LECTURE 12: MV POLYTOPES FROM LUSZTIG DATA AND FROM QUIVER VARIETIES

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In this last lecture, I will discuss MV polytopes, which are natural polytopes parameterizing $B(\infty)$ in finite type. Last week Dinakar Muthiah discussed how these polytopes arise from Mirkovic and Vilonen's work [MV] on the affine grassmannian. Today I'll discuss two other natural constructions of the same polytopes. The first comes directly from Lusztig's algebraic construction of $B(\infty)$, and the second (which is only valid in type ADE) comes from quiver varieties. That that the construction using Lusztig's parameterization agrees with the construction from Mirkovic and Vilonen's work is essentially due to Kamnitzer [Kam], and that this agrees with the construction from quiver varieties is due to Baumann and Kamnitzer [BK]. It is somewhat remarkable that al three constructions lead to the same polytopes, and I think this is a good argument that these polytopes are the "natural" combinatorial objects parameterizing finite type crystals.

At the end I'll briefly mention how some of this story can be generalized, at least to symmetric affine types. This generalization is the subject of some ongoing research with Pierre Baumann and Joel Kamnitzer.

1. Construction from Lusztig's parametrization of $B(\infty)$

Fix a reduced expression $w_0 = s_{i_1} \cdots s_{i_N}$. This gives an enumeration of the positive roots: $\beta_1 = \alpha_{i_1}$ and $\beta_j = s_{i_1} \cdots s_{i_{j-1}} \alpha_{i_j}$ for $2 \leq j \leq N$. Lusztig"s braid group action of $U_q(\mathfrak{g})$ also gives a set of root vectors

$$X_{\beta_1} = F_{i_1}, \quad X_{\beta_2} = T_{i_1} \alpha_{i_2}, \quad \ldots, \quad X_{\beta_j} = T_{i_1} \cdots T_{i_{j-1}} F_{i_j}.$$

Then $\{X_{\beta_N}^{(a_n)} \cdots X_{\beta_1}^{(a_1)}\}$ is a PBW type basis for $U_q^-(\mathfrak{g})$, and its residue mod q is a crystal basis.

Each $b \in B(\infty)$ then has many expressions of the form $X_{\beta_N}^{(a_n)} \cdots X_{\beta_1}^{(a_1)}$, one for each expression for w_0. It is natural to ask how these different expressions are related to each other.

Consider first $\mathfrak{g} = \mathfrak{sl}_3$. There are exactly two reduced expressions for w_0:

$$\mathbf{i}_1 := s_1 s_2 s_1 \quad \text{and} \quad \mathbf{i}_2 := s_2 s_1 s_2.$$

I will use a superscript of $\mathbf{i}_1, \mathbf{i}_2$ to denote the Lusztig root vectors constructed with respect to the expressions $\mathbf{i}_1$ and $\mathbf{i}_2$ respectively. By direct calculation, one finds that

$$(X_{\alpha_2}^{i_1})^{(1)} (X_{\alpha_1+\alpha_2}^{i_1})^{(2)} (X_{\alpha_1}^{i_1})^{(3)} = (X_{\alpha_1}^{i_2})^{(4)} (X_{\alpha_1+\alpha_2}^{i_2})^{(1)} (X_{\alpha_2}^{i_3})^{(2)} \mod q,$$

and thus these are expression for the same element $b \in B(\infty)$. These two basis vectors give paths in weight space $\mathfrak{h}^*$:

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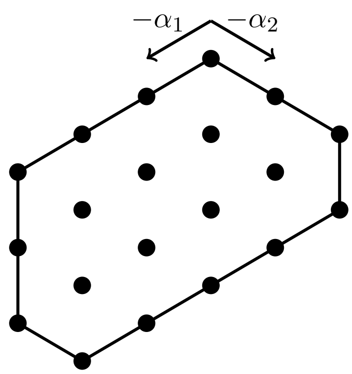
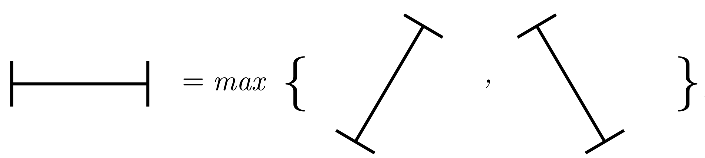


Here the lengths of the edges of the path from the top to the bottom along the left side of the polytope record the exponents in the expression for b with respect to $\mathbf{i}_1$, and the path on the right records the exponents of the expression in terms of $\mathbf{i}_2$. More generally, every expression for w_0 will give a path in weight space, and one can see that the union of these paths is the 1-skeleton of a polytope. This is the MV polytope P_b for b.

