LECTURE 10: HIGHEST WEIGHT CRYSTALS FROM QUIVER VARIETIES

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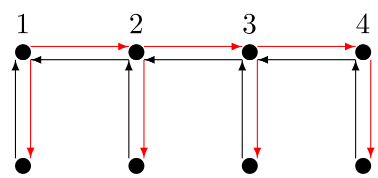
We saw in lectures 7 and 8 how Lusztig's nilpotent variety can be used to realize $U^-(\mathfrak{g})$ and the crystal $B(\infty)$. Last week we saw how to use quiver grassmannians to realize the highest weight modules $V(\lambda)$ as a quotient of $U^-(\mathfrak{g})$, and the same construction realizes the crystals $B(\lambda)$. This week we discuss a more standard approach to realizing $V(\lambda)$ and $B(\lambda)$, namely we will use Nakajima's quiver varieties. The two approaches essentially equivalent since the lagrangian quiver variety $\mathfrak{L}(v, W)$ whose irreducible components will index $B(\lambda)$ in todays story is actually isomorphic to a quiver grassmannian we used last week (see [L98, ST11, S10]). However, Nakajima's approach has one significant advantage, in that $\mathfrak{L}(v, W)$ is realized as a subvariety of a smooth variety $\mathfrak{M}(v, W)$ (and in fcat, is a lagrangiann subvariety with respect to a certain symplectic form).

In the second half of this lecture we discuss some methods of extracting known combinatorics (such as Young tableau) from the realization of $B(\lambda)$ and $B(\infty)$ using quiver varieties.

1. Nakajima's construction

Fix a Dynkin diagram Γ, and let$Q = (I, H)$ be its doubled quiver, with chosen orientation $\Omega \subset H$. Add "shadow" vertices, one for each vertex of Q, which in examples are shown below the original vertices. For each shadow vertex, add one arrow coming from the original vertex. This larger quiver will be denoted $Q' = (I', H')$. Orient the new edges towards from original vertices to get orientation $\Omega' \subset H'$, so that the red edges are negatively oriented. For example, the A_4 quiver becomes:

(1.1)



Let V and W be I-graded vector spaces. Let $M(V, W)$ be the variety of representations of Q' on $V \oplus W$ (V corresponds to the original vertices and W to the shadow vertices). Consider the symplectic form on M defined as in the case of the construction of Lusztig's nilpotent quiver variety, but for Q'. Thus if $p = (p_a)$ and $q = (q_a)$ are two representations of Q',

$$(p, q) = \sum_{a \in \Omega'} \operatorname{trace}(\varepsilon(a) q_{\overline{a}} p_a).$$

In order to distinguish the different arrows, we denote the horizontal arrows by x the maps corresponding to the down arrows by s, the maps corresponding to the up arrows by t. Thus a representation of Q' will be denoted by a triple (x, s, t) where x is a representation of Q, and $s : V \to W$ and $t : W \to V$ are maps of I-graded vector spaces.

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Let $\mathbf{GL}(V) = \prod_{i\in I} \mathrm{GL}(v_i)$ which acts on $M(V,W)$ as usual. Note that we do not include the $\mathrm{GL}(W_i)$. The moment map $\mu\colon M(V,W) \to \oplus \mathfrak{gl}(V_i)$ is

$$\mu(x,s,t) = \sum_{i\in I}\left(t_i s_i + \sum_{a:i\to j} \varepsilon(a) x_{\overline{a}} x_a \right).$$

Definition 1.2. $\mathfrak{M}(v,W) = \mu^{-1}(0)^s/\mathbf{GL}(v)$ where

$$\mu^{-1}(0)^s = \{(x,s,t) \in \mu^{-1}(0) \mid \text{no nontrivial } x\text{-invariant subspaces in } \ker t\}.$$

□

Remark 1.3. Since we mod out by $\mathbf{GL}(v)$, $\mathfrak{M}(v,W)$ does not depend on the choice of vector space V, but only the dimension. That is why we write v in lower case. Up to isomorphism it also only depends on the dimension of W, but the isomorphism is non-canonical. For that reason I'm being pedantic, and recording the actual vector space W.

Remark 1.4. One important point is tha the action of $\mathbf{GL}(v)$ on $\mu^{-1}(0)^s$ is free. However, $\mu^{-1}(0)^s$ is strictly smaller the the set of points in $\mu^{-1}(0)$ which have trivial stabilizer.

$\mathfrak{M}(v,W)$ is a symplectic quotient (Marsden-Weinstein reduction, see [MW74] of $M(V,W)$), so in particular:

Theorem 1.5. $(\cdot,\cdot)$ *descends to a symplectic form on* $\mathfrak{M}(v,W)$.

