Game Theory

A course in One Semester

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Preface

Why do countries insist on obtaining nuclear weapons? What is a fair allocation of property taxes in a community? Should armies ever be divided and in what way in order to attack more than one target? How should a rat run to avoid capture by a cat? Why do conspiracies almost always fail? What percentage of offensive plays in football should be passes and what percentage of defensive plays should be blitzes? These are the questions that game theory can answer. Game theory arises in almost every facet of human interaction (and inhuman interaction as well) because in almost every interaction objectives are either opposed or cooperation is an option. Modern game theory is a rich area of mathematics for economics, political science, military science, finance, biological science (because of competing species), and so on.

I have taught game theory to undergraduates off and on for 20 years. I was unable to find a text which was appropriate for the class. I needed a book which is meant for mathematics undergraduate students at junior or senior level and for a one semester course. In addition, I needed a book which did not spend three fourths of the class on linear programming because Maple can easily do the kinds of linear programs involved in game theory. Amazingly, I was not able to find any existing game theory book which met these requirements. Of course there are plenty of books on game theory but they were either too abstract or too applied. So, I started to prepare my own notes and realized that this might be useful for other instructors of game theory, especially in mathematics or engineering departments.

My experience is that the students who enroll in game theory are primarily mathematics students interested in applications, with about one third to one half of the class economics students. The modern economics and operations research curriculum requires more and more mathematics and game theory is typically a required course. For economics students with a more mathematical background, this is an ideal level of course. For mathematics students interested in a graduate degree in something other than mathematics, this course shows them another discipline in which they might be interested and be able to further their studies, or simply to learn some fun mathematics.

The prerequisites for this course involve very basic probability (definitions of discrete probability, expectation) and very basic linear algebra (matrix multiplication, transpose, inverses). These topics could even be covered in a short appendix. Exercises will be provided at the end of each chapter or section. Since I have lectured from these notes I have tried to present the material with logical breaks for each lecture. The length of the book is kept to a minimum amount to be able to cover the entire book in one semester (roughly 14 weeks).

These notes are far from exhaustive of the theory of games. Many very interesting topics are not included because there is no way to cover more topics in one semester. In addition, many precise definitions in their abstract form are omitted as are the proofs of many of the results used, such as for the properties of optimal strategies. This is not meant to be a *theorem-proof* type of course. Some proofs of the more basic results are at least sketched and they can be omitted. The sequence of topics is (1) 2 person zero sum matrix games; (2) Nonzero sum games and the reduction to nonlinear programming; (3) Cooperative games covering both the Nucleolus concept and the Shapley value; and (4) Bargaining, including threat strategies.

One of the unique features of this book is the use of Maple to find the values and strategies of games, both zero and nonzero sum, and noncooperative and cooperative. The major computational impediment to solving a game is the roadblock of solving a linear or nonlinear program. Maple gets rid of those problems and the theories of linear and nonlinear programming do not need to be presented to do the computations. The vast majority of colleges have Maple licenses and I feel that this is a reasonable technological requirement.

My experience with game theory for undergraduates is that students greatly enjoy both the theory and applications which are so obviously relevant and fun. I think that a book aimed at the audience I am targeting would be successful.

Chicago, Illinois,

December, 2006 *E.N. Barron*

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Matrix 2 person games

What is a game? This turns out to be not so easy to define even though everyone has played games from our earliest days. Relying on an intuitive idea, a 2 player game is a quantitative description of the rules of the game for each player. This includes the strategies they will use in playing the game in every situation, and the outcomes of the use of those strategies. The outcome of the game is represented by numbers which give the payoff to each player at the end of each play of the game. A strategy, loosely speaking, is a plan for what to do at each stage of the game depending on the current state. If you can imagine the game of chess, a strategy would be a very complicated thing.

Let's call the two players I and II. Suppose that player I has a choice of n possible strategies and player II has a choice of m possible strategies. If player I chooses a strategy, say strategy i and player II chooses a strategy j, then they play the game and the payoff to each player is computed. In a zero sum game, whatever one player wins the other loses, so if a_{ij} is the amount player I receives, then II gets $-a_{ij}$. Now we have a collection of numbers $\{a_{ij}\}, i = 1, \ldots, n, j = 1, \ldots, m$, and we can arrange these in a matrix. These numbers are called the payoffs to player I and the matrix is called the *payoff* or *game matrix*.

			Player II		
		Strategy 1	Strategy 2	$\cdots$	Strategy m
	Strategy 1	a_{11}	a_{12}	$\cdots$	a_{1m}
	Strategy 2	a_{21}	a_{22}		a_{2m}
Player I	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	Strategy n	a_{n1}	a_{n2}	$\cdots$	a_{nm}

By agreement we place player I as the row player and player II as the column player. We also agree that the numbers in the matrix represent the payoff to player I. In a zero sum game the payoffs to player II would be the negative of those in the matrix. Of course if player I has some payoff negative, then player II would have a positive payoff.

Summarizing, a two person zero sum game in matrix form means that there is a matrix $A = (a_{ij}), i = 1, \ldots, n, j = 1, \ldots, m$ of real numbers so that if Player I, the row player chooses to play row i and player II, the column player, chooses to play column j, then the payoff to Player I is a_{ij} and the payoff to Player II is $-a_{ij}$. Thus, whatever Player I wins, Player II loses.

Now lets be concrete and do this with an example.

Example 1.1. 2×2 **Nim** 4 pennys are set out in 2 piles of 2 pennys each. Player I chooses a pile and then decides to remove 1 or 2 pennys from the pile chosen. Then player II chooses a pile with at least one penny and decides how many pennys to remove. Then player I starts the second round with the same rules. When both piles have no pennys the game ends and the loser is the player who removed the last penny. The loser pays the winner 1 dollar.

Strategies for this game for each player must specify what each player will do depending on how many piles are left and how many pennys in each pile, at each stage. Let's draw a diagram of all the possibilities.

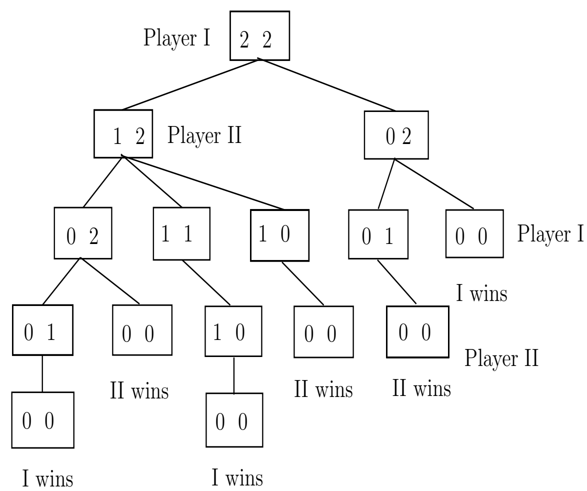


Fig. 1.1. 2 X 2 NIM

Next we need to write down the strategies for each player:

Strategies for Player I
(1) Play (1,2) then, if at (0,2) play (0,1).
(2) Play (1,2) then if at (0,2) play (0,0).
(3) Play (0,2).

You can see that a strategy for I must specify what to do no matter what happens. Strategies for II are even more involved:

Strategies for player II
(1)If at (1,2) → (0,2); if at (0,2) → (0,1)
(2)If at (1,2)→ (1,1); if at (0,2)→ (0,1)
(3) If at (1,2)→ (1,0); if at (0,2)→ (0,1)
(4) If at (1,2)→ (0,2); if at (0,2)→ (0,0)
(5) If at (1,2)→ (1,1); if at (0,2)→ (0,0)
(6) If at (1,2)→ (1,0); if at (0,2)→ (0,0)

Playing strategies for player I against player II results in the payoff matrix for the game, with the entries representing the payoffs to player I.

