LECTURE 3: GLOBAL BASES

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The plan for today is to give the algebraic construction of global bases (also called canonical bases). We will first give Kashiwara's construction, which starts with the crystal bases we have been discussing. For finite type cases, we will then give a similar construction due to Lusztig, which has the advantage that the global basis is obtained from a PBW type basis. Lusztig also has a geometric construction of canonical bases for all symmetric Kac-Moody algebras. This is important because it is needed to prove certain positivity results in those cases, but we will not cover it today.

1. Kashiwara's construction

In this section $\mathfrak{g}$ is a symmetrizable Kac-Moody algebra. The construction we present is due to Kashiwara [K1, K2], although we more closely follow [CP, Chapter 14.1] Recall that last time we constructed an $A_0 := \{f \in \mathbb{C}(q) : f \text{ is regular at } 0\}$-sublattice $\mathcal{L}(\infty) \subset U_q^-$, and a basis B of $\mathcal{L}(\infty)/q\mathcal{L}(\infty)$. These have very nice properties, including

- $\mathcal{L}(\infty)$ is closed under the Kashiwara operators $\widetilde{F}_i$.
- $\widetilde{F}_i$ acts by partial permutations.
- $(\mathcal{L}(\infty), B(\infty))/I_\lambda$ is a crystal basis of each $V(\lambda)$.
- Nice behavior with respect to tensor products.

The idea is to look for a basis that agrees with $B(\infty)$ at $q = 0$ and specializes everywhere (i.e., for all $q \in \mathbb{P}^1$). The intuition is that holomorphic functions on $\mathbb{P}^1$ are constant, so such a basis should be determined by its specialization at $q = 0$ (or any other single point).

Definition 1.1. $U_q^{\mathrm{res}}(\mathfrak{g})$ is the $A = \mathbb{C}[q, q^{-1}]$-subalgebra of $U_q(\mathfrak{g})$ generated by $E_i^{(n)} = E_i^n/[n]!$, $F_i^{(n)}$, and $K_i^{\pm 1}$. □

$U_q^{\mathrm{res}}(\mathfrak{g})$ is usually called the **restricted integral form**. It can also be defined just as well over $\mathbb{Z}[q, q^{-1}]$. One can show that $U_q^{\mathrm{res}}(\mathfrak{g}) \otimes_A \mathbb{C}(q) \cong U_q(\mathfrak{g})$.

Definition 1.2. The bar involution $^-$: $U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ is the $\mathbb{C}$-algebra involution defined by

$$E_i \leftrightarrow E_i \quad F_i \leftrightarrow F_i \quad K_i \leftrightarrow K_i^{-1} \quad q \leftrightarrow q^{-1}$$ □

Theorem 1.3. *We have an isomorphism of* $\mathbb{C}$*-vector spaces* $\mathcal{L}(\infty) \cap U_q^{\mathrm{res}}(\mathfrak{g}) \cap \overline{\mathcal{L}(\infty)} \cong \mathcal{L}(\infty)/q\mathcal{L}(\infty) \cong U^-(\mathfrak{g})$ *via the obvious maps.*

Definition 1.4. Letting π be the first isomorphism in Theorem 1.3, we define the **global basis** $G(\infty) = \pi^{-1}(B(\infty))$. □

Remark 1.5. There is an alternative characterization of $G(\infty)$ which is sometimes useful. For each $b \in B(\infty)$, there is a unique $b^g \in L(\infty) \cap U_q^{res}(\mathfrak{g})$ which is congruent to b mod q and such that $\overline{b^g} = b^g$.

The following remarkable property shows that this construction immediately gives a canonical basis for each $V(\lambda)$:

Theorem 1.6. *The set of non-zero elements in* $G(\infty)/I_\lambda$ *is a basis for* $V(\lambda)$ *for all dominant weights* λ.

Example 1.7. Here is what $G(\infty)$ looks like for some small examples.

(1) For $\mathfrak{sl}_2$, $G(\infty) = \{F^{(n)} \mid n \geq 0\}$.
(2) For $\mathfrak{sl}_3$, $G(\infty) = \{F_1^{(a)} F_2^{(b)} F_1^{(c)}, F_2^{(a)} F_1^{(b)} F_2^{(c)} \mid b \geq a + c\}$ (when $a + c = b$, the two expression are equal).

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(3) In $\mathfrak{sl}_4$, not all elements of $G(\infty)$ ca be expressed as monomials in the F_i. For instance, the element $F_2^{(2)}F_1F_3F_2 - F_2^{(3)}F_1F_3 = F_2F_1F_3F_2^{(2)} - F_1F_3F_2^{(3)}$ is in $G(\infty)$.

(4) The example of $\hat{\mathfrak{sl}}_2$ can be found in [Lu2, 14.5.5]. There as well, not all elements are monomials. □

If the Cartan matrix is symmetric, then $G(\infty)$ has an additional important property: the multiplicative structure constants are positive. That is, $bb' = \sum c_{b,b'}^{b''} b''$ where $c_{b,b'}^{b''} \in \mathbb{Z}_{\geq 0}[q, q^{-1}]$. This is why many people like to "categorify" by identifying $U_q^-(\mathfrak{g})$ with the Grothendieck group of a (graded) category, and having the global basis correspond to irreducible objects. Note however that this is false in other types. See [MO] for explicit counterexamples in types G_2, B_3, C_4.

