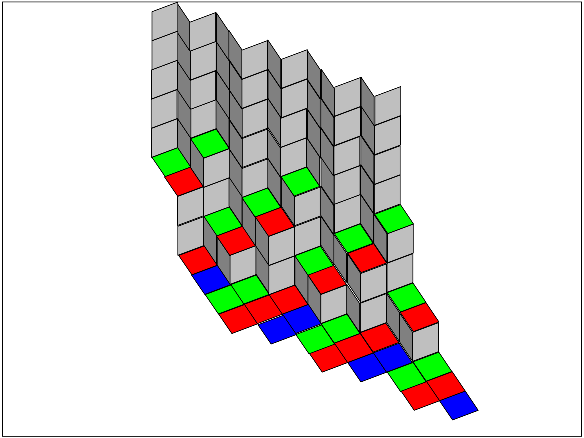

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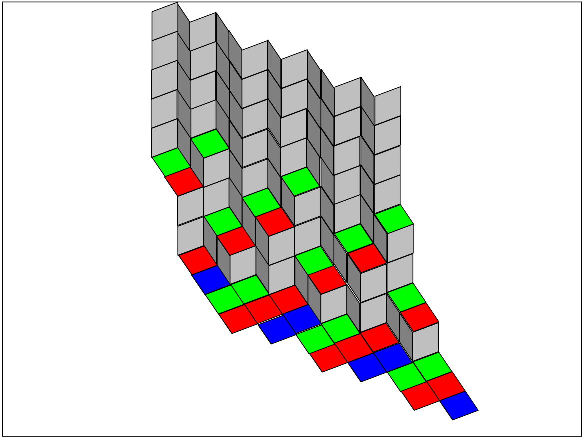

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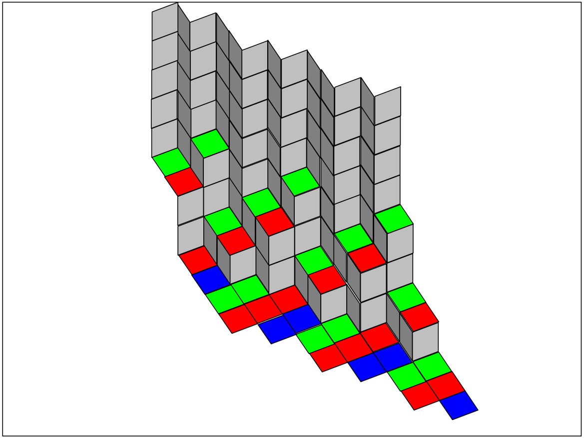

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$\bar{2}$					
$\bar{1}$	$\bar{1}$	$\bar{2}$			
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{2}$

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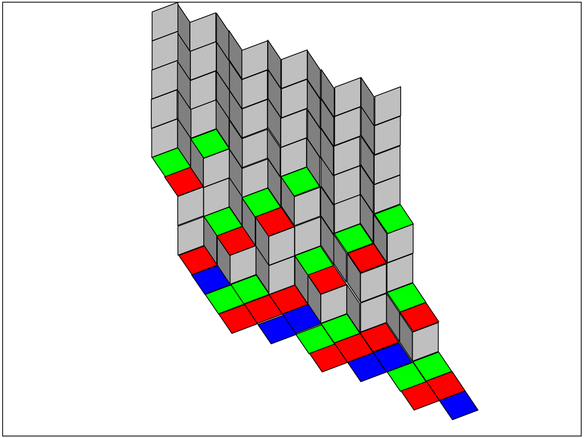

$$) \quad (\quad) \quad (\quad) \quad (\quad (\quad (\quad \cdots$$

$$\begin{array}{ccccc} \boxed{\bar{2}} & & & & \\ \boxed{\bar{1}} & \boxed{\bar{1}} & \boxed{\bar{2}} & & \\ \boxed{\bar{0}} & \boxed{\bar{0}} & \boxed{\bar{1}} & \boxed{\bar{1}} & \boxed{\bar{2}} \end{array}$$

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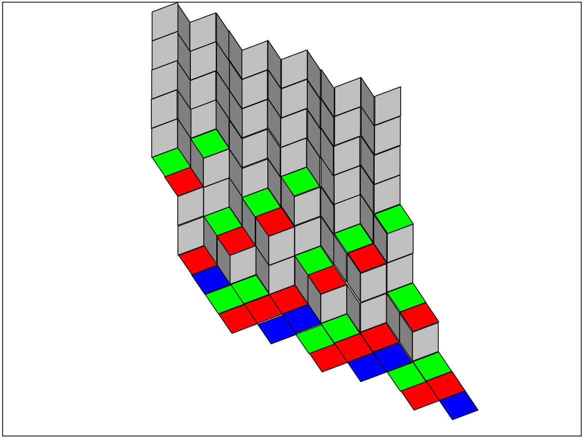

$$) \quad (\quad) \quad (\quad) \quad (\quad (\quad (\quad \cdots$$

$\bar{2}$				
$\bar{1}$	$\bar{1}$	$\bar{2}$		
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{1}$	$\bar{2}$

- Cylindric partitions are only needed to describe the image of B_Λ.

Application: generating functions/partition functions

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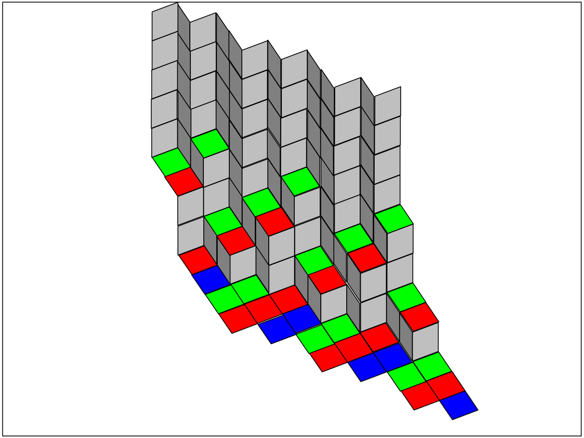

Application: generating functions/partition functions

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- The generating function for cylindric partitions on a given cylinder is a specialization of the Weyl character formula.