We would like to characterize which polytopes arise as P_b for some $b \in B(\infty)$. For the $\mathfrak{sl}_3$ case, the answer is given by the following, which follows from work of Lusztig [L, Chapter 42.2]:

Theorem 1.1. *A polytope P is an MV-polytope of type $\mathfrak{sl}_3$ if and only if the widths of the polytope in the three lattice directions satisfy*



For reasons explained in [Kam], the relation in the above theorem is called a "Tropical Plucker relation." In the other rank two types (i.e. B_2 and G_2), there are similar but somewhat more complicated relations that characterize MV polytopes (see [Kam]). For the second half of this talk we will restrict to simply laced cases, so we will only need to understand type $\mathfrak{sl}_3$ and $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ MV polytopes. In the $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ there are actually no conditions (i.e. all rectangles appear as MV polytopes). The following shows that understanding the rank two case is essentially enough. It is also proven in [L, Chapter 42].

Theorem 1.2. *In general, a polytope P is an MV-polytope if and only if every 2-dimensional face is an MV-polytope of the correct type.* □

Remark 1.3. This last theorem corresponds to the fact that any two reduced expressions are related by braid moves. Note that the roles of the three widths are not symmetric, so one needs to identify which vertex is the "top" of a general 2-face. The correct choice of top vertex is just the vertex corresponding to the shortest element of the Weyl group. □

In fact, it follows from Lusztig's results on parameterizing $B(\infty)$ that the path in the 1-skeleton corresponding to a single expression for w_0 uniquely determines the polytope, and furthermore any candidate for such a path does correspond to a polytope. This leads to a description of the crystal operators on MV polytopes: To apply f_i to P_b:

1. Find an expression $\mathbf{i}$ for w_0 such that the final root β_N in Lusztig's enumeration is α_i.
2. Change the path in the 1-skeleton corresponding to $\mathbf{i}$ by increasing the length of the final edge by 1.
3. Figure out how the rest of the polytope must change using the tropical plucker relations on the 2-faces.

2. Construction from quiver varieties

Earlier in this seminar we described a realization of $B(\infty)$ (for all simply laced types) in terms of the quiver varieties $\Lambda(v)$. Specifically, the vertices of $B(\infty)$ were identified with $\sqcup_v \operatorname{Irr}\Lambda(v)$, the set of irreducible components of these varieties as the dimension vector v varies. Since MV polytopes also parameterize $B(\infty)$, this gives a bijection between irreducible components and MV polytopes. We now describe how to construct the corresponding polytope directly in terms of the geometry of the quiver varieties (or perhaps more accurately in terms of the structure of the category of representations of the preprojective algebra), thus giving another way to approach these very natural polytopes.

First let me sketch how the polytope is constructed from the irreducible component:

- Fix a representation T of the preprojective algebra $\overline{\mathcal{P}}$.
- For each reduced expression $\mathbf{i}$: $w_0 = s_{i_1} \cdots s_{i_N}$, we will get a canonical filtration $T = T^{\mathbf{i}}_N \supset \cdots \supset T^{\mathbf{i}}_1 = 0$.
- It is always the case that $\{-\sum_{i,k} \dim T^{\mathbf{i}}_k\}$, as $\mathbf{i}$ and k vary, are the vertices of a polytope.
- If T is generic in an irreducible component $Z_b \in \operatorname{Irr}\Lambda(v)$, then this is the MV-polytope P_b.

To make this precise we need to describe the subrepresentations $T^{\mathbf{i}}_k$ that appear in these filtrations. We will give two definitions.