Player I/Player II	1	2	3	4	5	6
1	1	1	-1	1	1	-1
2	-1	1	-1	-1	1	-1
3	-1	-1	-1	1	1	1

Looking at this matrix we see that player II would never play strategy 5 because player I then wins no matter which row he plays. Any rational player would drop column 5 from consideration. By the same token if you look at column 3 this game is finished. Why? because no matter what player I does, if II plays column 3, he wins +1. Player I always loses as long as player II plays column 3. That is, if Player I goes to (1,2) player II should go to (1,0). If Player I goes to (0,2), player II should go to (0,1). This means that II can always win the game as long as I plays first.

We say that the value of this game is -1 and the strategies

$$(I1, II3), (I2, II3), (I3, II3)$$

are saddle points, or optimal strategies for the players. We will be more precise about what it means to be optimal in a little while, but for this example it means player I can improve if player II deviates from column 3 and player II should not deviate from column 3.

This game is not very interesting because there is always a winning strategy for player II and it is pretty clear what it is. Why would player I ever want to play this game? There are actually many games like this (tic-tac-dough is an obvious example) which are not very interesting because their outcome (the winner and the payoff) is determined as long as the players play optimally.

Chess is not so obvious an example because the number of strategies is so large that the game cannot, or has not, been analyzed in this way. It is a predetermined game if we knew what the optimal strategies were, but we don't. That is why it is still fun to play.

Example 1.2. **Evens or Odds** In this game, each player decides to show 1, 2 or 3 fingers. These are the strategies of each player. If the total number of fingers shown is even, player I wins +1 and player 2 loses −1. If the total number of fingers is odd, player I loses −1, and player II wins +1. We may represent the Payoff Matrix as follows:

			Player II Odds	
		1	2	3
	1	1	-1	1
Player I Evens	2	-1	1	-1
	3	1	-1	1

The row Player here and throughout this book will always want to **maximize** his payoff, while the column player wants to **minimize** the payoff to the row player, so that his own payoff is maximized (because it is a zero sum game).The rows are called the **pure strategies** for player I and the columns are called the **pure strategies** for player II.

The question is : How should each player decide what number of fingers to show? If the row player **always** chooses the same row, say 1 finger, then player II can **always** win by showing 2 fingers. No one would be stupid enough to play like that. So what do we do? In contrast to 2×2Nim, there is no obvious strategy that is winning for either player.

For now, consider what player I should do on a given play of the game. If I chooses row (=strategy) 1, then player II should choose strategy 2. Of course II won't know what I is going to show because they have to show fingers simultaneously.

In order to determine what the players should do in any zero sum matrix game we begin with figuring out a systematic way of seeing if there is an obvious solution.

We look at a game with matrix $A = (a_{ij})$ from player I's point of view. I assumes that player II is playing his best so II chooses a column j so as to

$$\text{Minimize } a_{ij} \text{ over } j = 1, \ldots, m$$

for any given row i. Then player I can guarantee that he can choose the row i that will Maximize this. So Player I can guarantee that in the worst possible situation he can get

$$v^- := \max_{i=1,\dots,n} \min_{j=1,\dots,m} a_{ij}$$

and we call this the **lower value of the game**.

Next, consider the game from II's point of view. II assumes that I is playing his best so that I will choose a row so as to

$$\text{Maximize } a_{ij} \text{ over } i = 1, \dots, n$$

for any given column $j = 1, \dots, m$. Player II can therefore choose his column j so as to lose no more than

$$v^+ := \min_{j=1,\dots,m} \max_{i=1,\dots,n} a_{ij}$$

and we call this the **upper value of the game**.

Here is how to find the upper and lower values:

In a 2 person zero sum game with a finite number of strategies for each player, we write the game matrix is written as

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$$

For each row, find the minimum payoff in each column (written in the last column). Then the lower value is the largest number in that column, i.e., the maximum over rows of the minimum over columns. Similarly, in each column find the maximum of the payoffs (written in the last row). The upper value is the the largest of those numbers.

$$\begin{array}{cccc|cc} a_{11} & a_{12} & \cdots & a_{1n} & \longrightarrow & \min_j a_{1j} \\ a_{21} & a_{22} & \cdots & a_{2n} & \longrightarrow & \min_j a_{2j} \\ \vdots & \vdots & \cdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & \longrightarrow & \min_j a_{mj} \\ \hline \downarrow & \downarrow & \cdots & \downarrow & & \\ \max_i a_{i1} & \max_i a_{i2} & \cdots & \max_i a_{in} & v^- = \text{largest min} & \\ & & & & v^+ = \text{smallest max} & \end{array}$$

Definition 1.0.1 *A matrix game with matrix $A = (a_{ij})$ has the lower value*

$$v^- := \max_{i=1,\dots,n} \min_{j=1,\dots,m} a_{ij}$$

and v^- is the smallest amount Player I is guaranteed to receive, and the upper value

$$v^+ := \min_{j=1,\dots,m} \max_{i=1,\dots,n} a_{ij}$$

and v^+ is the largest amount Player II can be guaranteed to lose. The game has a value if $v^- = v^+$ and we write it as $v = v(A) = v^+ = v^-$.

It should be clear that the most I can be guaranteed to win should be less than (or equal) to the least II can be guaranteed to lose, i.e., $v^- \le v^+$. Here is a quick verification of this fact.

For any column j we know that $\min_j a_{ij} \le a_{ij}$ and so $v^- = \max_i \min_j a_{ij} \le \max_j a_{ij}$. This is true for any row $i = 1, \dots, n$. The left side is just a number independent of i and it is smaller than the right side for any i. But this means that $v^- \le \min_i \max_j a_{ij} = v^+$, and we are done.

Closely related to the concept of value is the concept of a saddle point which basically tells the player what to do in order to obtain the value of the game. We call a particular row i^* and column j^* a **saddle point in pure strategies** of the game if

$$a_{ij^*} \le a_{i^*j^*} \le a_{i^*j}, \quad \text{for all rows } i \text{ and columns } j. \tag{1.0.1}$$

A game will have a saddle point in pure strategies if and only if

$$v^- = \max_i \min_j a_{ij} = \min_j \max_i a_{ij} = v^+.$$

Why? Well, if (1.0.1) is true, then

$$v^+ = \min_i \max_j a_{i,j} \le \max_i a_{i,j^*} \le a_{i^*,j^*} \le \min_j a_{i^*,j} \le \max_i \min_j a_{i,j} = v^-.$$

But $v^- \le v^+$ always and so we have equality throughout and $v = v^+ = v^- = a_{i^*,j^*}$. We leave it as an exercise for the reader to show that if the game has a value then it also has a saddle point.

When a saddle point exists in pure strategies it says that if any player deviates from playing his part of the saddle then the other player can take advantage and do better.

We know that $v^+ \ge v^-$ is always true. The question is: What do we do if $v^+ > v^-$?

1.0.1 The Von Neumann Minimax Theorem

Saddle points don't only apply to matrix games. Indeed let's give the definition for an arbitrary function f.

Definition 1.0.2 *A function $f : X \times Y \to \mathbb{R}$ has at least one saddle point (x^*, y^*) with $x^* \in X$ and $y^* \in Y$ if and only if*

$$f(x, y^*) \le f(x^*, y^*) \le f(X^*, y) \textit{ for all } x \in X, y \in Y.$$

Once again we could define the upper and lower values for the game defined using the function f, called a continuous game, by

$$v^+ = \min_{x \in X} \max_{y \in Y} f(x,y), \text{ and } v^- = \max_{y \in Y} \min_{x \in X} f(x,y).$$

You can check as before that $v^- \le v^+$. The Von Neumann minimax theorem gives conditions on $f, X,$ and Y so that $v^+ = v^-$ and the game value v then exists.

In order to state the theorem we need to introduce some definitions.

