LECTURE 2: CRYSTAL BASES

STEVEN SAM AND PETER TINGLEY

Today I'll define crystal bases, and discuss their basic properties. This will include the tensor product rule and the relationship between the crystals $B(\lambda)$ of highest weight modules and the infinity crystal $B(\infty)$. I'll then discuss an intrinsic characterization of crystal lattices and crystal bases in terms of a natural bilinear form (sometimes called "polarization"). Unless otherwise stated, the results here are due to Kashiwara, and proofs can be found in [K1].

For today, $\mathfrak{g}$ is a symmetrizable Kac–Moody algebra, $U_q(\mathfrak{g})$ is its quantized universal enveloping algebra, that V is a representation which is a (not necessarily finite) direct sum of integrable highest weight modules $V(\lambda)$ (i.e., an object in $\mathcal{O}^{int}$).

1. Definition of crystal bases

The goal is to find a basis of a representation V of $U_q(\mathfrak{g})$ such that the Kashiwara operators $\widetilde{E}_i$, $\widetilde{F}_i$ act by partial permutations. This will allow us to "draw" $V(\lambda)$ as a colored directed graph. However, simple calculations show that this is impossible, even for the adjoint representation of $\mathfrak{sl}_3$. But, in a sense that will be made precise today, this will work "at $q = 0$".

Note: we have switched conventions from last week, so that our crystals are at $q = 0$ rather then $q = \infty$. The first part of the theory works just as well at either 0 or ∞, but once we start discussing tensor products our choice of coproduct will determine this choice. So, the reason we have changed this convention is that we are changing our convention for coproduct. I did this to match Kashiwara's conventions in [K1, K2], which are the main references for the next few lectures. Perhaps this is a good time to note that there are four choices of coproduct that work equally well in this theory:

$$\begin{array}{lll}
\text{(i)} & \Delta(E_i) = E_i \otimes K_i^{-1} + 1 \otimes E_i; & \Delta(F_i) = F_i \otimes 1 + K_i \otimes F_i \\
\text{(ii)} & \Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i; & \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i \\
\text{(iii)} & \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i; & \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i \\
\text{(iv)} & \Delta(E_i) = E_i \otimes 1 + K_i^{-1} \otimes E_i; & \Delta(F_i) = F_i \otimes K_i + 1 \otimes F_i.
\end{array}$$

One must make a choice, and all four choices have been used in the literature. We will be using the first one from now on, whereas last week we used the second one.

Let us also recall the definition of Kashiwara operators on a representation V of $U_q(\mathfrak{g})$. For each i, $\langle E_i, F_i, K_i^{\pm 1}\rangle$ is a copy of $U_q(\mathfrak{sl}_2)$ inside $U_q(\mathfrak{g})$. The irreducible representations of $U_q(\mathfrak{sl}_2)$ are root strings

$$\begin{tikzcd}
\circ \arrow[r, bend left, "1"] & \circ \arrow[l, bend left, "{[3]}"] \arrow[r, bend left, "{[2]}"] & \circ \arrow[l, bend left, "{[2]}"] \arrow[r, bend left, "{[3]}"] & \circ, \arrow[l, bend left, "1"]
\end{tikzcd}$$

where the arrows to the right are the matrix coefficients of F_i, and the ones to the left are the matrix coefficients of E_i. Decompose V into a direct sum of irreducible representations of $U_q(\mathfrak{sl}_2)$. The Kashiwara operator $\widetilde{F}_i$ is defined as the operator that moves 1 step down in each root string, without multiplying by any quantum integer. Similarly, $\widetilde{E}_i$ moves 1 up the root string. This is independent of the chosen decomposition.

Let $A_0 = \{f(q) \in \mathbb{C}(q) \mid f \text{ regular at } 0\}$. This is a local ring, and $A/qA \cong \mathbb{C}$.

Date: February 18, 2011.