2.1. First characterization of $T^{\mathbf{i}}_k$.

Definition 2.1. Given T, define $T^{\mathbf{i}}_k$ to be the maximum (by containment) submodule of T such that the dimension of all quotients Q of $T^{\mathbf{i}}_k$ are in the $\mathbb{N}$-span of $\beta_1, \dots, \beta_k$. $\square$

It turns out that $T^{\mathbf{i}}_k / T^{\mathbf{i}}_{k-1} \cong (R^{\mathbf{i}}_k)^{\oplus d_k}$ where $R^{\mathbf{i}}_{\beta_k}$ is a certain indecomposable representation labeled by β_k, although with the current definition that is not immediately obvious.

Example 2.2. $\mathfrak{g} = \mathfrak{sl}_4$, write $w_0 = s_1 s_2 s_3 s_2 s_1 s_2$. The corresponding ordering of the positive roots is $\{\alpha_1, \alpha_1+\alpha_2, \alpha_1+\alpha_2+\alpha_3, \alpha_3, \alpha_2+\alpha_3, \alpha_2\}$, and the indecomposable representations $R^{\mathbf{i}}_\beta$ are

$$\begin{array}{lcccccc}
R^{\mathbf{i}}_{\alpha_2} & = & & & 2 & & \\
R^{\mathbf{i}}_{\alpha_2+\alpha_3} & = & & & 2 & \longleftarrow & 3 \\
R^{\mathbf{i}}_{\alpha_3} & = & & & & & 3 \\
R^{\mathbf{i}}_{\alpha_1+\alpha_2+\alpha_3} & = & 1 & \longrightarrow & 2 & \longrightarrow & 3 \\
R^{\mathbf{i}}_{\alpha_1+\alpha_2} & = & 1 & \longrightarrow & 2 & & \\
R^{\mathbf{i}}_{\alpha_1} & = & 1 & . & & &
\end{array}$$

One can easily see that each $R^{\mathbf{i}}_\beta$ is characterized by the property that it has dimension β_k, and has no submodules isomorphic to an earlier $R^{\mathbf{i}}_{\beta'_k}$ (i.e. no submodules isomorphic to the modules drawn lower in the chart). For any choice of a multiplicity for each $R^{\mathbf{i}}_{\beta_k}$, the subset of $\Lambda(v)$ consisting of modules with that prescribed filtration is dense in some irreducible component. $\square$

2.2. Definition of $T^{\mathbf{i}}_k$ using reflection functors.

The previous section does not make it clear that the modules $R^{\mathbf{i}}_{\beta_k}$ exist, or that they are unique. Here we discuss how the submodules $T^{\mathbf{i}}_k$, along with the $R^{\mathbf{i}}_{\beta_k}$, can be explicitly constructed. This is done using the following reflection functors on the category of representations of the completed preprojective algebra $\overline{\mathcal{P}}$.

Definition 2.3. Fix a representation T of $\overline{\mathcal{P}}$. Then the define two "reflection functors" by:

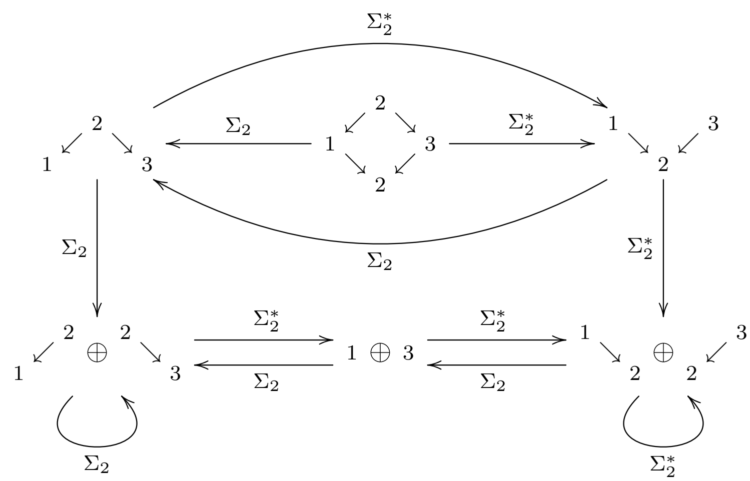
(1) $\Sigma_i T$ is obtained by taking a generic extension
$$0 \to T \to T' \to S_i^{\oplus K} \to 0$$
for large K, then taking the quotient of T' by the largest submodule isomorphic to a direct sum of S_i. This stabilizes for large K, as if more S_i are added they contribute to the socle of T', and hence are removed at the next step.