Application: generating functions/partition functions

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Corollary

$\sum_{\pi \text{on a given cylinder}} q^{|\pi|} = \dim_q(W_\Lambda)$, *where* W_Λ *is an irreducible representation of* $\widehat{\mathfrak{gl}}_n$ *at level* ℓ. *(Calculated by A. Borodin in a different form).*

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Theorem

(Borodin 2006) The partition function for cylindric plane partitions is given by:

$$Z := \sum_{\substack{\text{cylindric partitions } \pi \\ \text{on a given cylinder}}} q^{|\pi|} = \prod_{k \geq 1} \frac{1}{1-q^{kN}} \prod_{\substack{i \in \overline{1,N} : A[i]=1 \\ j \in \overline{1,N} : A[j]=0}} \frac{1}{1-q^{(i-j)(N)+(k-1)N}}.$$

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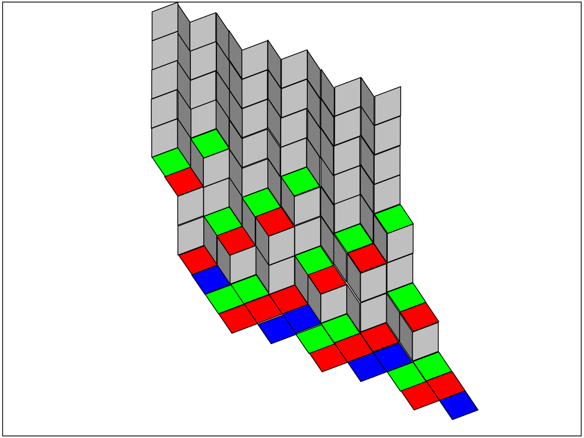

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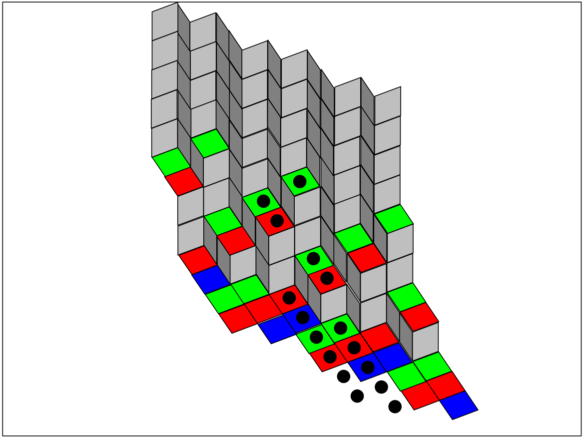

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Application: Level-rank duality

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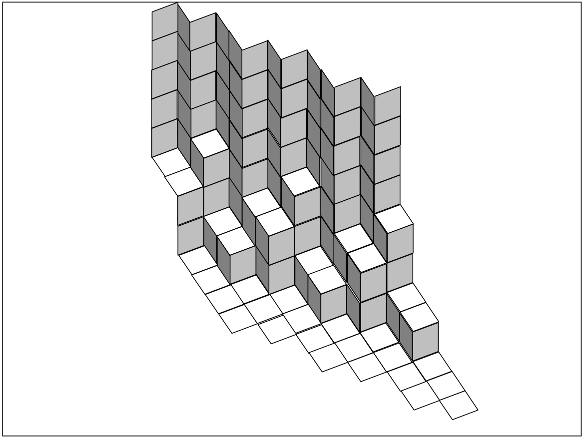

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Relation to the Kyoto path model

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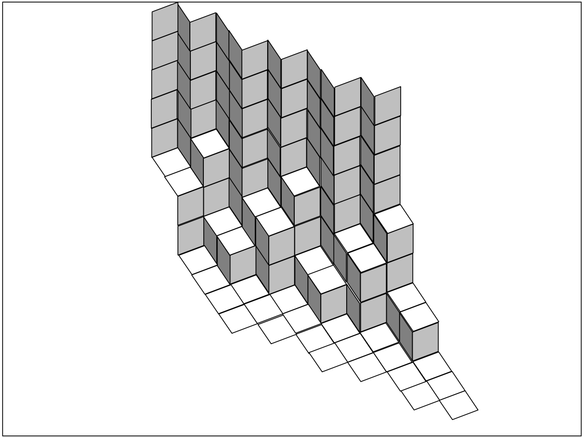

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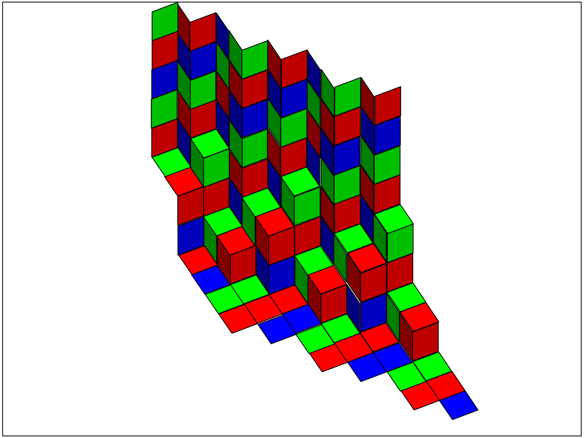

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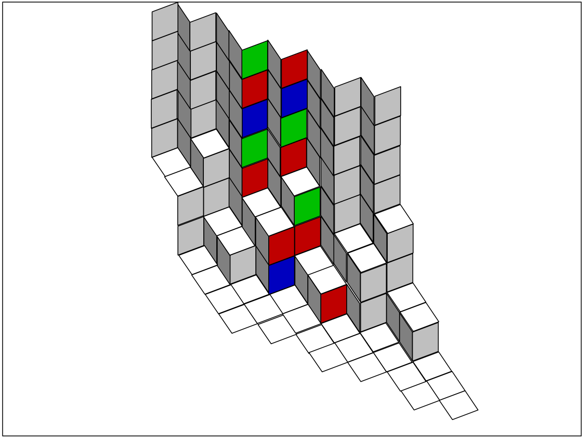

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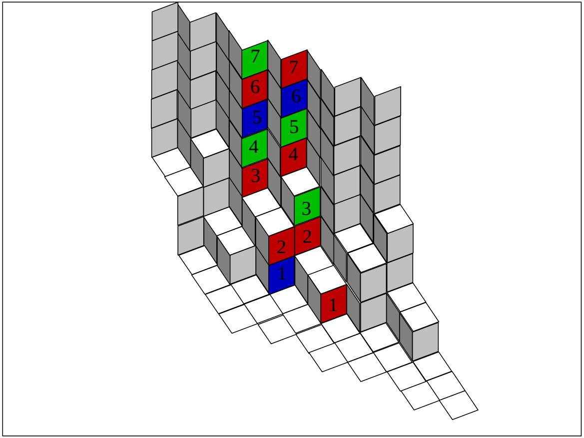

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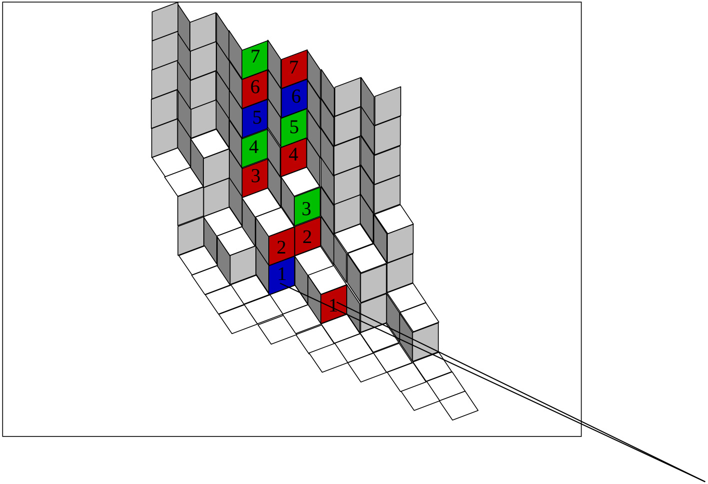