Definition 1.1. Let V be an integrable module over $\mathbb{C}(q)$. A **crystal lattice** in V is an A_0-submodule $\mathcal{L} = \bigoplus_{\lambda \in P} \mathcal{L}_\lambda \subset V$ compatible with the weight decomposition of V, which is closed under $\widetilde{F}_i$ and $\widetilde{E}_i$. A **crystal basis** is a basis for $\mathcal{L}/q\mathcal{L}$ such that the induced actions of $\widetilde{F}_i$ and $\widetilde{E}_i$ on $\mathcal{L}/q\mathcal{L}$ act by partial permutations on B. □

Theorem 1.2. (1) *Any integrable V has a crystal basis $(\mathcal{L}, B)$.*
(2) *Given $(\mathcal{L}^{(1)}, B^{(1)})$, and $(\mathcal{L}^{(2)}, B^{(2)})$, there is an automorphism Ψ of V such that $\Psi(\mathcal{L}^{(1)}) = \mathcal{L}^{(2)}$ and $\Psi(B^{(1)}) = B^{(2)}$.*
Note that (2) implies in particular that, if $V = V(\lambda)$ is irreducible, then the structure of $B(\lambda)$ is unique.

Remark 1.3. Given an integrable module V, let $\{v_1, \ldots, v_k\}$ be a weight basis for the space of highest weight vectors of V. Set $B = \{\widetilde{F}_{i_N} \cdots \widetilde{F}_{i_1} v_j\}$ and $\mathcal{L}$ to be the A_0-span of B. Then $\mathcal{L}$ is a crystal lattice and the non-zero elements in the image of B in $\mathcal{L}/q\mathcal{L}$ form a crystal basis. □

Remark 1.4. If v is a highest weight vector, then $\widetilde{F}_i^n v = F_i^{(n)} v$ where $F_i^{(n)} := F_i^n/[n]_q!$ is the nth divided power of F_i. This is one explanation for why divided powers show up so often. □

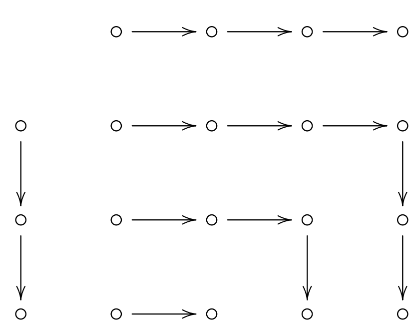
Remark 1.5. $U_\hbar(\mathfrak{g})$ is a trivial deformation of $U(\mathfrak{g})$ as an algebra. So the representation theory of both algebras is the same (until we start taking tensor product). $U_q(\mathfrak{g})$ is not quite a deformation, but its representation theory is still very similar (essentially there are some new "non type **1**" representations, but the remaining representations look the same). So it is a bit suspicious that the crystal basis (which is new) appears out of something which is old. In fact, the crystal basis can be constructed without considering the deformation, see [BK]. However, some of the important properties of crystals are less visible in this picture. In particular, as one would expect, the relationship with tensor products described below is more difficult to see. □

2. Tensor products of crystal bases

Theorem 2.1. *If $(\mathcal{L}^{(1)}, B^{(1)})$ and $(\mathcal{L}^{(2)}, B^{(2)})$ are crystal bases for V and W, then $(\mathcal{L}^{(1)} \otimes \mathcal{L}^{(2)}, B^{(1)} \otimes B^{(2)})$ is a crystal basis for $V \otimes W$.*

Remark 2.2. Until now, the choice of coproduct was not important, and in fact we could have worked with crystal bases at ∞ instead of at 0 just as easily. Theorem 2.1 holds at $q = 0$ for coproducts (i) and (iv), and at $q = 0$ for the other two. The tensor product rule as we give it below only holds for coproduct (i), but a simple modification works for coproduct (iv).