Definition 1.0.3 *A set $C \subset \mathbb{R}^n$ is convex if for any two points $a, b \in C$ and all scalars $\lambda \in [0,1]$, the line segment connecting a and b is also in C, i.e., $\lambda a + (1-\lambda)b \in C, 0 \le \lambda \le 1$. C is closed if it contains all limit points of sequences in C and C is bounded if it can be shoved inside a ball for a large enough radius.*

A function $g : C \to \mathbb{R}$ is convex if $g(\lambda a + (1-\lambda)b) \le \lambda g(a) + (1-\lambda)g(b)$ for any $a, b \in C, 0 \le \lambda \le 1$. This says that The line connecting $f(a)$ with $f(b)$ must always lie above the function.The function is concave if $g(\lambda a + (1-\lambda)b) \ge \lambda g(a) + (1-\lambda)g(b)$ for any $a, b \in C, 0 \le \lambda \le 1$.

Here is the basic von Neumann minimax theorem

Theorem 1.0.4 *Let $f : X \times Y \to \mathbb{R}$ be a continuous function. Let X and Y be convex, closed, and bounded. Suppose that $x \mapsto f(x,y)$ is concave, and $y \mapsto f(x,y)$ is convex. Then*

$$\min_{y \in Y} \max_{x \in X} f(x,y) = \max_{x \in X} \min_{y \in Y} f(x,y).$$

Proof. This is a sketch of the proof using only elementary properties of convex functions.

1. Assume first that f is *strictly* concave-convex, meaning that

$$\begin{gathered} f(\lambda x + (1-\lambda)z, y) > \lambda f(x,y) + (1-\lambda) f(z,y) \text{ and} \\ f(x, \mu y + (1-\mu)w) < \mu f(x,y) + (1-\mu) f(x,w). \end{gathered}$$

The advantage of doing this is that for each $x \in X$ there is one and only one $y = y(x) \in Y$ (y depends on the choice of x) so that

$$f(x, y(x)) = \min_{y \in D} f(x,y) := g(x).$$

This defines a function $g : X \to \mathbb{R}$ which is continuous (since f is continuous on the closed bounded sets $X \times Y$ and so uniformly continuous) and $g(x)$ is concave since

$$g(\lambda x + (1-\lambda)z) \ge \min_y (\lambda f(x,y) + (1-\lambda) f(x,z)) \ge \lambda g(x) + (1-\lambda) g(z).$$

So, there is a point $x^* \in X$ at which g achieves its maximum, i.e.,

$$g(x^*) = \max_{x \in X} \min_{y \in Y} f(x, y).$$

2. Let $x \in X$ be arbitrary. Then

$$f(\lambda x + (1-\lambda)x^*, y) > \lambda f(x, y) + (1-\lambda) f(x^*, y) \geq \lambda f(x, y) + (1-\lambda) g(x^*)$$

Now set

$$\begin{aligned} &y_\lambda := y(\lambda x + (1-\lambda)x^*) \in Y, \text{ so} \\ &\quad f(\lambda x + (1-\lambda)x^*, y_\lambda) = \min_{y \in Y} f(\lambda x + (1-\lambda)x^*, y) = g(\lambda x + (1-\lambda)x^*). \end{aligned}$$

Since g is concave, we have

$$g(\lambda x + (1-\lambda)x^*) \geq \lambda g(x) + (1-\lambda) g(x^*) \geq \lambda f(x, y_\lambda).$$

It is also true that $g(x^*) \geq g(\lambda x + (1-\lambda)x^*)$ since $g(x^*) \geq g(x), \forall\, x \in X$. Therefore,

$$f(x, y_\lambda) \leq g(x^*) = f(x^*, y(x^*)). \tag{1.0.2}$$

3. Sending $\lambda \to 0$ we see that $y_\lambda = \lambda x + (1-\lambda)x^* \to x^*$ as well as $y_\lambda \to y(x^*)$. Using (1.0.2) we see that

$$f(x, y(x^*)) \leq f(x^*, y(x^*)) := v, \text{ for any } x \in X.$$

Consequently, with $y^* = y(x^*)$

$$f(x, y^*) \leq f(x^*, y^*) := v.$$

But, by construction of $y(x)$ we immediately have $f(x^*, y^*) \leq f(x^*, y), \forall\, y \in Y$, so we conclude that

$$f(x, y^*) \leq f(x^*, y^*) = v \leq f(x^*, y), \forall\, x \in X, y \in Y.$$

This says that (x^*, y^*) is a saddle point and the minimax theorem holds, since

$$\min_y \max_x f(x, y) \leq \max_x f(x, y^*) \leq v \leq \min_y f(x^*, y) \leq \max_x \min_y f(x, y).$$

and so we have equality throughout.

4. The last step would be to get rid of the assumption of strict concavity and convexity, which we omit.

□

So, this very important theorem tells us what we need in order to guarantee that our game has a value. It is critical that we are dealing with a concave convex function, and that the strategy sets be convex. Given a matrix game, how do we do that? That is the subject of the next section.

1.0.2 Mixed strategies

In most 2 person zero sum game a saddle point in pure strategies will not exist because that would say that the players should *always* do the same thing. Such games, which include 2×2Nim, Tic-Tac-Dough, and many others, are not interesting when played over and over. The question we answer in this section is to decide how to play the game when there is not a saddle point in pure strategies, such as in the simple game of odds and evens. Indeed most games of interest in real life do not have saddle points in pure strategies.

It seems that if a player should not always play the same strategy, then there should be some randomness involved, because otherwise the opponent will be able to figure out what the player is doing. When a player plays randomly it means that he chooses a row or column according to some probability vector. These vectors are called mixed strategies.

Definition 1.0.5 *A mixed strategy is a vector $X = (x_1, \ldots, x_m)$ for player I and $Y = (y_1, \ldots, y_n)$ for player II where, $x_i \geq 0, \sum_i^n x_i = 1$ and $y_j \geq 0, \sum_j^m y_j = 1$. The components x_i represent the probability that row i will be used by I and y_j the probability column j will be used by II. Denote the set of mixed strategies with k components by*

$$S_k := \{(z_1, z_2, \ldots, z_k) | z_i \geq 0, i = 1, 2, \ldots, k, \sum_{i=1}^{k} z_i = 1\}.$$

So, if player I uses the mixed strategy $X = (x_1, \ldots, x_n) \in S_n$ then he will use row i on each play of the game with probability x_i. Every pure strategy is a mixed strategy by choosing all the probability to be concentrated at the pure strategy, i.e., if I wants to always play row 3, for example, then the mixed strategy he would choose is $X = (0, 0, 1, 0, \ldots, 0)$. Therefore allowing the players to choose mixed strategies permits more choices for the players and so this is an extension of the game.

Now if the players use mixed strategies the payoff can only be calculated in the expected sense. That means the game payoff will represent what each player can be expected to receive. More precisely we calculate as follows.

Given a choice of mixed strategy $X \in S_n$ for I and $Y \in S_m$ for II, the **expected payoff** to player I of the game is

$$\begin{aligned} E(X,Y) &= \sum_{i,j} a_{ij} P(\text{I uses i and II uses j}) \\ &= \sum_{i,j} a_{ij} P(\text{I uses i}) P(\text{II uses j}) = \sum_{i=1}^{n} \sum_{j=1}^{m} x_i a_{ij} y_j = X\ A\ Y^T. \quad (1.0.3) \end{aligned}$$

In a zero sum 2 person game the expected payoff to player II would be $-E(X,Y)$. Recall that player I wants to maximize his expected payoff and player II wants to minimize the expected payoff to I.

Now we can define in a similar way what we mean by a saddle point in mixed strategies.