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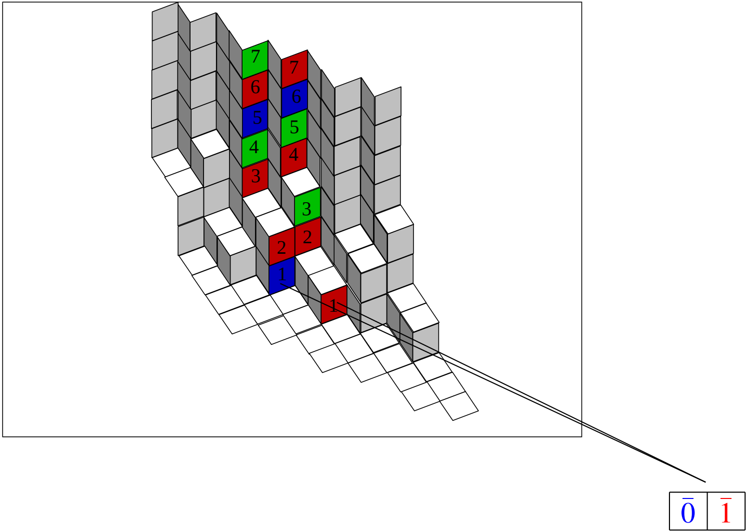

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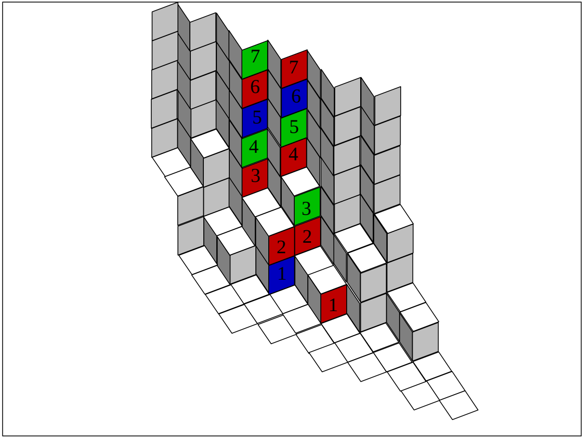

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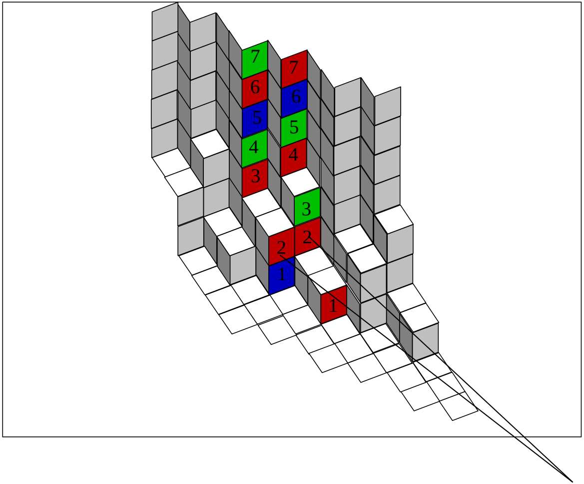

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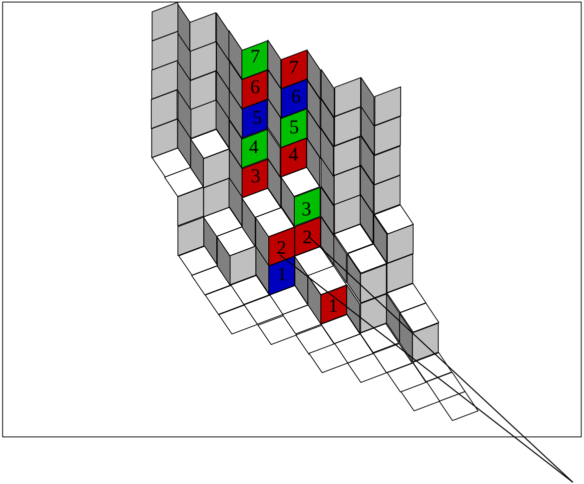

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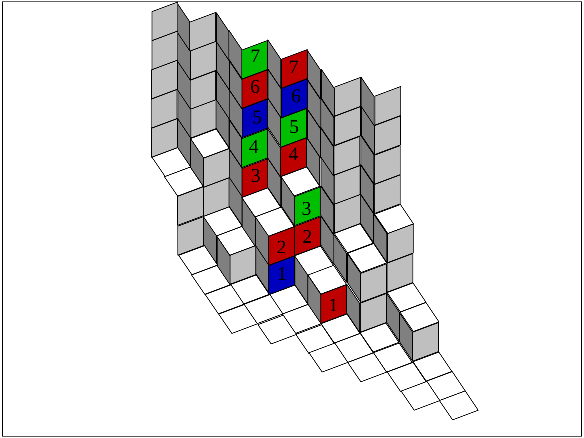

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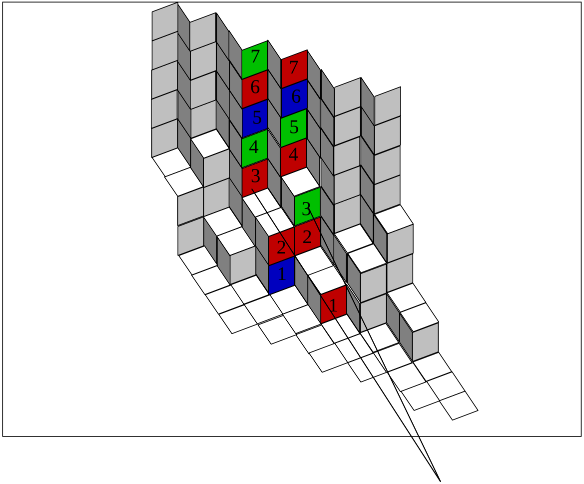