Theorem 2.1 induces a combinatorial tensor product $\otimes$, which we now describe. When $\mathfrak{g} = \mathfrak{sl}_2$, crystals are just directed intervals $B(n)$ with $n+1$ nodes, for each $n \in \mathbb{Z}_{\geq 0}$. The tensor product rule for $B(3) \otimes B(2)$ is expressed graphically as follows:



The first tensor factor is placed at the top, and the second on the left side. The vertices of the tensor product are all pairs of vertices one in each factor, which can be arranged in a grid as shown. The top row and right column of the grid make one irreducible component. Then the nodes second from the top and second from the right. Continue until all the nodes are used up.

For other $\mathfrak{g}$, simply treat the arrows coming from each copy of $U_q(\mathfrak{sl}_2)$ separately. Decompose the crystal into connected components for the arrows corresponding to that $\widetilde{F}_i$, and take the tensor product of each pair. Once this has been done for all the different $\widetilde{F}_i$ (i.e., all different colors of arrows in the crystal graph), the result is the tensor product of the $\mathfrak{g}$ crystals.

Example 2.3. Consider $\mathfrak{g} = \mathfrak{sl}_3$. Then the two fundamental crystals are $B(\omega_1) = \circ \xrightarrow{1} \circ \xrightarrow{2} \circ$ and $B(\omega_2) = \circ \xrightarrow{2} \circ \xrightarrow{1} \circ$. Their tensor product is

$$\begin{tikzcd}
\circ \arrow[r, "1"] \arrow[d, "2"] & \circ \arrow[r, "2"] & \circ \arrow[d, "2"] \\
\circ \arrow[r, "1"] & \circ \arrow[d, "1"] & \circ \arrow[d, "1"] \\
\circ & \circ \arrow[r, "2"] & \circ
\end{tikzcd}$$

which illustrates that $V(\omega_1) \otimes V(\omega_2) \cong V(\omega_1 + \omega_2) \oplus V(0)$. □

Theorem 2.4. *$B(\lambda)$ is connected.*

This implies that for any representation V, its irreducible components correspond to the connected components of its crystal graph. Note that this gives a combinatorial way to study the Clebsch–Gordon rule (i.e., to find multiplicities of various $V(\gamma)$ in $V(\lambda) \otimes V(\mu)$).

The tensor product rule can also be expressed algebraically: set

$$\varepsilon_i(b) := \max\{m \mid e_i^m(b) \neq 0\},$$
$$\varphi_i(b) := \max\{m \mid f_i^m(b) \neq 0\}.$$

Then for $b \otimes c \in B \otimes C$,

$$e_i(a \otimes b) = \begin{cases} e_i(a) \otimes b & \text{if } \varepsilon_i(a) > \varphi_i(b), \\ a \otimes e_i(b) & \text{otherwise}, \end{cases}$$
$$f_i(a \otimes b) = \begin{cases} f_i(a) \otimes b & \text{if } \varepsilon_i(a) \geq \varphi_i(b), \\ a \otimes f_i(b) & \text{otherwise}. \end{cases} \tag{2.5}$$

3. $B(\infty)$

For the moment I will forget the relationship with representations, and think of crystals combinatorially. When I do this, I will denote the crystal operators by e_i, f_i instead of $\widetilde{E}_i$ and $\widetilde{F}_i$.

By looking at the top "row" of $B(\lambda) \otimes B(\mu)$, we see a copy of $B(\lambda)$. This is not quite a subcrystal, but it will be closed under the operators e_i. So the embedding $B(\lambda) \subset B(\lambda) \otimes B(\mu)$ is "e_i-equivariant". It is also clear that the top-left element in the tensor product generates a copy of $B(\lambda + \mu)$. Thus we have a commutative diagram

$$\begin{tikzcd}
B(\lambda) \arrow[r] \arrow[dr] & B(\lambda) \otimes B(\mu) \\
& B(\lambda + \mu). \arrow[u]
\end{tikzcd}$$

The inclusions $B(\lambda) \to B(\lambda + \mu)$ turn $\{B(\lambda) \mid \lambda \in P^+\}$ into a directed system.