Remark 1.8. Unlike in the crystal ($q = 0$) case, the tensor product of global bases does not give a global basis for each irreducible component of the product. In fact, the tensor product of global bases does not even give a basis respecting the decomposition into isotypic. □

2. Lusztig's construction

In this section $\mathfrak{g}$ is of finite type. The results here can be found in [Lu1] for type ADE, and the extension to other types can be found in [S]. See also [CP, Chapter 14.2].

Let $\widetilde{U_q(\mathfrak{g})}$ be the completion of $U_q(\mathfrak{g})$ in the weak topology defined by all matrix elements of all finite dimensional representations. So, an element of $\widetilde{U_q(\mathfrak{g})}$ can be defined by describing how it acts on all finite dimensional representations.

Definition 2.1. $T_i \in \widetilde{U_q(\mathfrak{g})}$ acts on V by

$$T_i v = \sum_{\substack{a+c+b\geq 0 \\ a-b+c=(\mathrm{wt}(v),\alpha_i)}} (-1)^b q_i^{b-ac} E_i^{(a)} F_i^{(b)} E_i^{(c)}.$$

Define $C_{T_i} \in \operatorname{End} U_q(\mathfrak{g})$ via

$$E_i \mapsto -F_i K_i^{-1}, \quad F_i \mapsto -K_i E_i, \quad K_H \mapsto K_{s_i(H)}$$

$$E_j \mapsto \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} K_i^r E_i^{(-a_{ij}-r)} E_j E_i^{(r)} \quad (j \neq i)$$

$$F_j \mapsto \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} K_i^{-r} F_i^{(-a_{ij}-r)} F_j F_i^{(r)} \quad (j \neq i)$$

where $H = \sum c_i H_i$ is an element of the coweight lattice, and $K_H = \prod K_i^{c_i}$. □

The notation is explained because conjugation by T_i in $\widetilde{U_q(\mathfrak{g})}$ preserves $U_q(\mathfrak{g})$, and acts by the given formulas. The C_{T_i} define an action of the braid group on $U_q(\mathfrak{g})$. So we can define $T_w = T_{i_k} \cdots T_{i_1}$ given a reduced expression $s_{i_k} \cdots s_{i_1} = w$ for $w \in W$, and this is independent of the choice of reduced expression.

Fix a reduced expression for the longest word $w_0 = s_{i_N} \cdots s_{i_1}$, and let $\beta_k = T_{i_1} \cdots T_{i_{k-1}}(F_{i_k})$. Then $\{\mathrm{wt}(\beta_k) \mid 1 \leq k \leq N\} = \Delta_-$.

The following was essentially proven by Lusztig [Lu1] in type ADE, and was shown in all finite type cases by Saito [S]. This precise statement can be found at the end of the introduction of [S].

Theorem 2.2. $\{\beta_N^{a_N} \cdots \beta_1^{(a_1)} \mid a_1, \ldots, a_N \geq 0\}$ *is a basis for* $\mathcal{L}(\infty)$. *Furthermore the residue to this set modulo* q *is equal to Kashiwara's local basis* $B(\infty)$.

Call the above basis the **PBW basis** PBW_σ corresponding to the reduced expression σ for w_0. One key property is that the ordering $\{\beta_1, \ldots, \beta_N\}$ is **convex**: the nonnegative spans of $\{\beta_1, \ldots, \beta_k\}$ and $\{\beta_{k+1}, \ldots, \beta_N\}$ are empty for all k.

One can then define the canonical basis element b^c corresponding to $b \in B$ to be the unique element in $\mathrm{span}_{\mathbb{Z}[q]}\mathrm{PBW}_\sigma$ which is equal to b modulo q, and such that $\bar{b^c} = b^c$.

Remark 2.3. Lusztig originally developed this theory in type ADE only. He developed the theory of canonical bases independently of Kashiwara's construction, using a connection with quiver varieties. □

Remark 2.4. It may seem natural to ask (as Chris Dodd did in class) if the transition matrix between Lusztig's PBW basis and the global basis $G(\infty)$ has any nice properties. In fact, at least in type ADE, for certain well chosen words σ for w_0, the transition matrix is triangular with respect to an appropriately chosen order on the PBW basis (see [Lu1, Chapter 9]). The words where this is true are words adapted to some orientation of the Dynkin-quiver (we'll define some of these words later on). The appropriate order is defined geometrically (see [Lu1, Chapter 9.1]), and I'm not sure how good a combinatorial definition there is. I've been looking through the literature to try and find out how well this generalizes, but the search has been inconclusive so far. Hopefully I'll be able to give an update on this soon.

Related to this, if one is careful there is also a triangularity result between the canonical basis and a well chosen set of expressions in the divided powers $F_i^{(k)}$ of the Chevalley generators. See [Li, Proposition 10.3].

Anyway, if you are reading this and happen to know the precise situation, I'd be grateful if you tell me. □

References

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[MO] Mathoverflow question, posed by Ben Webster, answered by Shunsuke Tsuchioka, When does Lusztig's canonical basis have non-positive structure coefficients?, available at `http://mathoverflow.net/questions/39934/when-does-lusztigs-canonical-basis-have-non-positive-structure-coefficients`.

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