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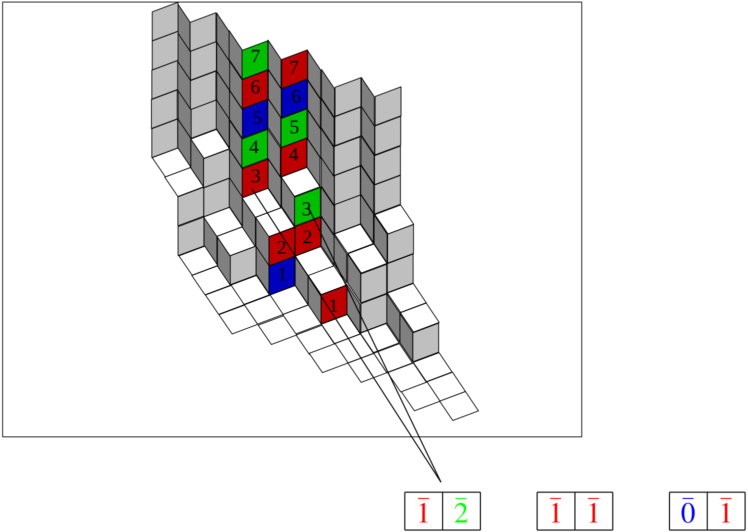

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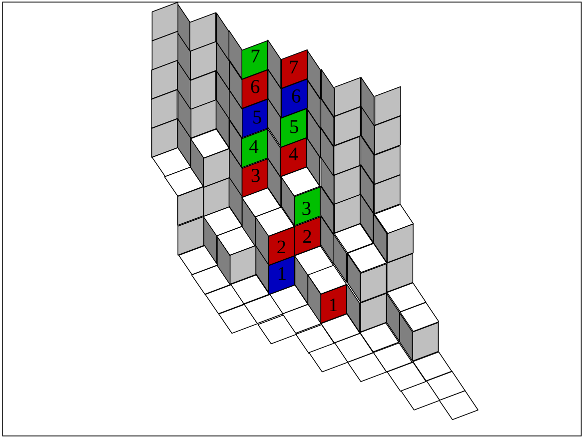

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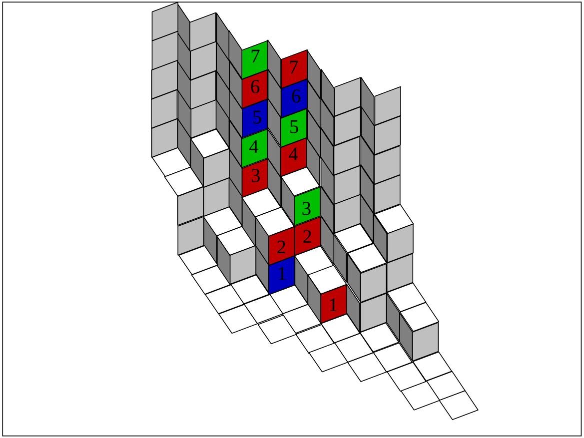

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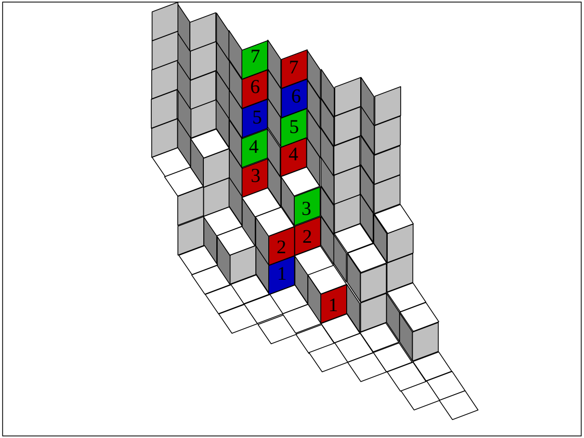

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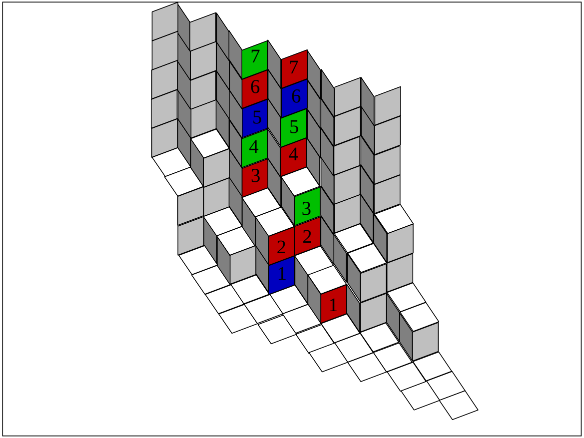

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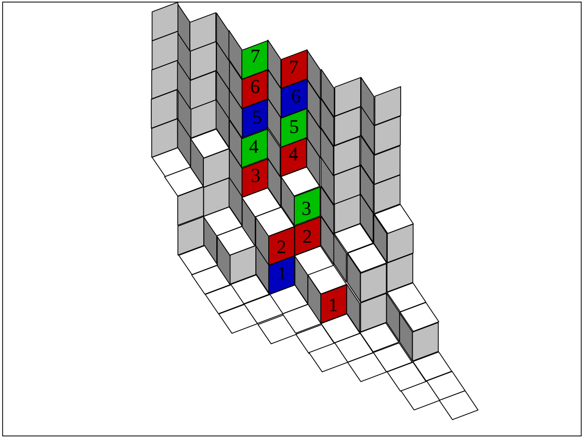

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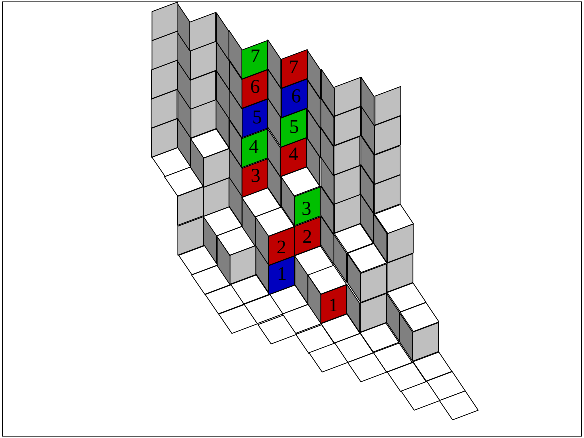

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Recent developement: Berg/Fayers' crystals

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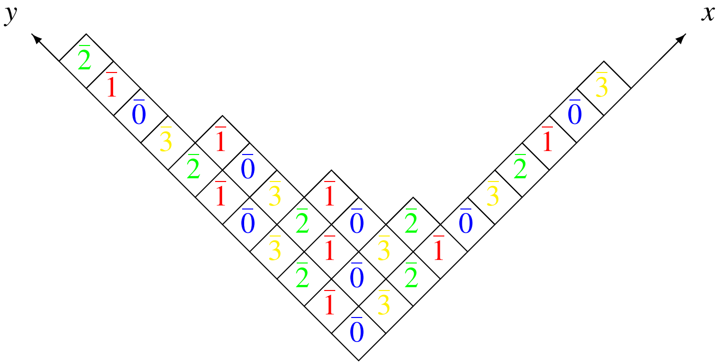

