GELFAND–TSETLIN BASES AND CRYSTALS

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1. Gelfand–Tsetlin bases

1.1. **General construction.** Recall: the finite-dimensional irreducible polynomial representations of $\mathbf{GL}_n(\mathbb{C})$ are in bijection with partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$, which we represent as Young diagrams with at most n rows. Call this latter set Y_n. Let V_λ denote the representation corresponding to λ. Our goal is to construct nice bases for V_λ that are well-behaved with respect to restrictions and tensor products.

The idea is work by induction. Suppose that we have a Gelfand–Tsetlin (G-T) basis for all irreducible representations of $\mathbf{GL}_{n-1}$. We consider the restriction of V_λ to $\mathbf{GL}_{n-1}$, decompose it as a direct sum of irreducible representations, and take the Gelfand–Tsetlin basis of each of these. This gives a basis for V_λ itself. Of course, we have to make a choice when we decompose V_λ into irreducible representations of $\mathbf{GL}_{n-1}$, so the notion of Gelfand-Tsetlin basis can only be well defined up to such choices. However, in this case the decomposition is multiplicity free, so in the end we get a basis for $V(\lambda)$ which is well defined up to rescaling each basis vector.

1.2. **Combinatorics.**

Definition 1.1. For $\lambda \in Y_n$ and $\lambda' \in Y_{n-1}$ with $\lambda' \subset \lambda$, say that λ/λ' is a **horizontal strip** if each column in λ/λ' has at most 1 element. $\square$

Then we have

$$V_\lambda \cong \bigoplus_{\lambda/\lambda' \text{ horizontal strip}} V_{\lambda'}$$

as $\mathbf{GL}_{n-1}$-representations.

Definition 1.2. A **GT-pattern** is a triangular array of numbers $(\lambda_{ij})_{n\geq i\geq j\geq 1}$ such that $\lambda_{ij} \geq \lambda_{i-1,j} \geq \lambda_{i,j+1}$. These are in bijection with semistandard Young tableaux by considering the successive shapes

$$\lambda_{1,\bullet} \subseteq \lambda_{2,\bullet} \subseteq \cdots \subseteq \lambda_{n,\bullet}$$

and labeling the boxes in $\lambda_{i,\bullet} \setminus \lambda_{i-1,\bullet}$ with the number i. Call this bijection τ. $\square$

The G-T basis of V_λ is parametrized by GT-patterns with $\lambda_{n,\bullet} = \lambda$. To describe the restriction $V_\lambda \downarrow_{\mathbf{GL}_{n-1}}$,take the union of GT-bases for $\mathbf{GL}_{n-1}$ representations, where you forget about the top row.

1.3. **Orthogonal Lie algebras.** The representations of $\mathfrak{so}_{2n+1}$ are parametrized by sequences $\lambda_1 \geq \cdots \lambda_n \geq 0$ which are either all integers or all half-integers and for $\mathfrak{so}_{2n}$, they are parametrized by $\lambda_1 \geq \cdots \lambda_{n-1} \geq |\lambda_n|$ which are either all integers or all half-integers.

The branching rules are

$$V_\lambda \downarrow^{\mathfrak{so}_{2n+1}}_{\mathfrak{so}_{2n}} \cong \bigoplus V_\mu$$

where $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_n \geq |\mu_n|$ and

$$V_\lambda \downarrow^{\mathfrak{so}_{2n}}_{\mathfrak{so}_{2n-1}} \cong \bigoplus V_\mu$$

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where $\lambda_1 \geq \mu_1 \geq \cdots \geq \mu_{n-1} \geq |\lambda_n|$.

Once again the branchings are multiplicity free, so one can define analogues of G-T bases and G-T patterns in this case.

2. G-T bases compatible with crystal structure

Now consider $U_q(\mathfrak{gl}_n(\mathbb{C}))$. One can construct GT bases for the irreducible representations V_λ, just as for $\mathbf{GL}_n$, which are well defined up to individual rescaling of the basis vectors. Also, consider the vector representation V. It is well known that $V^{\otimes N} \cong \bigoplus_T V_{\mathrm{shape}(T)}$, where T ranges over standard Young tableaux with N nodes and at most n rows.