Definition 1.0.6 *A saddle point in mixed strategies is a pair (X^*, Y^*) of probability vectors which satisfies*

$$E(X, Y^*) \leq E(X^*, Y^*) \leq E(X^*, Y), \;\; \forall \, (X \in S_n, Y \in S_m).$$

If player I decides he is going to use a strategy other than X^* but player II still uses Y^*, then I receives a smaller expected payoff than what he could get if he stuck with X^*. A similar statement holds for player II. So (X^*, Y^*) is an **equilibrium** *in this sense.*

Does a game with matrix A have a saddle point in mixed strategies? Von Neumann's minimax theorem tells us the answer is yes. All we need to do is define the function $f(X, Y) := E(X, Y) = XAY^T$. For any $n \times m$ matrix A this function is concave in X and convex in Y. Actually, it is even **linear** in each variable when the other variable is fixed. Recall that any linear function is both concave and convex so our function f is concave convex and certainly continuous. The second requirement of Von Neumann's theorem is that the sets S_n and S_m be convex sets. This is very easy to check and we leave that as an exercise for the reader. These sets are also closed and bounded. Consequently, we may apply the general Von Neumann Theorem to conclude the following.

Theorem 1.0.7 *For any $n \times m$ matrix A,*

$$\min_{Y \in S_m} \max_{X \in S_n} XAY^T = \max_{X \in S_n} \min_{Y \in S_m} XAY^T.$$

The common value is denoted $v(A)$, and that is the value of the game. In addition there is at least one point $X^* \in S_n, Y^* \in S_m$ so that

$$E[X, Y^*] \leq E[X^*, Y^*] = v(A) \leq E[X^*, Y], \; \textit{for all } X \in S_n, Y \in S_m.$$

So, (X^*, Y^*) is a saddle point, i.e., optimal mixed strategies for the game.

Remark 1.0.8 *While this theorem says that there is always a saddle point in mixed strategies it does not give a way to find them. The next theorem is a step in that direction.*

Notation 1.0.9 *For an $n \times m$ matrix $A = (a_{ij})$ we denote the jth column vector of A by A_j and the ith row vector of A by ${}_iA$. So*

$$A_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix} \;\; \textit{and } {}_iA = (a_{i1}, a_{i2}, \cdots, a_{im})$$

If player I decides to use the pure strategy $X = (0, \ldots, 1, \ldots, 0)$ with row i used 100% of the time and II uses the mixed strategy Y we denote the expected payoff by $E(i, Y) =_i A \cdot Y^T$. Similarly, if player II decides to use the pure strategy $Y = (0, \ldots, 1, \ldots, 0)$ with column j used 100% of the time we denote the expected payoff by $E(X, j) = XA_j$. So we may also write $E[i, j] = a_{ij}$.

Our next theorem gives us a way of finding the value and the optimal mixed strategies.

Theorem 1.0.10 *Let $A = (a_{ij})$ be an $n \times m$ game with value $v(A)$.*
$X^* \in S_n$ is optimal for player I iff

$$v(A) \leq E(X^*, j) = X^* A_j = \sum_{i=1}^{n} x_i^* a_{ij}, j = 1, \ldots, m.$$

So X^* played against any column for player II will result in a payoff at least as large as $v(A)$.
$Y^* \in S_n$ is optimal for II iff

$$E(i, Y^*) =_i A\ Y^{*T} = \sum_{j=1}^{m} a_{ij} y_j^* \leq v(A), i = 1, 2, \ldots, n.$$

So Y^* played against any row for player I results in a payoff (to player I) no bigger than $v(A)$.

Proof. Suppose $v(A) \leq X^* A_j, j = 1, 2, \ldots, m$. Let X^0, Y^0 be a saddle point of the game (which we know exists by Von Neumann's theorem) so that $XAY^{0T} \leq X^0 AY^{0T} \leq X^0 AY^T$ for any $X \in S_n, Y \in S_m$.
Claim: (X^*, Y^0) is also a saddle point.
Let $Y = (y_j)$ be any mixed strategy for II. Then

$$\sum_j v(A)\ y_j = v(A) \leq \sum_{j=1}^{m} E(X^*, j) y_j = \sum_{j=1}^{m} X^* A_j y_j = X^* AY^T.$$

In particular we could certainly choose $Y = Y^0$ to get also $v(A) \leq X^* AY^{0T}$.
Next, we obtain in a similar way that $X^* AY^{0T} \leq X^0 AY^{0T} = v(A)$. Combining these two gives $X^* AY^{0T} = X^0 AY^{0T} = v(A)$. We conclude that

$$XAY^{0T} \leq X^* AY^{0T} \leq X^* AY^T, \text{ for all } X \in S_n, Y \in S_m,$$

so that (X^*, Y^0) is a saddle point for the game and so X^* is optimal for player I. The rest of the proof follows in the same way. □
Now let's use the theorem to see how we can compute the value and strategies for some games.

Example 1.3. **Evens and Odds revisited** In the game of odds or evens we came up with the game matrix

			Player II	
		1	2	3
	1	1	-1	1
Player I	2	-1	1	-1
	3	1	-1	1

Suppose that v is the value of this game and $(X^* = (x_1, x_2, x_3), Y^* = (y_1, y_2, y_3))$ is a saddle point. According to the theorem these quantities should satisfy

$$E(i, Y^*) =_i A\, Y^{*T} = \sum_{j=1}^{3} a_{ij} y_j$$
$$\leq v \leq E(X^*, j) = X^* A_j = \sum_{i=1}^{3} x_i a_{ij}, \ \ i = 1,2,3, \ j = 1,2,3.$$

Using the values from the matrix we have the system of inequalities

$$y_1 - y_2 + y_3 \leq v, \ -y_1 + y_2 - y_3 \leq v, \ \text{ and } y_1 - y_2 + y_3 \leq v$$
$$x_1 - x_2 + x_3 \geq v, \ -x_1 + x_2 - x_3 \geq v, \ \text{ and } x_1 - x_2 + x_3 \geq v.$$

Let's go through finding only the strategy X^* since finding Y^* is similar. We are looking for numbers x_1, x_2, x_3 and v satisfying $x_1 - x_2 + x_3 \geq v$, $-x_1 + x_2 - x_3 \geq v$, as well as $x_1 + x_2 + x_3 = 1$ and $x_i \geq 0, i = 1,2,3$. But then $x_1 = 1 - x_2 - x_3$ and so

$$1 - 2x_2 \geq v, \quad \text{and } -1 + 2x_2 \geq v \implies -v \geq 1 - 2x_2 \geq v.$$

If $v \geq 0$ this says that in fact $v = 0$ and then $x_2 = 1/2$. Let's assume then that $v = 0$ and $x_2 = 1/2$. This would force $x_1 + x_3 = 1/2$ as well. Instead of substituting for x_1 let's do it for x_2. You would see that again we would get $x_1 + x_3 = 1/2$. Something is going on with x_1 and x_3 and we don't seem to have enough information to find them. But we can see from the fact that in the matrix it doesn't matter if player I shows 1 or 3 fingers! The payoffs in all cases are the same. This means that row 3 (or row 1) is a redundant strategy and we might as well drop it. We can say the same for column 1 or column 3. If we drop row 3 we go through the same set of calculations but we quickly find that $x_2 = 1/2 = x_1$. Of course we assumed that $v \geq 0$ to get this but now we have our candidates for the saddle points and value, namely $v = 0$, $X^* = (1/2, 1/2, 0)$ and also, in a similar way $Y^* = (1/2, 1/2, 0)$. Check that with these candidates the inequalities of the theorem are satisfied and so they are the actual value and saddle.

But, it is important to remember that with all three rows and columns, the theorem does not give a single characterization of the saddle point. Indeed, there are an infinite number of saddle points, $X^* = (x_1, 1/2, 1/2 - x_1), 0 \leq x_1 \leq 1/2$ and $Y^* = (y_1, 1/2, y_1 - 1/2), 0 \leq y_1 \leq 1/2$. Nevertheless, there is only one value for this, or any matrix game, and it is $v = 0$ in the game of odds and evens.