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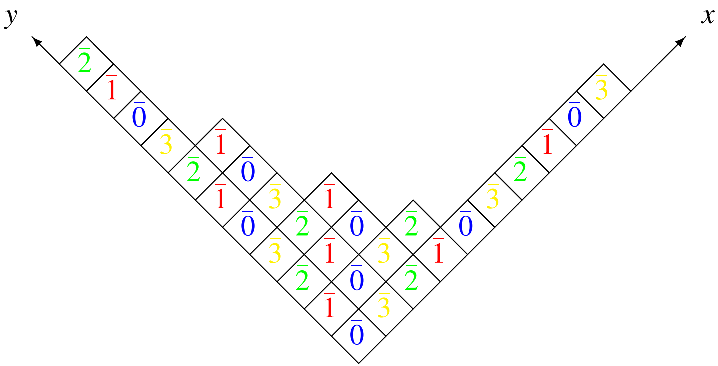

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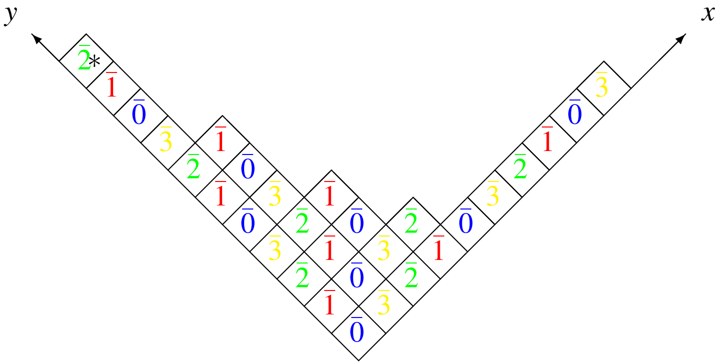

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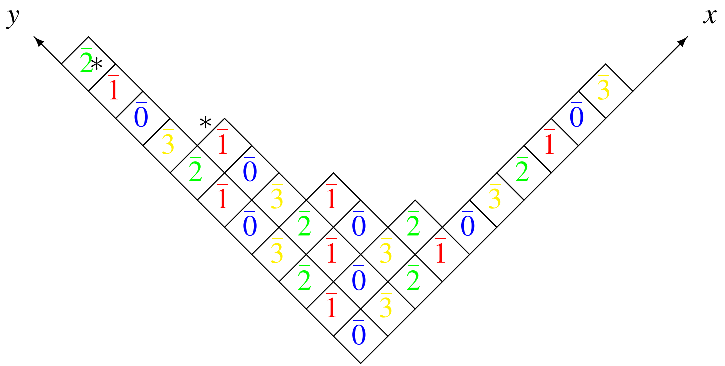

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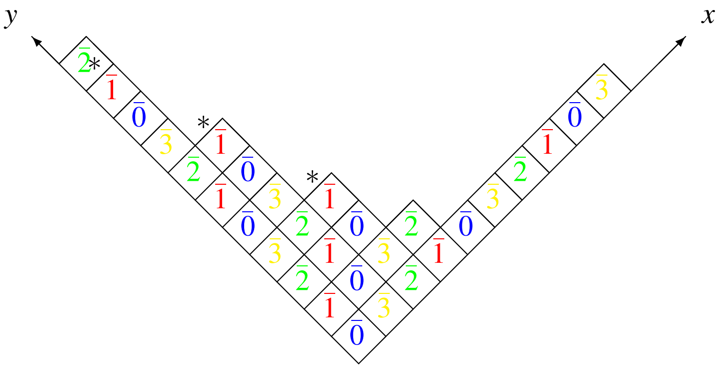

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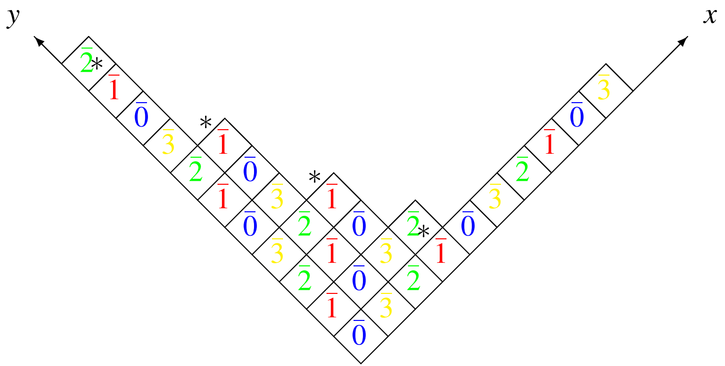

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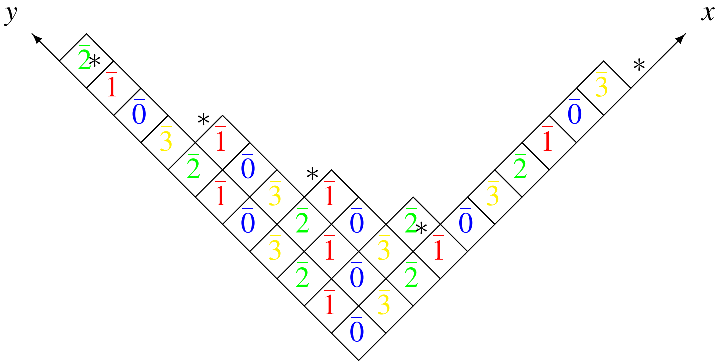

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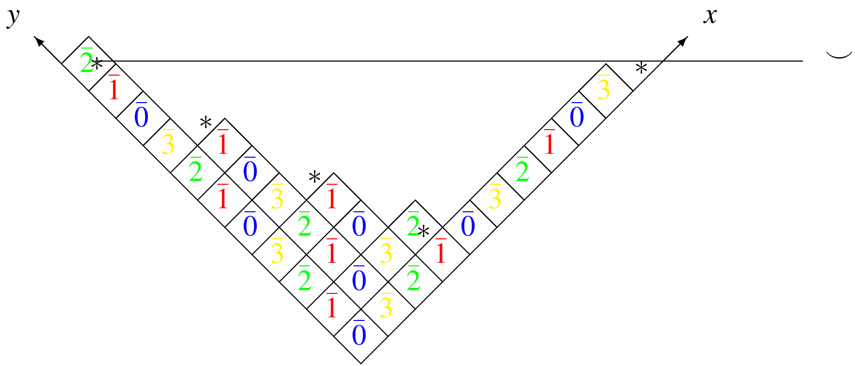