The main result states the following: there is an appropriate decomposition of choice of $V^{\otimes N}$ into irreducible representations $V(T_R)$ corresponding to each standard Young tableau T, and an appropriate normalization of the G-T bases for each of these representations, such that when $q \to 0$, the G-T basis vector approach the standard basis vectors of $V^{\otimes N}$. For $q^{-1} \to 0$, the same is true, but we use different decompositions $V(T_L)$, and a different normalization of the G-T basis elements. In modern language, this occurs because both the G-T bases (correctly normalized) and the standard basis of $V^{\otimes N}$ are crystal bases. In fact, the results discussed in this section, due to Date, Jimbo and Miwa [3], were an important precursor to the notion of a crystal basis.

The remainder of this section is occupied with making these statements precise and providing a proof.

2.1. Action of $U_q(\mathfrak{gl}_n(\mathbb{C}))$ on G-T basis. $U_q(\mathfrak{gl}_n(\mathbb{C}))$ is generated by $q^{\varepsilon_i/2}$, $q^{-\varepsilon_i/2}$, X_j^+, X_j^-. Then denoting the Gelfand-Tsetlin basis elements of the $U_q(\mathfrak{gl}_n(\mathbb{C}))$-module V_λ by $|m\rangle$, the action of the above generators is given as follows:

$$q^{\varepsilon_i/2}|m\rangle = q^{\sum_{i=1}^{j} m_{ij} - \sum_{i=1}^{j-1} m_{i,j-1}}|m\rangle$$

$$X_j^+|m\rangle = \sum^{(j)} c_j(m,m')|m'\rangle$$

$$X_j^-|m\rangle = \sum^{(j)} c_j(m,m')|m'\rangle,$$

where $c_j(m,m') \neq 0$ only if there exists i such that $m'_{ij} = m_{ij} - 1, m'_{ab} = m_{ab} \forall (a,b) \neq (i,j)$, in which case the coefficients are rather complicated to write down. The highest weight vector is given by the Gelfand-Tsetlin pattern with first row $(\lambda_1, \cdots, \lambda_n)$, second row $(\lambda_1, \cdots, \lambda_{n-1})$ and so on.

2.2. The embedding $V_W \subset V_Y \otimes V$. Say $Y \xrightarrow{\mu} W$ if W is obtained from Y by adding a box in the μth row. We will now describe explicitly the decomposition $V_Y \otimes V \cong \oplus_{Y \xrightarrow{\mu} W} V_W$. Given $|m\rangle \in GT(W)$, define $|m'\rangle = |m; i_n, \cdots, i_j\rangle \in GT(Y)$ (note the slight abuse of notation: $|m'\rangle$ is not a single element), where for $j \leq k \leq n, 1 \leq i_k \leq k$, $m'_{ik} = m_{i_k,k} - 1$ if $j \leq k \leq n, i = i_k$ and $m'_{ik} = m_{ik}$ otherwise. Then the above branching rule is determined explicitly by the following, where the coefficients $w_q(m; i_n, \cdots, i_j)$ are known as Wigner coefficients.

$$|m\rangle = \sum_{j=1}^{n} \sum_{i_n=\mu, i_{n-1}, \cdots i_1} w_q(m; i_n, \cdots, i_j)|m; i_n, \cdots, i_j\rangle \otimes v_j$$

2.3. RSK. We'll define two bijections α and β between : $\{1, \ldots, n\}^N$ and $\coprod_Y S(Y) \times T(Y)$ ranging over all Young diagrams Y with N nodes and at most n rows, where $S(Y)$ is the set of semistandard Young tableaux of shape Y, and $T(Y)$ is the set of standard Young tableaux of shape Y.

First, given a SSYT S and a number x, define the α-insertion $S \leftarrow x$ to be the jdt rectification of the shape obtained by adjoining x to the lower left corner of the tableau S. Given a word $w = w_1 \cdots w_N$, define $\alpha_S(w) = ((((w_1 \leftarrow w_2) \leftarrow w_3) \cdots) \leftarrow w_N)$, and let $\alpha_T(w)$ record the growth of the subsequent shapes. The bijection α is then $w \to (\alpha_S(w), \alpha_T(w))$.

