QUANTUM GROUPS AT ROOTS OF UNITY

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This talk is summarizes parts of chapters 9 and 11 of Chari and Pressley [1], where everything can be found in more detail.

1. Notation and review

Throughout this talk $\mathfrak{g}$ will denote a Kac-Moody algebra with symmetrizable Cartan matrix $(a_{i,j})$, diagonal matrix (d_i) and $q_i = q^{d_i}$. Moreover $U_q(\mathfrak{g})$ will denote the quantized universal enveloping algebra over $\mathbb{Q}(q)$. More precisely $U_q(\mathfrak{g}) =< X_i^+,\, X_i^-,\, K_i^{\pm 1} >$ modulo the relations:

$$\begin{cases} K_i's \quad \text{commute} \\ K_i X_j^{\pm} K_i^{-1} = q_i^{\pm a_{ij}} X_j^{\pm 1} \\ X_i^+ X_j^- - X_j^- X_i^+ = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \\ \sum_{r=0}^{1-a_{ij}} (-1)^r [1 - a_{ij}/r]_{q_i} (X_i^{\pm})^{1-a_{ij}-r} X_j^{\pm} (X_i^{\pm})^r = 0. \end{cases}$$

We can think of $K_i = \exp(d_i \hbar H_i)$ and $q_i = \exp(d_i \hbar)$. We want to construct an integral form of $U_q(\mathfrak{g})$, that is a $\mathbb{Z}[q, q^{-1}]$ subalgebra $U_q^{int}(\mathfrak{g})$ of $U_q(\mathfrak{g})$ such that

$$U_q^{int} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}(q) = U_q(\mathfrak{g}). \tag{1.1}$$

This allows us to specialize to any nonzero value of q. There is more then one way to do this. When q is generic they will lead to the same specialized algebra, but when q is a root of unity they will be honestly different.

2. Non-restricted specialization

Consider the $\mathbb{Q}[q, q^{-1}]$ subalgebra $U_q^{\mathcal{A}}(\mathfrak{g})$ of $U_q(\mathfrak{g})$ defined by:
Notation: $[K; 0]_{q_i} = \frac{K_i - K_i^{-1}}{q - q_i^{-1}}$.
Generators: same as before + $[K; 0]_{q_i}$.
This is an integral form, which is to say it satisfies (1.1).

Definition 2.1. Fix $\epsilon \in \mathbb{C}^*$. Set $U_\epsilon(\mathfrak{g}) = U_q^{\mathcal{A}} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}$, using the homomorphisms $\mathbb{Z}[q, q^{-1}] \to \mathbb{C}$ defined by $q \to \epsilon$.

We often assume $(*)$ ϵ is not a primitive ℓ^{th}root of unity for ℓ even or $\ell < d_i$ for any i. $\square$

For the rest of this section, fix ϵ a primitive ℓ^{th} root of unity such that ℓ is odd and $\ell > d_i$ for all i.

For $\alpha = \Delta^+$, consider $X_\alpha^{\pm}, X_\alpha^{\pm} = X_i^{\pm 1}$ and denote by $\mathcal{Z}_0$ the subalgebra generated by

$$\mathcal{Z}_0 = \langle (X_\alpha^{\pm})^\ell,\, (K_i)^\ell \rangle \subset \mathcal{Z}_\epsilon \text{ (the center of } \mathcal{U}_\epsilon \text{)}.$$

Assume $\mathfrak{g}$ is finite dimensional. Then the following facts hold:

(1) U_ϵ is finite dimensional over $\mathcal{Z}_0$ (and hence also over the center $\mathcal{Z}_\epsilon$).

(2) $Spec(\mathcal{Z}_\epsilon) \to Spec(\mathcal{Z}_0)$, finite of degree ℓ^n, and $Spec(\mathcal{Z}_0) \cong \mathbb{C}^{2N} \times (\mathbb{C}^*)^n$, $N = |\Delta^+|$.