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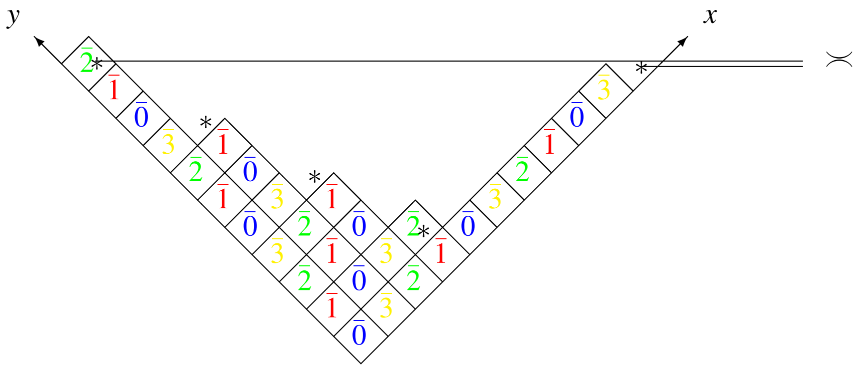

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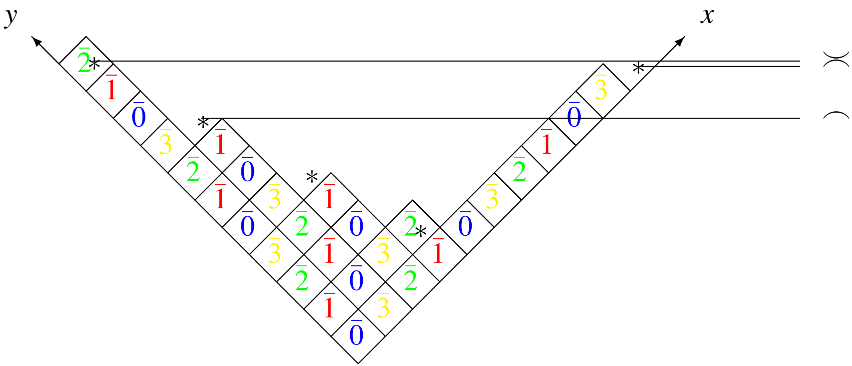

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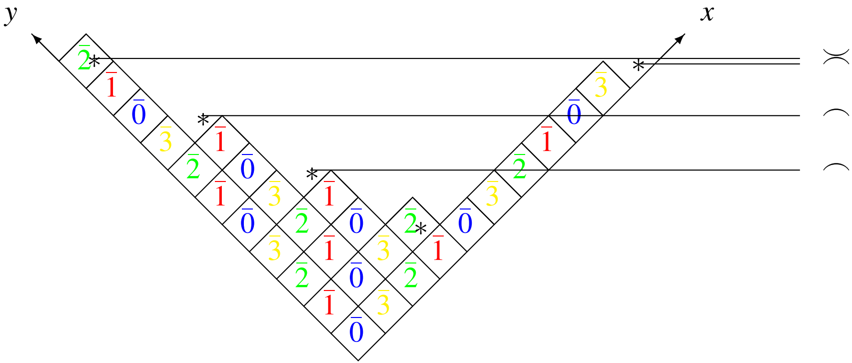

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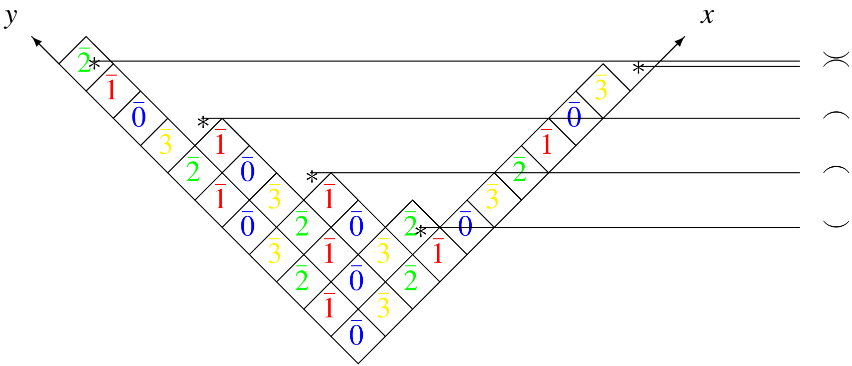

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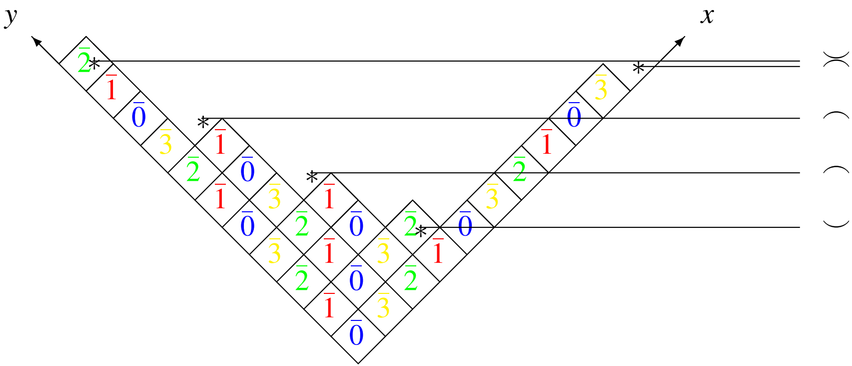

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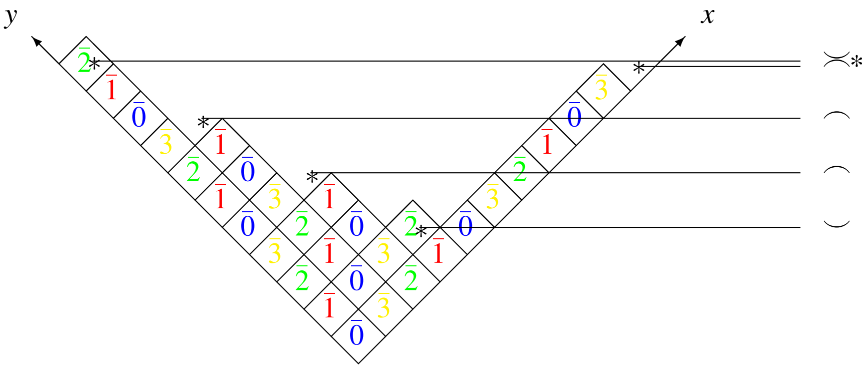
Recent developement: Berg/Fayers' crystals