Later we will see that this theorem gives a method for solving any matrix game if we pair it up with linear programming which is a way to optimize a linear function over a set with linear constraints.

Properties of optimal strategies Before we go any further we summarize various properties of 2 person zero sum games and their optimal strategies. We skip the proofs but they are not that hard to obtain.

1. Let $value(A)$ be the value of a game with matrix A. If X is any strategy for player I such that $w \le E(X, j), j = 1, \dots, m$, for a given number w, then $w \le value(A)$. Similarly, if Y is a strategy for player II such that $w \ge E(i, Y), i = 1, \dots, n$, then $w \ge value(A)$. In other words, if w is a number less than the expected payoff when X is played against any column, then it must be that w is at least as large as $value(A)$. A similar statement holds for Y. That is, $w \le E(X, j), \forall j, \implies w \ge v(A)$. Also $w \ge E(i, Y), \forall\, i, \implies w \le v(A)$.

2. If w is any number such that $E(i, Y) \le w \le E(X, j), i = 1, \dots, n, j = 1, \dots, m$, where X is a strategy for I and Y is a strategy for II, then $w = value(A)$ and (X, Y) must be a saddle point.

3. If X is a strategy for I and $value(A) \le E(X, j), j = 1, \dots, n$, then X is optimal for I. If Y is a strategy for II and $value(A) \ge E(i, Y), i = 1, \dots, m$, then Y is optimal for II.

4. If Y is optimal for II and $y_j > 0$ then $E(X, j) = value(A)$ for any optimal mixed strategy X for I. Similarly, if X is optimal for I and $x_i > 0$ then $E(i, Y) = value(A)$ for any optimal Y for II. Thus, if any optimal mixed strategy for a player has a strictly positive probability of using a row or a column, then that row or column played against any optimal opponent strategy will yield the value.

5. If X is any optimal strategy for I and $E(X, j_0) > value(A)$ for some column j_0, then for any optimal strategy Y for II, we must have $y_{j_0} = 0$. Similarly, if Y is any optimal strategy for II and $E(i_0, Y) < value(A)$, then any optimal strategy X for I must have $x_{i_0} = 0$.

6. If for any optimal strategy Y for II, $y_{j_0} = 0$, then there is an optimal strategy X for I so that $E(X, j_0) > value(A)$. If for any optimal strategy X

for I, $x_{i_0} = 0$, then there is an optimal strategy Y for II so that $E(i_0, Y) < value(A)$.

7. If a row (or column) is strictly dominated by a convex combination of other rows (or columns), then this row (column) can be dropped from the matrix.

Remark 1.0.11 *Property (7) says that sometimes we can reduce the size of the matrix A by eliminating rows or columns (i.e., strategies) that will never be used because there is always a better row or column to use. This is elimination by* **dominance.**

For example, if every number in row i is bigger or equal to every corresponding number in row k, i.e., $a_{ij} \geq a_{kj}, j = 1, \dots, m$, then the row player would never play row k (since he wants the biggest possible payoff) and so we can drop it from the matrix. Similarly,if every number in column j is less than or equal to every corresponding number in column k, i.e., $a_{ij} \leq a_{ik}, i = 1, \dots, n$, then the column player would never play column k (since he wants player I to get the smallest possible payoff) and so we can drop it from the matrix. If we can reduce it to a $2 \times m$ or $n \times 2$ game then we can solve it by graphical procedure which we will consider shortly. If we can reduce it to a 2×2 matrix we can use formulas we derive in the next section.

Property (7) means that if, for example row k is dominated by a convex combination or two other rows, say p and q, then we can drop row k. This means that if there is a constant $\lambda \in [0, 1]$ so that

$$a_{kj} \leq \lambda a_{pj} + (1 - \lambda) a_{qj}, \quad j = 1, \dots, m$$

then row k is dominated and can be dropped. Of course if the constant $\lambda = 1$ then row p dominates row k and we can drop row k. If $\lambda = 0$ then row q dominates row k.

Example 1.4. Consider the 3×3 game

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

There is no obvious dominance of one row by another or one column by another. However, if we suspect dominance by a convex combination we can check as follows.

If row 1 is dominated by a convex combination of rows 2 and 3 then we must have, for some $0 \leq \lambda \leq 1$ the inequalities

$$1 \leq \lambda(1) + (1 - \lambda)(1), \; 1 \leq 2(\lambda) + 0(1 - \lambda), \; 1 \leq 0(\lambda) + 2(1 - \lambda).$$

Simplifying, $1 \leq 1, 1 \leq 2\lambda, 1 \leq 2 - 2\lambda$. But this says $\lambda = 1/2$. So, there is a $\lambda = 1/2$ which works, and row 1 may be dropped from the matrix (i.e., any mixed strategy will play row 1 with probability 0). So now the new matrix is

$$A' = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

By the same technique, it is easy to see that column 1 is dominated by a combination of columns 2 and 3 and so column 1 may be dropped. We are left with the matrix

$$A'' = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

from which it should be clear that each row and column should be played 1/2 the time. So the optimal strategies for the original game are $X^* = (0, 1/2, 1/2) = Y^*$ and the value$=E(X^*, Y^*)$ is 1.

This example shows that games do not necessarily have unique optimal strategies. For instance $X^* = (1, 0, 0) = Y^*$ is another saddle point, and in fact $X^* = (\lambda, \frac{1}{2}(1-\lambda), \frac{1}{2}(1-\lambda))$ and $Y^* = (\mu, , \frac{1}{2}(1-\mu), \frac{1}{2}(1-\mu))$ for any choice of $\lambda, \mu \in [0, 1]$ is another saddle point. However, the value of the game is always unique and it is 1 in this example.

1.0.3 Best Response Strategies

If you are playing a game and you determine, in one way or another, that your opponent is using a particular strategy, then what should you do? To be specific, suppose you are player I and you know that player II is using the mixed strategy Y, optimal or not for player II. In this case you should play the mixed strategy X which maximizes $E(X, Y)$. This strategy that you use would be a best response to the use of Y by player II. The best response strategy to Y may not be the same as what you would use if you knew that player II was playing optimally. That is, it may not be a part of a saddle point. Here is the precise definition.

Definition 1.0.12 *A mixed strategy X^* for player I is a best response strategy to the strategy Y for player II if it satisfies*

$$\max_{X \in S_n} E(X, Y) = \max_{X \in S_n} \sum_{i=1}^{n} x_i^* a_{ij} y_j = E(X^*, Y).$$

A mixed strategy Y^* for player II is a best response strategy to the strategy X for player I if it satisfies

$$\min_{Y \in S_m} E(X, Y) = \min_{Y \in S_m} \sum_{j=1}^{m} x_i a_{ij} y_j^* = E(X, Y^*).$$

Example 1.5. In the preceding example we determined that for the matrix Consider the 3×3 game

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

the saddle point is $X^* = (0, 1/2, 1/2) = Y^*$ and $v(A) = 1$. Now suppose that player II, for some reason, thinks they can do better by playing $Y = (1/4, 1/4, 1/2)$. What is an optimal response strategy for player I?

Let $X = (x_1, x_2, 1 - x_1 - x_2)$. Calculate

$$E(X, Y) = XAY^T = -x_1/4 - x_2/2 + 5/4.$$

We want to maximize this with the constraints $0 \le x_1, x_2 \le 1$. We don't have to do anything more. We see right away that $E(X, Y)$ is maximized by taking $x_1 = x_2 = 0$ and then $x_3 = 1$. Hence, the best response strategy for player I if player II uses $Y = (1/4, 1/4, 1/2)$is $X^* = (0, 0, 1)$. Then, the expected payoff to I is $E(X^*, Y) = 5/4$ which is larger than the value of the game $v(A) = 1$. So that is how player I should play if player II decides to deviate from the optimal Y.