Definition 1.6. $\mathfrak{L}(V,W) = \{(x,s,0) \in \mathfrak{M}(V,W) \mid x \text{ is nilpotent}\}$ □

Theorem 1.7. $\mathfrak{L}(v,W)$ *is a Lagrangian subvariety of* $\mathfrak{M}(V,W)$*. In particular, it is of pure dimension*

$$\dim \mathfrak{L}(V,W) = (\alpha,\lambda) - \frac{(\alpha,\alpha)}{2}$$

where $\alpha = \sum_i v_i\alpha_i$ *and* $\lambda = \sum_i w_i\omega_i$.

Stability is an open condition, so in particular:

Lemma 1.8. *For* $Z \in \operatorname{Irr}\Lambda(v)$*, then either*

(i) For generic $x \in Z$*, and generic* s*,* $(x,s,0)$ *is stable, or*
(ii) $(x,s,0)$ *is never stable for any* $x \in Z$ *or any* s.

The following combines results of Nakajima [Nak94, Nak98] and Saito [Sai02].

Theorem 1.9. *Assume we are in condition* (i) *from Theorem 1.8. Then*

(i) $\{[x,s,0] \mid x \in Z,\ (x,s,0) \text{ stable}\}$ *is an irreducible component of* $\mathfrak{L}(v,w)$.
(ii) All irreducible components of $\mathfrak{L}(v,W)$ *are of this form.*
(iii) $\coprod_v \operatorname{Irr}\mathfrak{L}(v,w) = B(\lambda) \subset B(\infty)$*, where as in Lecture 8* $B(\infty)$ *is identified with* $\coprod_v \operatorname{Irr}\Lambda(v)$.

The action of the crystal operators f_i on $\coprod_v \operatorname{Irr}\mathfrak{L}(v,W)$ is then inherited from the action on $\coprod_v \Lambda(v)$, where $f_i(Z)$ is set to 0 if it lands in case (ii) of Theorem 1.8.

2. Extracting combinatorics

We would like to understand how the realization on $B(\lambda)$ in terms of quiver varieties relates to the combinatorial models we've already considered. There are (at least) two fruitful approaches to this question.

2.1. **Conormal bundle approach.** The following is explained in [Sav06]. Fix Γ of finite type. Consider the "single" quiver $Q_\Omega = (I, \Omega)$, which is just an orientation of the Dynkin diagram Γ. In Example (1.1), this is the quiver obtained by deleting the "shadow" vertices, all edges to and from those vertices, and all red edges. The key observation that there is a bijection

$$\{\text{isomorphism classes of dimension } v \text{ representations } Q_\Omega\} \cong \{\text{irreducible components of } \Lambda(v)\},$$

where the irreducible component of $\Lambda(v)$ corresponding to an isomorphism class R of Q_Ω representations in the closure of the subset of $\Lambda(v)$ consisting of these representations whose restriction to Q_Ω is isomorphic to R.

Now restrict to type A, with the orientation $1 \leftarrow 2 \leftarrow \cdots \leftarrow n$. The indecomposable representations Q_Ω are of the form

$$i \leftarrow (i+1) \leftarrow \cdots \leftarrow j$$

for various $1 \leq i < j \leq n$, and isomorphism classes of Q_Ω representations are just sums of these "segments."

Recall that the crystal $B(\lambda) \subset B(\infty)$ is parameterized by column-strict Young tableau. Thus we should have a map from column strict Young tableau to isomorphism classes of representations of Q_Ω. This is given as follows:

$$\begin{array}{|c|c|c|c|c|} \cline{1-2} 4 & 5 \\ \cline{1-3} 3 & 4 & 4 \\ \hline 1 & 1 & 2 & 4 & 5 \\ \hline \end{array} \qquad \longleftrightarrow \qquad \begin{array}{cccccccc} 4 & & & & & & & \\ \downarrow & & & & & & & \\ 3 & 3 & & & & & & \\ \downarrow & \downarrow & & & & & & \\ 2 & 2 & 3 & & 4 & 3 & & \\ \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & & \\ 1 & 1 & 2 & 1 & 3 & 2 & 3 & 2. \end{array} \tag{2.1}$$

Each box in the Young tableau, at height i and containing j, corresponds to the Q_Ω module $i \leftarrow \cdots \leftarrow (j-1)$. The Young tableau corresponds to the direct sum of these modules for each box. An i at height i contributes nothing.

Overall, we have described a pair of maps, first from $\coprod_v \operatorname{Irr} \mathfrak{L}(v, W)$ to isomorphism classes of Q_Ω representations, and then from isomorphism classes of Q_Ω representations to Young tableau with a certain shape. We have also previously described crystal operators e_i and f_i on both $\coprod_v \operatorname{Irr} \mathfrak{L}(v, W)$ on the set of Young tableau. In fact our map is an isomorphism of crystals. One can find a proof of this in [Sav06].