The second bijection β is defined in the same way, but where β-insertion, given a SSYT S and a number x, $S \downarrow x$ denotes the jdt rectification of the shape obtained by adjoining x to the upper right corner of the tableau S. Then $\beta_S(w)$ and $\beta_T(w)$ are defined as above. The importance of α-insertion and β-insertion to study the embedding $V_W \subset V_Y \otimes V$ is detailed in the below proposition:

Proposition 2.1. *Given $Y \xrightarrow{\mu} W$, node added in the νth column. Fix $R \in S(W)$, and let $|m\rangle \in GT(W)$ be the corresponding Gelfand-Tsetlin pattern. Set*

$$|m'\rangle = \begin{cases} \tau^{-1}(R \to \nu) & q \to 0 \\ \tau^{-1}(R \uparrow \mu) & q^{-1} \to 0 \end{cases}.$$

Here $R \to \nu$ (resp $R \uparrow \mu$) are the deletion procedures inverse to $S \leftarrow \mu$ and $S \downarrow \mu$. If the deletion throws away the letter j, then under the embedding $V_W \subset V_Y \otimes V$, we have:

$$\lim_{q^{\pm 1} \to 0} |m\rangle = \lim_{q^{\pm 1} \to 0} (\pm 1)^{\mu - 1} |m'\rangle \otimes v_j$$

2.4. **Proof of Main Theorem.** First we explicitly describe the decomposition $V^{\otimes N} = \oplus_T V_T$ in two different ways: recall that T ranges over all standard tableau with N nodes and $\leq n$ rows. Given a fixed tableau T, suppose the entry k entry occurs in the position (μ_k, ν_k). To describe the first decomposition $V^{\otimes N} \cong \oplus_T V(T_R)$, the embedding $V(T_R) \to V^{\otimes N}$ is defined inductively: let T_i be the subtableau of T consisting of entries $\leq i$, then $V_{T_1} \cong V$, embed $V_{T_2} \hookrightarrow V_{T_1} \otimes V$, and so on until we get $V(T) = V(T_N) \hookrightarrow V_{T_{N-1}} \otimes V$; composing we get an embedding $V_T \to V^{\otimes N}$, and the direct sum decomposition $V^{\otimes N} \cong \oplus V(T_R)$ follows inductively from the decomposition $V_Y \otimes V \cong \oplus_{Y \xrightarrow{\mu} W} V_W$. The second decomposition $V^{\otimes N} \cong \oplus_T V(T_L)$ is defined similarly, but using a slightly different embedding $V(T_2) \hookrightarrow V \otimes V(T_1)$ and so on, using a modification of the Wigner coefficients.

With the notation developed above, for emphasis we now state in full detail the Main Theorem that was quickly mentioned above; and we will note that its proof follows directly from the Proposition above by induction on N.

Theorem 2.2. *If $w = i_1 i_2 \cdots i_N$, then $v_{i_1} \otimes \cdots \otimes v_{i_N} \in V^{\otimes N} \cong \oplus_T V(T_R)$, at the limit $q \to 0$, lies in the copy of V_{T_R} where $T = \alpha_T(w)$, and is the Gelfand-Tsetlin basis element corresponding to the SSYT $S = \alpha_S(w)$. A similar statement holds in the limit $q^{-1} \to 0$, where we instead use the decomposition $V^{\otimes N} \cong \oplus_T V(T_L)$. Thus in both cases, the union of the Gelfand-Tsetlin bases coincides with the "obvious" bases of $V^{\otimes N}$.*

This is clear using the proposition and using induction on N. Indeed, assume the statement for $N-1$, consider what the vector $v_{i_1} \otimes \cdots \otimes v_{i_{N-1}}$ corresponds to under the decomposition, and then use the Proposition above to deduce the required statement.

References

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[3] 'Date, Jimbo & Miwa, Representations of $U_q(\mathfrak{gl}_n(\mathbb{C}))$ at $q = 0$ and the Robinson-Schensted correspondence, Physics and mathematics of strings, 185211, World Sci. Publ., Teaneck, NJ, 1990