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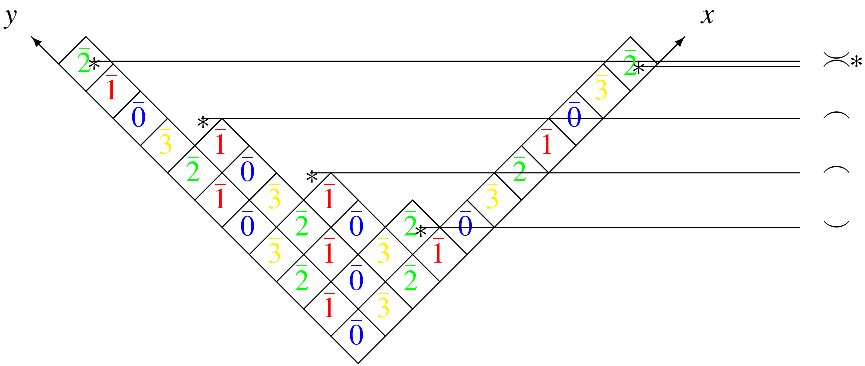

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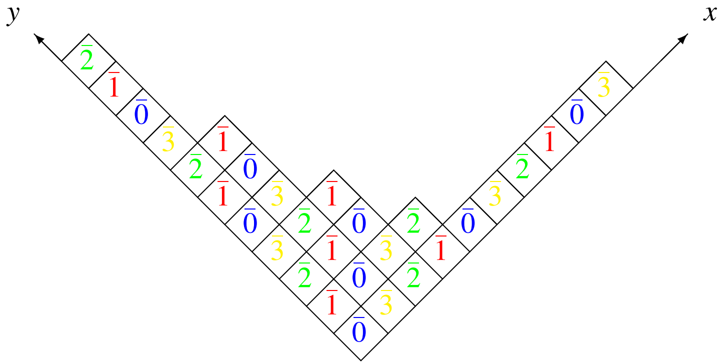

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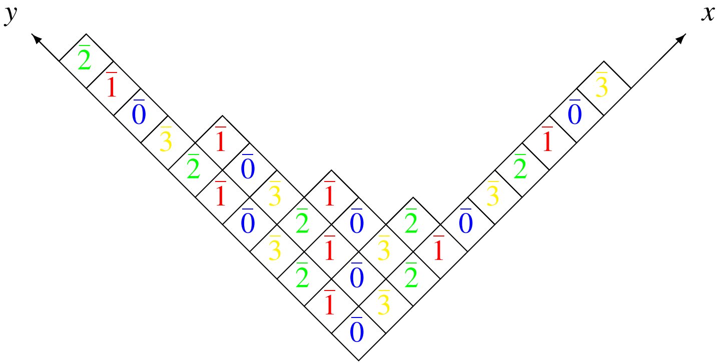

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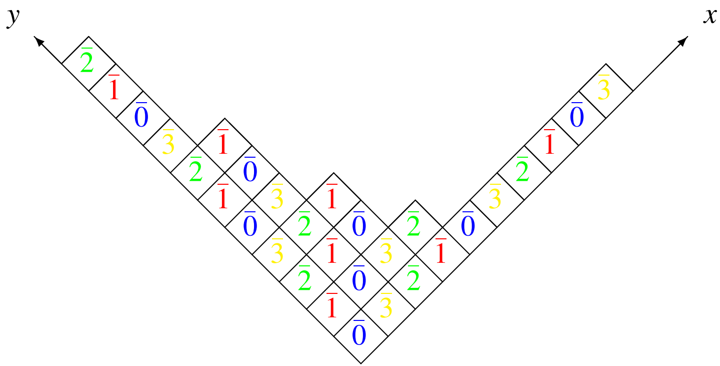

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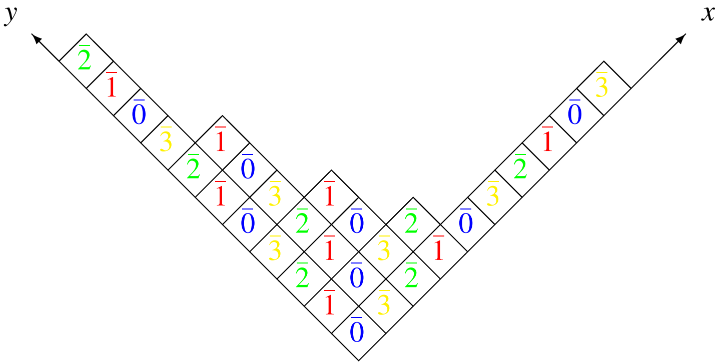

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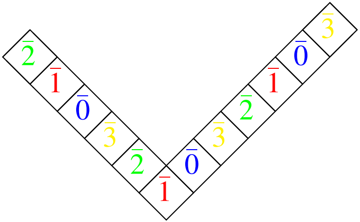

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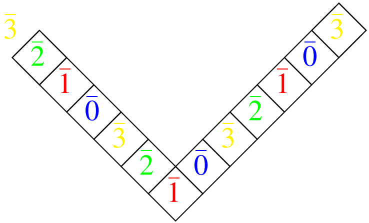

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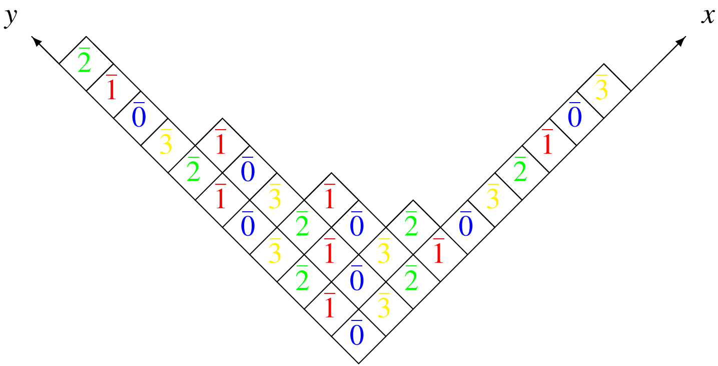

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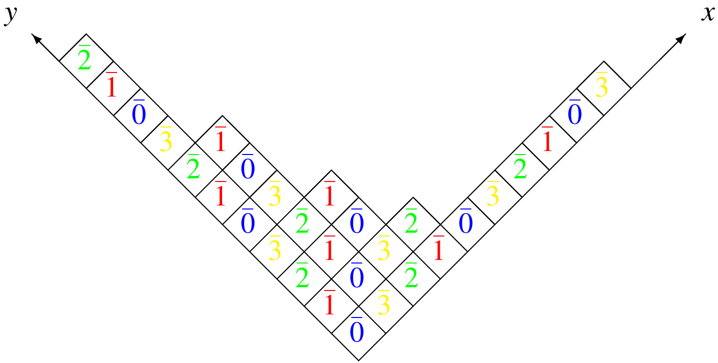