Definition 3.1. $B(\infty) = \varinjlim B(\lambda)$. □

For each $\lambda \in P^+$, we will have a surjection of crystals $B(\infty) \to B(\lambda) \cup \{0\}$ such that the elements not mapping to 0 map e_i-equivariantly onto $B(\lambda)$.

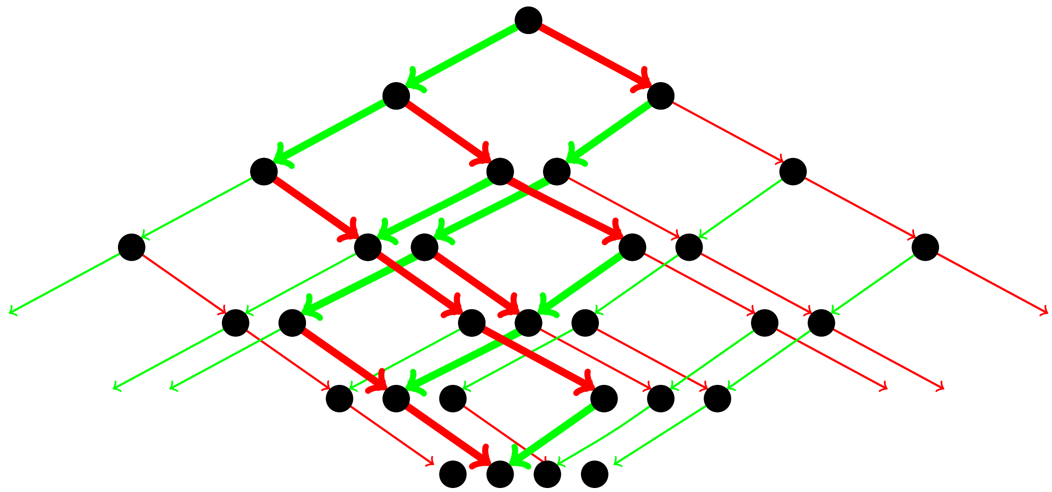


FIGURE 1. The infinity crystal $B(\infty)$ for $\mathfrak{sl}_3$. The red arrow represent f_1, and always go to the leftmost available node (in the right weight space). The green arrows represent f_2, and always go to the rightmost available node. Any of the nodes directly below the top vertex can be the lowest weight element of a $B(\lambda)$. The thicker arrows show the copy of $B(\omega_1 + 2\omega_2)$.

3.1. **Finding $B(\infty)$ algebraically.** Recall that, as a $U_q(\mathfrak{g})^-$ module, $V(\lambda) \simeq U_q(\mathfrak{g})^-/I_\lambda$, for some ideal I_λ. Let π_λ denote the projection from $U_q(\mathfrak{g})^-$ to $V(\lambda)$. If $\lambda - \mu$ is dominant, then $I_\lambda \subset I_\mu$, so these projections fit together nicely.

Say that $(\mathcal{L}, B)$ is a **local basis** of $U_q^-(\mathfrak{g})$ if $\mathcal{L}$ is an A_0-lattice and $B(\infty)$ is a basis for $\mathcal{L}/q\mathcal{L}$. Note that here we require no compatibility with the algebra structure of $U_q(\mathfrak{g})^-$.