1.1 Solution of some special games

Now we will consider some special types of games for which we actually have a formula giving the value and the saddle points. Let's start with the easiest possible class of games which can always be solved explicitly.

1.1.1 2 × 2 games

Here

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{Player I} : X = \begin{pmatrix} x \\ 1 - x \end{pmatrix} \quad \text{Player II} : Y = \begin{pmatrix} y \\ 1 - y \end{pmatrix}$$

So for any mixed strategies we have $E(X, Y) = XAY^T$, which, written out, is

$$E(X, Y) = xy(a_{11} - a_{12} - a_{21} + a_{22}) + x(a_{12} - a_{22}) + y(a_{21} - a_{22}) + a_{22}.$$

Theorem 1.1.1 *In the 2×2 game with matrix A, assume that there are no pure optimal strategies. Let $X^* = (x^*, 1 - x^*)$ be optimal for I and $Y^* = (y^*, 1 - y^*)$ optimal for II. If we set*

$$x^* = \frac{a_{22} - a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}}, \qquad y^* = \frac{a_{22} - a_{12}}{a_{11} - a_{12} - a_{21} + a_{22}}.$$

then $X^* = (x^*, 1-x^*), Y^* = (y^*, 1-y^*)$ *are optimal mixed strategies. The value of the game is*

$$v(A) = E(X^*, Y^*) = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}}.$$

Remark 1.1.2 *A more compact way to write this and easier to remember is*

$$x^* = \frac{(1\ 1)A^*}{(1\ 1)A^* \begin{pmatrix} 1 \\ 1 \end{pmatrix}} \quad \textit{and} \quad y^* = \frac{A^* \begin{pmatrix} 1 \\ 1 \end{pmatrix}}{(1\ 1)A^* \begin{pmatrix} 1 \\ 1 \end{pmatrix}} \tag{1.1.1}$$

$$value(A) = \frac{det(A)}{(1\ 1)A^* \begin{pmatrix} 1 \\ 1 \end{pmatrix}} \tag{1.1.2}$$

$$where A^* = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \textit{ and } det(A) = a_{11}a_{22} - a_{12}a_{21}. \tag{1.1.3}$$

The matrix A^* *is called the (transpose) of the cofactor matrix. Recall that the inverse of a* 2×2 *matrix is found by swapping the main diagonal numbers, and putting a minus sign in front of the other diagonal numbers, and finally by dividing by the determinant of the matrix.* A^* *is exactly the first two steps but we don't divide by the determinant. The matrix we get is is defined even if the matrix* A *doesn't have an inverse. Remember however, we need to make sure the matrix doesn't have optimal pure strategies first. If there are optimal pure strategies then we can't use these formulas. (Exercise: give an example to show that when optimal pure strategies exist, the formulas won't work).*

Notice too, that if $det(A) = 0$ *the value of the game is zero.*

Here is why the formulas hold. Write $f(x, y) = E(X, Y)$ where $X = (x, 1-x), Y = (y, 1-y), 0 \leq x, y \leq 1$. By assumption there are no optimal pure strategies and so the extreme points of f will be found inside the intervals $0 < x, y < 1$. The function of two variables $f(x, y)$ will have a maximum and a minimum but because we assume there are no pure strategy points, when we use calculus to find the critical points the partial derivatives set to zero and solved will yield the only possibility for a concave convex function, namely a saddle. The function f has to look like the following:

This is the graph of a concave convex function which has a saddle point at $(1/2, 1/2)$ and you can see now why it is called a saddle.

Returning to our function $f(x, y)$, take the partial derivatives and set to zero:

$$\frac{\partial f}{\partial x} = y\alpha + \beta = 0, \text{ and } \frac{\partial f}{\partial y} = x\alpha + \gamma = 0,$$

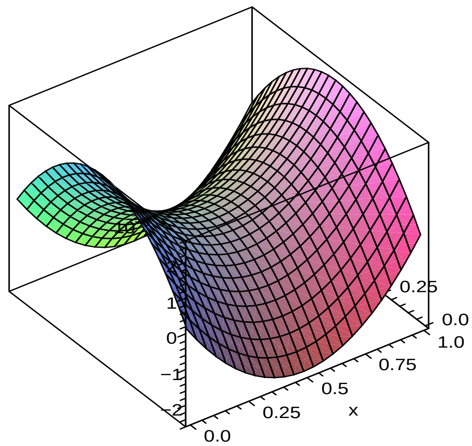


Fig. 1.2. A Saddle Function

where

$$\alpha = (a_{11} - a_{12} - a_{21} + a_{22}) \;\; \beta = (a_{12} - a_{22}) \;\; \gamma = (a_{21} - a_{22}).$$

We solve where the partials are zero to get

$$x^* = -\frac{\gamma}{\alpha} = \frac{a_{22} - a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}}, \quad \text{and } y^* = -\frac{\beta}{\alpha} = \frac{a_{22} - a_{12}}{a_{11} - a_{12} - a_{21} + a_{22}},$$

which is the same as what the theorem says. □

So the procedure is: you look for pure strategy solutions first (calculate v^- and v^+ and see if they're equal), but if there aren't any you may use the formulas to find the mixed strategy solutions.

1.1.2 Invertible matrix games

In this section we will solve the class of games in which the matrix A is square, say $n \times n$ and invertible, so $det(A) \neq 0$ and A^{-1} exists and satisfies $A^{-1}A = AA^{-1} = I_{n \times n}$. For the present let us suppose that

> Player I has an optimal strategy which is completely mixed, i.e. $X = (x_1, \ldots, x_n)$, and $x_i > 0, i = 1, 2, \ldots, n$. So player I plays every row with positive probability.

By the properties of strategies 1.0.2(4) we know that this implies that every optimal strategy for player II, Y must satisfy

$$E(i, Y) =_i AY^T = value(A), \text{ for every row } i = 1, 2, \ldots, n.$$

So Y played against any row will give the value of the game. If we write $J_n = (1\ 1\ \cdots 1)$ for the row vector consisting of all 1's we can write

$$AY^T = v(A)J_n^T. \tag{1.1.4}$$

Now, if $v(A) = 0$ then $AY^T = 0J_n^T = 0$ and this is a system of n equations in the n unknowns $Y = (y_1, \ldots, y_n)$. It is a homogeneous system. Since A is invertible this would have the one and only solution $Y = 0$. But that is impossible if Y is a strategy (the components must add to 1.) So, the value of the game cannot be zero, and we get, by multiplying both sides of (1.1.4) by A^{-1},

$$A^{-1}AY^T = Y^T = v(A)A^{-1}J_n^T.$$

This gives Y if we knew $v(A)$. How do we get that. The extra piece of information we have is that the components of Y add to 1, i.e., $\sum_{j=1}^n y_j = J_n Y^T = 1$. So,

$$J_n Y^T = 1 = v(A)J_n A^{-1}J_n^T$$

and therefore

$$v(A) = \frac{1}{J_n A^{-1} J_n^T} \text{ and } Y^T = \frac{A^{-1}J_n^T}{J_n A^{-1} J_n^T}. \tag{1.1.5}$$

We have found the only candidate for the optimal strategy for player II assuming that every component of X is greater than 0. But, if it turns out that this formula for Y has at least one $y_j < 0$, something would have to be wrong and what's wrong is that it would not be true that X was completely mixed. But, if $y_j \geq 0$ for every component, we could try to find an optimal X for player I by the exact same method (exercise). This would give us

$$X = \frac{J_n A^{-1}}{J_n A^{-1} J_n^T}.$$

This method will work if the formulas we get for X and Y end up satisfying the condition that they are strategies. If either X or Y has a negative component, then it fails. But notice that the strategies do not have to be completely mixed as we assumed from the beginning.