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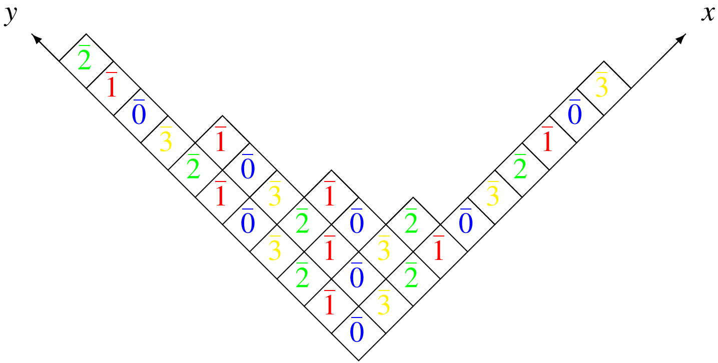

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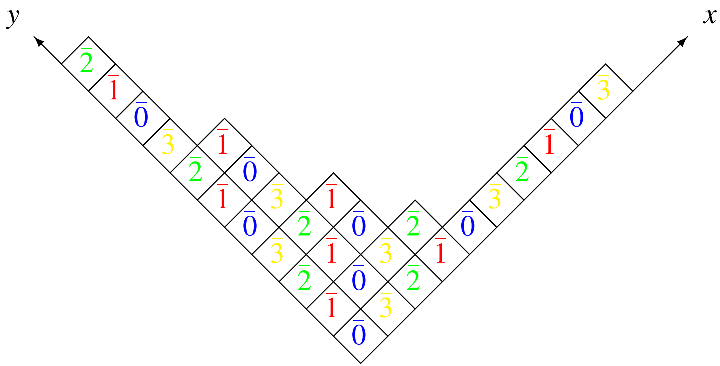

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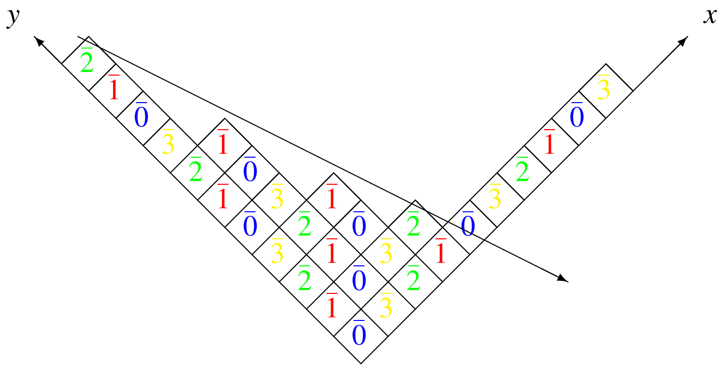

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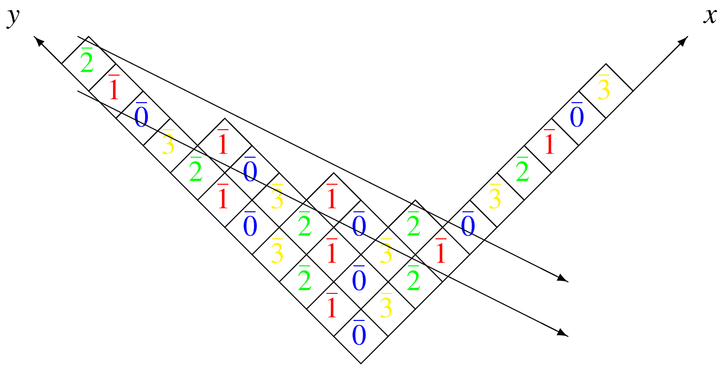

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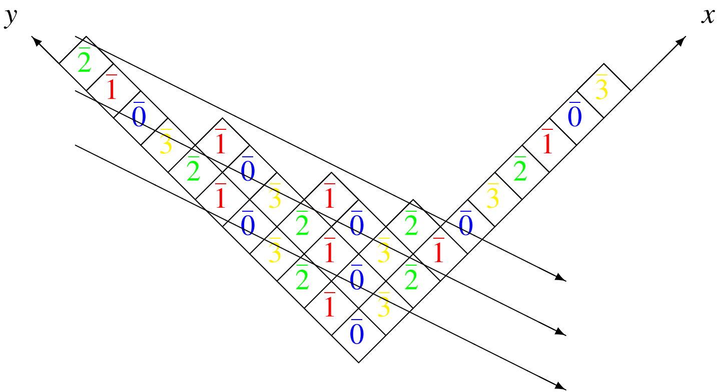

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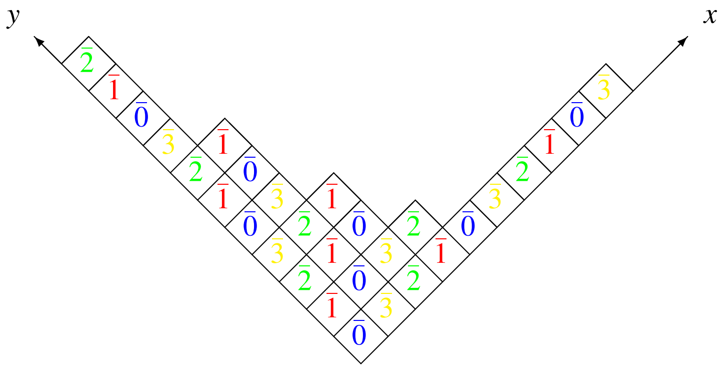

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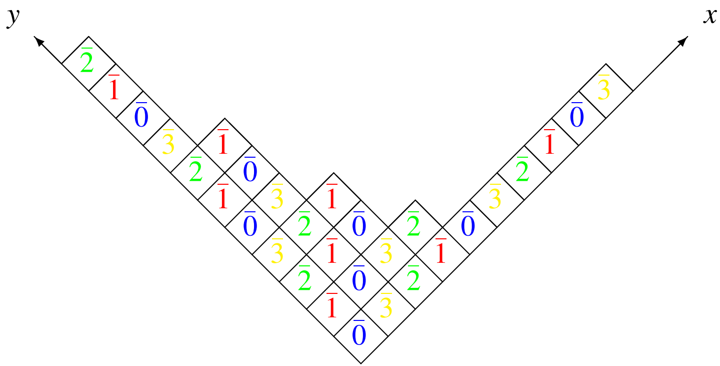

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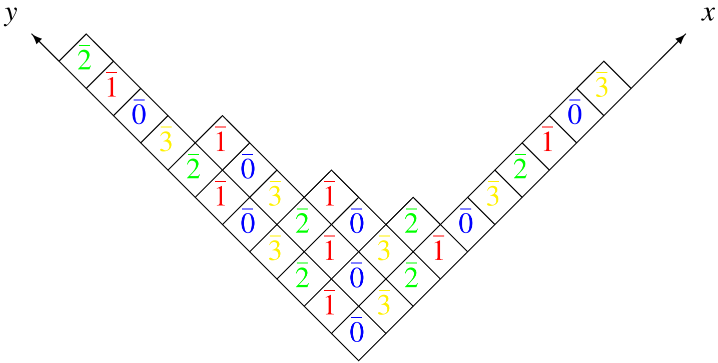

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