Theorem 3.2. *There exists a unique local basis $(\mathcal{L}(\infty), B(\infty))$ of $U_q^-(\mathfrak{g})$ such that the highest weight space of $\mathcal{L}$ is spanned by $1 \in U_q^-(\mathfrak{g})$, and, for all λ, $\pi_\lambda(\mathcal{L}(\infty), B(\infty))$ is a crystal basis for $V(\lambda)$.*

Question 3.3. *Can you characterize $(\mathcal{L}(\infty), B(\infty))$ using $\widetilde{E}_i$ and $\widetilde{F}_i$? Also, how should we define $\widetilde{E}_i$ and $\widetilde{F}_i$ on $U_q^-(\mathfrak{g})$?*

We will give a positive answer to this question in Remark 4.11 below, but it is convenient to first consider an alternative characterization of both $B(\lambda)$ and $B(\infty)$.

4. Intrinsic characterizations

This section is based on [K1, Section 5]. Let D be the diagonal matrix with entries d_i such that DA is symmetric, where A is the Cartan matrix. Set $q_i = q^{d_i}$.

Definition 4.1 (Adjoint map)**.** Define θ by

$$\theta(E_i) = q_i F_i K_i^{-1}$$
$$\theta(F_i) = q_i^{-1} K_i E_i$$
$$\theta(K_i) = K_i.$$

One can check that this extends to an algebra anti-involution which is also a coalgebra isomorphism. □

Remark 4.2. There appears to be a typo in [K1, Section 5], in that the map used there does not actually define an antiautomorphism. We have modified $\theta(E_i)$ to give θ this property. In most calculations we will only apply θ to elements of U^-, so these calculations are unaffected by the change.

Let $v_\lambda \in V(\lambda)$ be a fixed highest weight vector. Define a bilinear form $(,)$ on $V(\lambda)$ via $(v_\lambda, v_\lambda) = 1$ and $(au, v) = (u, \theta(a)v)$ for all $u, v \in V(\lambda)$, $a \in U_q(\mathfrak{g})$.

Since θ is a coalgebra isomorphism, If we define $(,)_{V(\lambda)\otimes V(\mu)} = (,)_{V(\lambda)}(,)_{V(\mu)}$, then θ still acts as an adjoint.

Theorem 4.3. *Notation as above.*
(1) $\mathcal{L}(\lambda) = \{u \in V(\lambda) \mid (u, \mathcal{L}(\lambda)) \in A_0\} = \{u \in V(\lambda) \mid (u, u) \in A_0\}$.
(2) $B(\lambda)$ *is an orthonormal basis of* $\mathcal{L}(\lambda)/q\mathcal{L}(\lambda)$ *with the induced form.*

Remark 4.4. We are working over $\mathbb{C}(q)$, but we really could work with an appropriate integral form. Then we could characterize $B(\lambda) \cup -B(\lambda)$ as the set of vectors with norm 1. □

4.1. **Bilinear forms on $U_q^-(\mathfrak{g})$.** If we want to define as inner product as we did for highest weight representations, we need to have an action of $U_q(\mathfrak{g})$ on $U_q^-(\mathfrak{g})$. In fact, there is a whole family of such actions, one corresponding to each highest weight (i.e. the Verma modules). These can be described as follow. Recall the triangular decomposition of $U_q(\mathfrak{g})$: As a vector space,

$$U_q(\mathfrak{g}) \simeq U_q^-(\mathfrak{g}) \otimes U_q^0(\mathfrak{g}) \otimes U_q^+(\mathfrak{g}).$$

For each λ, let define

$$\begin{aligned} \psi_\lambda : U_q^0(\mathfrak{g}) \otimes U_q^+(\mathfrak{g}) &\to \mathbb{C}(q) \\ E_i &\to 0 \\ K_i &\to q^{\langle H_i, \lambda\rangle}. \end{aligned} \tag{4.5}$$

We get an action $\cdot_\lambda$ of $U_q(\mathfrak{g})$ on $U_q^-(\mathfrak{g})$ by left multipliction composed with applying ψ_λ to the two rightmost factors in the triangular decomposition (4.5).