Here is a summary of what we know:

Theorem 1.1.3 *Assume*

1. $A_{n\times n}$ has an inverse A^{-1}.

2. $J_n A^{-1} J_n^T \neq 0$.

Then, setting

$$v := \frac{1}{J_n A^{-1} J_n^T}, Y^T = \frac{A^{-1}J_n^T}{J_n A^{-1} J_n^T}, X = \frac{J_n A^{-1}}{J_n A^{-1} J_n^T},$$

if $x_i \geq 0, i = 1, \ldots, n$ and $y_j \geq 0, j = 1, \ldots, n$, we have that $v = v(A)$ is the value of the game with matrix A and (X, Y) is a saddle point in mixed strategies.

Now the point is that when we have an invertible game matrix we can always use the formulas in the theorem to calculate the number v and the vectors X and Y. If the result gives vectors with nonnegative components then by the properties 1.0.2 of games, v must be the value, and (X, Y) is a saddle point. Notice that from the formulas directly $J_n Y^T = 1$ and $X J_n^T = 1$ so the components will automatically sum to 1; it is only the sign of the components which must be checked.

In order to guarantee that the value is not zero we may add a constant to every element of A which is large enough to make all the numbers of the matrix positive. In this case the value of the new game could not be zero. Since $v(A+b) = v(A)+b$, where b is the constant added to every element (Exercise), we can find the original $v(A)$ by subtracting b. Adding the constant to all the elements of A will not change the probabilities of using any particular row or column, i.e., the optimal mixed strategies are not affected by doing that.

Even if our original matrix A does not have an inverse, if we add a constant to all the elements of A we get a new matrix $A + b$ and the new matrix *may* have an inverse (of course it may not as well). Here is an example.

Example 1.6. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & -2 \\ 1 & -2 & 3 \\ -2 & 3 & -4 \end{pmatrix}$$

This matrix has negative entries so it's possible that the value of the game is zero. The matrix does not have an inverse because the determinant of A is 0. So, let's try to add a constant to all the entries to see if we can make the new matrix invertible. Since the largest negative entry is -4 let's add 5 to everything to get

$$A + 5 = \begin{pmatrix} 5 & 6 & 3 \\ 6 & 3 & 8 \\ 3 & 8 & 1 \end{pmatrix}$$

This matrix has an inverse given by

$$B = \frac{1}{80} \begin{pmatrix} 61 & -18 & -39 \\ -18 & 4 & 22 \\ -39 & 22 & 21 \end{pmatrix}$$

Next we calculate using the formulas $v = 1/(J_3 B J_3^T) = 5$, and

$$X = v(J_3 B) = (1/4\ 1/2\ 1/4) \quad \text{and } Y = (1/4\ 1/2\ 1/4).$$

Since both X and Y are strategies (they have nonnegative components) the theorem tells us that they are optimal and the value of our original game is $v(A) = 0$.

Here are the Maple commands to work this out. You can try many different constant to add to the matrix to see if you get an inverse if the original matrix does not have an inverse.

```
>restart:with(LinearAlgebra):
>A:=Matrix([[0 ,1 ,-2 ],[1 ,2 ,-3],[-2 ,3 ,-4 ]]);
>MatrixDeterminant(A);
>A:=MatrixAdd( ConstantMatrix(5,3,3), A );
>B:=A^(-1);
>J:=Vector[row]([1 ,1 ,1 ]);
>J.B.Transpose(J);
>v:=1/J.B.Transpose(J);
>X:=v*(J.B);
>Y:=v*(B.Transpose(J));
```

The first line loads the Linear Algebra package. The next line enters the matrix. The MatrixDeterminant command finds the determinant of the matrix so you can see if it has an inverse. The MatrixAdd command adds the constant 3×3 matrix which consists of all 4's to A. The ConstantMatrix(5,3,3) says the matrix is 3×3 and has all 5's. The inverse of A is put into the matrix B and the row vector consisting of 1's is put into J. Finally we calculate $v = value(A)$ and the optimal mixed strategies X, Y.

1.1.3 Graphical solution of $2 \times m$ games

When we have an invertible matrix we can use the method of the preceding section to solve the game. Of course a game with a $2 \times m$ matrix cannot have an inverse but now we can use a very instructive graphical method to solve the game.

Consider the game given by

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \end{pmatrix}$$

We denote, as before, by A_j the jth column of the matrix and ${}_iA$ the ith row.

Suppose that Player I chooses a mixed strategy $X = (x, 1-x)$ and Player II chooses column j. The payoff to player I is $E(X, j) = XA_j$ or, written out,

$$E(X, j) = x\ a_{1j} + (1-x)a_{2j}.$$

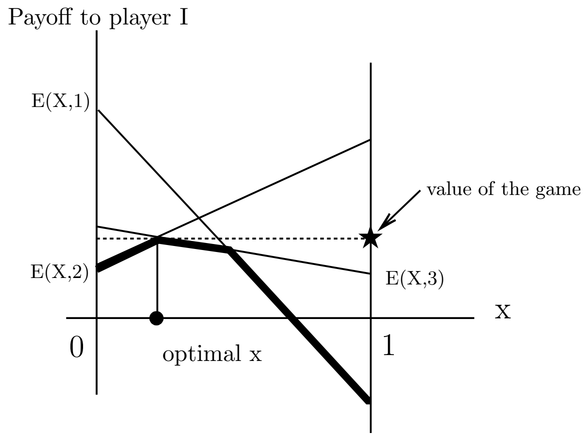
Now, because there are only two rows, a mixed strategy is determined by the choice of the single variable $x \in [0, 1]$. This is perfect for drawing a plot. On a graph (with x on the horizontal axis), $y = E(X, j)$ is a straight line through the two points $(0, a_{2j})$ and $(1, a_{1j})$. Now do this for each column j and look at

$$f(x) = \min_{1 \le j \le n} X\ A_j = \min_j x\ a_{1j} + (1-x)a_{2j}.$$

This is the lower envelope of all the straight lines associated to each strategy j for player II. Then let $0 \le x^* \le 1$ be the point where the maximum of f is achieved:

$$f(x^*) = \max_{0 \le x \le 1} f(x) = \max_x \min_j x\, a_{1j} + (1-x)a_{2j}.$$

This is the maximum minimum of f. Then $X^* = (x^*, 1-x^*)$ is the optimal strategy for player I. This is shown in the figure for a 2×3 game.



Graphical Solution for 2 x 3 Game

Each line represents the payoff that player I would receive if player I plays the mixed strategy $X = (x, 1-x)$ and player II always plays a fixed column. In the figure you can see that if player I decides to play the mixed strategy $X_1 = (x_1, 1-x_1)$ where x_1 is to the left of the optimal value, then player II would choose to play column 2. If player I decides to play the mixed strategy $X_2 = (x_2, 1-x_2)$ where x_2 is to the right of the optimal value, then player II would choose to play column 3 (up to the point of intersection where $E(X,1) = E(X,3)$), and then switch to column 1. This shows how player I should choose $x, 0 \le x \le 1$, namely, player I would choose the x that guarantees player I will receive the maximum of all the lower points of the lines. By choosing this optimal value, say x^*, it will be the case that player II would play some combination of columns 2 and 3. It would be a mixture (a convex combination) of the columns because if player II always chose to play, say column 2, then player I could do better by changing his mixed strategy to a point to the right of the optimal value. Remember that 'optimal' means that it is optimal under the assumption that the other player is doing his best.

Now we can find the optimal strategy for II because the graph shows that player II would only use columns 2 and 3, so we can eliminate column 1 and reduce to a 2×2 matrix. Let's work through a specific example.

Example The matrix payoff to player I is

$$A = \begin{pmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \end{pmatrix}$$

Consider the graph for player I first.