(2) $\Sigma_i^* T$ is obtained by taking a generic extension
$$0 \to S_i^{\oplus K} \to T' \to T \to 0$$
for large K, then setting $\Sigma_i^* T$ to be the largest submodule of T' such that the quotient $T'/\Sigma_i^* T$ is a direct sum of copies of S_i.

□

Remark 2.4. Definition 2.3 may not seem completely functorial, but it can be described in a completely functorial way, as is done in [BK]. The current description certainly gives the right isomorphism class of representation, which is all we really need. Note also that Σ_i and Σ_i^* are adjoint functors. □

Example 2.5. The action of these "reflection functors" can be more complicated then one might at first guess. The following depicts how these functors act on various modules in the $\mathfrak{sl}_4$ case. This diagram was taken from [BK]:



□

Example 2.5 may cause the reader to wonder why these operations are called "reflection" functors. In fact, on certain modules they do act more like reflections. Specifically:

Theorem 2.6. *(see [BK]) The reflection functors Σ_i and Σ_i^* define inverse equivalence of categories*
$$\left\{ \begin{matrix} \overline{\mathcal{P}}\text{modules} \\ \text{with trivial } i\text{-head} \end{matrix} \right\} \underset{\Sigma_i^*}{\overset{\Sigma_i}{\rightleftarrows}} \left\{ \begin{matrix} \overline{\mathcal{P}}\text{-modules} \\ \text{with trivial } i\text{-socle} \end{matrix} \right\}.$$
Furthermore, in these cases both Σ_i and Σ_i^* act on $\gamma := \sum_i (\dim V_i)\alpha_i$ by the reflection s_i.

One can then explicitly write expressions for the $V_k^{\mathbf{i}}$: If $\mathbf{i} = (i_1, \ldots, i_N)$, so that $w_0 = s_{i_1} \cdots s_{i_N}$. Let $i^{\mathrm{op}} = (i_N, \ldots i_1)$. Then
$$V_{N-k}^{\mathbf{i}^{\mathrm{op}}} = \Sigma_{i_1}^* \cdots \Sigma_{i_k}^* \Sigma_{i_k} \cdots \Sigma_{i_1} V.$$

There is also an explicit expression for the root modules $R^{\mathbf{i}}_{\beta_k}$:

$$R^{\mathbf{i}}_k = \Sigma_{i_1} \cdots \Sigma_{i_{k-1}} S_{i_k}.$$

2.3. **Affine MV polytopes.** In this case there is no longest element of the Weyl group, so it does not make sense to speak of expressions for w_0. However, all we really needed from such an expression was the ordering if gave on the positive roots. The essential property of this ordering is that it is **biconvex**: the $\mathbb{N}$-span of any initial segment and the $\mathbb{N}$-span of any final segment are disjoint.

If we use this notion, then, to have a hope of finding a biconvex order in affine type, we must treat all imaginary roots as equivalent, since they are proportional. All imaginary roots point is the direction δ, and it is in fact possible to find convex total orderings on the set of positive real roots together with δ.

Much of the analysis from this lecture goes through in the affine case, and in particular one finds modules $R^{\mathbf{i}}_\beta$ for each real root β. But at the level of the filtration where the sub-quotient has dimension a multiple of δ things are considerably more complicated: for each k, there are r families of indecomposable representations of dimension $k\delta$ which have no submodules isomorphic to earlier $R^{\mathbf{i}}_\beta$, where r is the rank of the underlying finite type Lie algebra. These families correspond to the imaginary roots with their multiplicities, and can be used to define an analogue of MV polytopes in the symmetric affine case. See the upcoming work [BKT]. The affine MV polytopes will not just be polytopes though, as we will need to include "decorations" recording the imaginary root data.

References

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