For each λ, we then get a bilinear form on $U_q^-(\mathfrak{g})$ defined by

$$\begin{aligned} &(1,1) = 1 \\ &(a \cdot_\lambda u, v) = (u, \theta(a) \cdot_\lambda v) \quad \text{for all } \ u, v \in U_q^-(\mathfrak{g}),\ a \in U_q(\mathfrak{g}). \end{aligned} \tag{4.6}$$

If λ is dominant, the quotient of $U_q^-(\mathfrak{g})$ by the kernel of this form is isomorphic to $V(\lambda)$. Thus it is natural that, to study $B(\infty)$, we are really interested in the limit of the inner product as $\lambda \to \infty$, and as $q \to 0$.

Recall that $E_iF_ju = F_jE_iu + \delta_{i,j}\dfrac{K_i - K_i^{-1}}{q - q^{-1}}u$. As the highest weight $\lambda \to \infty$, K_i will act on u by some large power of q and K_i^{-1} acts by some large negative power of q. Since we are also taking the limit as $q \to 0$, the K_i^{-1} will dominate. For this reason, we modify E_i to get an operator E_i^∞ by:

Definition 4.7. E_i^∞ acts on $U_q^-(\mathfrak{g})$ by $E_i^\infty \cdot 1 = 0$ and $E_i^\infty F_j = F_jE_i^\infty + \delta_{i,j}qK_i^{-1}$. □

Theorem 4.8. *There is a unique bilinear form* $(,)$ *on* $U_q^-(\mathfrak{g})$ *such that*
(1) $(1,1) = 1$,
(2) $(F_iu, v) = (u, q_i^{-1}K_iE_i^\infty v)$.

Remark 4.9. It is important that, one you commute the E_i^∞ to the front, all terms with no E_i^∞ also have no K_i. One can check that this does in fact happen. Kashiwara handles this issue by defining $E_i' = q_i^{-1}K_iE_i^\infty$. Then $E_i'F_j = q^{-\langle H_i, \alpha_j\rangle}F_jE_i' + \delta_{i,j}$, and it is clear that no K_i appear in this commutation relation.

Theorem 4.10. *Notation as above.*

(1) $\mathcal{L}(\infty) = \{u \in V(\infty) \mid (u, \mathcal{L}(\infty)) \in A_0\} = \{u \in V(\infty) \mid (u,u) \in A_0\}$.
(2) $B(\infty)$ *is an orthonormal basis of* $\mathcal{L}(\infty)/q\mathcal{L}(\infty)$ *with the induced form.*

Again, by working over an appropriate integral form, we could characterize $B(\lambda) \cup -B(\lambda)$ as the set of vectors with norm 1.

Remark 4.11. We can also now see the correct definition of $\widetilde{E}_i, \widetilde{F}_i$ on $U_q^-(\mathfrak{g})$: Define $a \in U_q^-(\mathfrak{g})$ to be i-singular if $E_i'(a) = 0$. For all i-singular $v \in U_q^-(\mathfrak{g})$, define $\widetilde{F}_i^n(v) = F_i^{(n)}(v)$. These $\widetilde{F}_i$ play the role of the Kashiwara operators on $U_q^-(\mathfrak{g})$, leading to a crystal basis. See [K1, Theorem 4].

References

[BK] Arkady Berenstein and David Kazhdan, Geometric and unipotent crystals II: From unipotent bicrystals to crystal bases, *Quantum groups*, 13–88, *Contemp. Math.* **433**, Amer. Math. Soc., Providence, RI, 2007, `arXiv:math/0601391v4`.

[CP] V. Chari and A. Pressley, *A guide to quantum groups*, Cambridge university press, Cambridge (1994).

[K1] Masaki Kashiwara. On crystal bases of the q-analogue of universal enveloping algebras, *Duke Math. J.* **63** no. 2, 465–516, 1991.

[K2] Masaki Kashiwara. On crystal bases, *Representations of groups (Banff, AB, 1994)*, 155-197, CMS Conf. Proc. **16**, Amer. Math. Soc., Providence, RI, 1995.