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(3) If G simple Lie group with Lie algebra $\mathfrak{g}$ then $Spec(\mathcal{Z}_0) \cong G^*$ as a Poisson Lie group.

Think of the irreducible representations of U_ϵ by looking at the fibers over each χ. It turns out that each fiber is non-empty.
1) If let $I_\epsilon^\chi :=$ ideal generated by$\{z - \chi(z)\, z \in \mathcal{Z}_0\}$

$$\left\{ \begin{array}{c} \text{Representations of } U_\epsilon \\ \text{with central character } \chi \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Represenations of} \\ U_\epsilon^\chi := U_\epsilon / I_\epsilon^\chi \end{array} \right\}$$

Theorem 2.2. • *Over an open set $U \subset Spec(\mathcal{Z}_\epsilon)$, there is exactly one representation over each $\chi \in U$, and this has dimension ℓ^N.*
• *For $\chi \in Spec(\mathcal{Z}_\epsilon) \backslash U$ there are $< \ell^N$ representations over χ.*

In particular, this means that over a generic point in $Spec(\mathcal{Z}_\epsilon)$, $U_\epsilon(\mathfrak{g})$ is isomorphic to a matrix algebra.

3. Restricted specialization

$U_q^{\mathcal{A},res}$ is the $\mathbb{Z}[q, q^{-1}]$ subalgebra of $U_q(\mathfrak{g})$ generated by $(X_i^\pm)^{(r)} = \frac{(X_i^\pm)^r}{[r]_{q_i!}}$ and $K_i^\pm$.

Remark 3.1. It is non trivial to show that this is indeed an integral form in the sense of (1.1)

Definition 3.2. Fix $\epsilon \in \mathbb{C}^*$. Set $U_\epsilon^{res} := U_q^{\mathcal{A},res} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}$, where again the tensor product is defined using the homomorphism from $\mathbb{Z}[q, q^{-1}]$ to $\mathbb{C}$ which sends q to ϵ. □

For the rest of this section, fix ϵ to be a primitive ℓ^{th} root of unity, where ℓ is odd and greater then all d_i. Then one can see
1. $(X_\alpha^\pm)^\ell = 0$, (Proof: $(X_i^+)^\ell = (X_i^+)^{(\ell)}[\ell]_{q_i}!$, and the latter factorial is zero).
2. $(K_i)^\ell$ central, $(K_i)^{2\ell} = 1$.

Let V be an irreducible representation. By tensoring with a 1-dimensional representation we can assume without loss of generality that K_i^ℓ acts by 1 (we call these representations of type 1).
Highest weight theory: We want to define the weight space of a representation V for $U_\varepsilon(\mathfrak{g})$ by $V_\lambda := \{v \in V | K_i v = \epsilon^{(\lambda, \hat{\alpha})} v\}$. Problem: if $\lambda - \mu \in \ell \cdot P$, then $V_\lambda = V_\mu$. The problem here is that the $K_i s$ don't generate a maximal abelian subalgebra. In order to fix this, we define

$$V_\lambda = \{v \in V | K_i v = \epsilon^{(\lambda, \hat{\alpha_i})} v, \begin{bmatrix} K; 0 \\ \ell \end{bmatrix}_{\epsilon_i} v = \begin{bmatrix} (\lambda, \hat{\alpha_i}) \\ \ell \end{bmatrix}_{\epsilon_i} v, i = 1 \dots n\}.$$

Now define the character of a representation V to be

$$chV = \sum \dim(V_\lambda) e^\lambda \in \mathbb{Z}[P]$$

then we have:

Theorem 3.3. *$ch(V \otimes W) = ch(V) \cdot ch(W)$.*

So, the theory of weights and characters works, and representations at a root of unity can be studied with many of the same tools as the generic case. However, there are significant differences. For instance, while there is still an irreducible representation of each highest weight, these have different dimensions then in the generic case. More seriously, the category of finite dimensional representations at a root of unity is not semi-simple.

References

[1] V. Chari and A. Pressley, *A guide to quantum groups*, Cambridge university press, Cambridge (1994).