2.2. **Torus fixed-point approach.** Choose a basis $w_1, \dots w_k$ for W, which is compatible with the decomposition $W = \oplus_{i \in I} W_i$. Choose integers n_r for $1 \leq r \leq k$. Define $A_z \in \mathbf{GL}(V)$ to be the element which scales each w_r by z^{n_r}. Define a torus action on $\mathfrak{M}(v, W)$ by

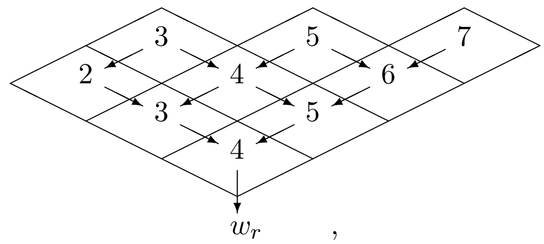
$$z \cdot (x, s, t) = (tx, tA_t s, tA_t^{-1} t).$$

By e.g. [CG97, Lemma 5.11.1], the fixed point locus F of this action consists of a disjoint union of smooth subvarieties of $\mathfrak{L}(v, W) \subset \mathfrak{M}(v, W)$. One can consider the map

$$\begin{aligned} \operatorname{Irr} \mathfrak{L}(v, W) &\to \operatorname{Irr} F \\ X &\to \lim_{z \to \infty} z \cdot (x, s, t) \qquad \text{for } (x, s, t) \in X \text{ generic.} \end{aligned} \tag{2.2}$$

This map is always 1-1 (this follows from the fact that z acts with positive weight on the sympletic form), so one can enumerate $\operatorname{Irr} \mathfrak{L}(v, W)$ by certain fixed point components. This is at least starting to look combinatorial.

Now restrict to type A_n. Assuming that $n_1, \dots n_k$ are sufficiently generic, the torus action in fact has isolated fixed points. These fixed points are all direct sums of representations of the following type:



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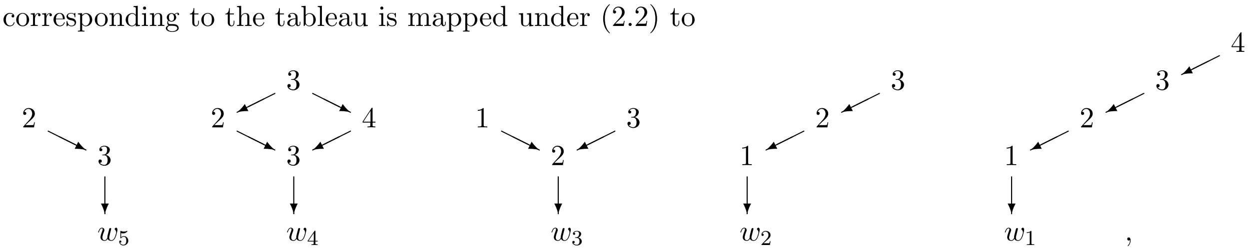
where in this example $w_r \in W_4$. Here the numbers i stand for basis vectors in V_i, and the arrows are matrix coefficients of i for the appropriate x_a. By "of this type," I mean that there is a single w_r, and one can find a basis and an arrangement of that basis into the squares of a partition as in the figure such that there is a matrix element of 1 mapping the basis element corresponding to one box to the basis elements corresponding to the boxes immediately to the southeast and southwest of that box, and all other matrix coefficients are 0.

The representation shown above can be recorded as a column-strict Young tableau on a single column of height r, by recording $s+1$ each time s appears as the top-right entry in a southwest-northeast diagonal (the $+1$ is needed to match standard conventions). If there are less then r non-trivial diagonals, fill up the column with $1, 2, \ldots$. The above example becomes

$$\begin{array}{|c|}\hline 8\\\hline 6\\\hline 4\\\hline 1\\\hline\end{array}\,.$$

If the integers n_r are chosen carefully, the columns corresponding to the fixed points in the image the map (2.2) form a column-strict Young tableau. For instance, in the example from (2.1), one should take $w_1, w_2 \in W_1, w_3 \in W_2, w_4, w_5 \in W_3$ and choose $n_1 >> n_2 >> n_3 >> n_4 >> n_5$ (in fact, having each difference be at least 2 would be enough). Then the irreducible component corresponding to the tableau is mapped under (2.2) to



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which gives the 5 columns of the original tableau. Note that one could choose $n_1, \ldots n_5$ differently (although subject to some conditions to ensure that the fixed points remain isolated), and one should see non-trivially different combinatorics.

In cases with non-isolated fixed points the correspondence between the fixed point locus and other combinatorics is less trivial. However, I believe that in many cases it still gives a useful way to approach this issue.

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