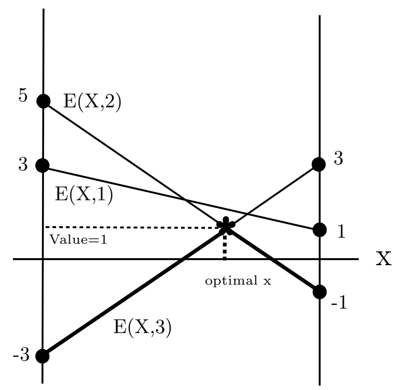
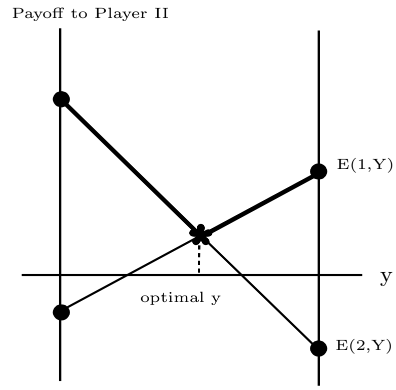


Fig. 1.3. Graph for player I

Looking at Figure 3, we see that the optimal strategy for I is the x value where the two lower lines intersect and yields $X^* = (2/3, 1/3)$. Also $v(A) = E(X^*, 3) = E(X^*, 2) = 1$. To find the optimal strategy for II we see that II will never use the first column. So consider the subgame with the first column removed

$$A1 = \begin{pmatrix} -1 & 3 \\ 5 & -3 \end{pmatrix}$$

Now we can solve this graphically assuming that II uses $Y = (y, 1-y)$. Again we have the payoff to player I if II uses Y but now we look for the $y \in [0, 1]$ for II. Player II wants to choose y so that no matter what I does he (player II) is guaranteed the smallest maximum. This is now the lowest point of the highest part of the lines.



We see that the lines intersect with $y^* = 1/2$. Hence the optimal strategy for II is $Y^* = (0, 1/2, 1/2)$.

The graphical solution of an $n \times 2$ game is similar to the preceding except that we begin by finding the optimal strategy for player II first. Here is an $n \times 2$ game matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ \vdots & \vdots \\ a_{n1} & a_{n2} \end{pmatrix}$$

Assume that II uses the mixed strategy $Y = (y, 1-y), 0 \leq y \leq 1$. Then II wants to choose $0 \leq y \leq 1$ to *minimize* the quantity

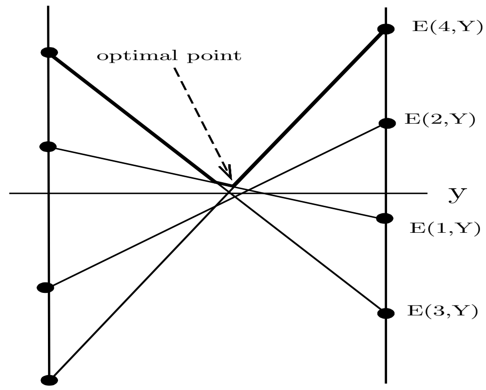
$$\max_{1 \leq i \leq n} E(i, Y) = \max_{1 \leq i \leq n} \, {}_iAY^T = \max_{1 \leq i \leq n} y(a_{i1}) + (1-y)(a_{i2}).$$

For each row $i = 1, 2, \ldots, n$ the graph of the payoffs (to player I) $E(i, Y)$ will be a straight line. So we will end up with n lines. Player I will want to go as high as possible and there is not much player II can do about stopping him from doing that. So, player II, playing conservatively, will play the mixed strategy Y which will give the lowest maximum. The optimal y^* will be the point giving the minimum of the upper envelope. Notice that this is a guaranteed optimal strategy because if player II deviates from the lowest maximum, player I can do better. Let's work out an example.

Example 1.7. Let's consider

$$A = \begin{pmatrix} -1 & 2 \\ 3 & -4 \\ -5 & 6 \\ 7 & -8 \end{pmatrix}$$

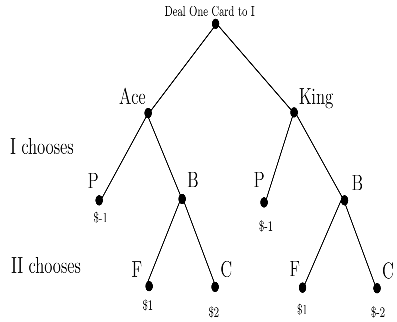
This is a 2×4 game without a saddle point in pure strategies since $v^- = -1, v^+ = 6$. There is also no obvious dominance, so we try to solve the game graphically. Suppose player II uses the strategy $Y = (y, 1-y)$, then we graph the payoffs $E(i, Y), i = 1, 2, 3, 4$.



You can see the difficulty with solving games graphically, i.e., you have to be very accurate with your graphs. Carefully reading the information, it looks like the optimal strategy for Y will be determined at the intersection point of $E(3, Y) = -5y + 6(1-y)$ and $E(1, Y) = -y + 2(1-y)$. This occurs at the point $y^* = 5/9$ and the corresponding value of the game will be $v(A) = 1/3$. Since this uses only rows 1 and 4, we may now drop rows 2 and 3 to find the optimal strategy for player I. We obtain that row 1 should be used 5/6 of the time and row 4 1/6 of the time, so $X^* = (5/6, 0, 0, 1/6), Y^* = (5/9, 4/9)$.

Example 1.8. This is a modified version of the endgame in poker. Here are the rules. Player I is dealt a card which may be an *Ace* or a $King$. Player I sees the result but II does not. I may then choose to Pass or Bet. If I passes he has to pay II \$ 1. If I bets, player II may choose to Fold or Call. If II folds, he pays I \$ 1.

Here is a graphical representation of the game.



The End Game in Poker

Now I has four strategies FF =fold on Ace and fold on King, FB=fold on Ace and Bet on King, BF =bet on Ace and Fold on King, and BB =bet on Ace and Bet on King. Player II has only two strategies, namely F =fold or C =call.

Assuming that the probability of being dealt a King or an Ace is 1/2 we may calculate the expected reward to I and get the matrix as follows:

I/II	C	F
FF	-1	-1
FB	-3/2	0
BF	1/2	0
BB	0	1

For example if I plays BF and II plays C we calculate the expected payoff to I as $(1/2)2 + (1/2)(-1) = 1/2$. This is a 4×2 game which we can solve graphically.

1. The lower and upper values are $v^- = 0, v^+ = 1/2$ so there is no saddle point in pure strategies. In addition row 1, namely FF, is a strictly dominated strategy, so we may drop it. Row 2 is also dominated and it can be dropped. So we are left with considering the 2×2 matrix

$$A' = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}$$

We may use formulas to solve this but we are going to do it graphically.

2. Suppose II plays $Y = (y, 1-y)$. Then

$$E(BF, Y) = A'Y^T = \frac{1}{2}y \text{ and } E(BB, Y) = (1-y)$$

There are only two lines to consider and these two lines intersect at $\frac{1}{2}y = 1-y$ so that $y^* = 2/3$. The optimal strategy for II is $Y^* = (2/3, 1/3)$ so II should Call 2/3 of the time and Bet 1/3 of the time. The value of the game is then at the point of intersection $v = 1/3$.

3. For player I, suppose he plays $X = (x, 1-x)$. Then

$$E(X, C) = XA' = \frac{1}{2}x, \text{ and } E(X, F) = 1 - x.$$

Since there are only two lines we again calculate the intersection point and so the optimal strategy for I is $X^* = (0, 0, 2/3, 1/3)$.

4. Player II is at a distinct disadvantage since the value of this game is $v = 1/3$. Player II in fact would never be induced to play the game unless player I pays II \$1/3 before the game begins. That would make the value zero and hence a fair game.

Another point to notice is that the optimal strategy for I has him betting 1/3 of the time when he has a losing card (King). This means in an optimal strategy he should bluff.

1.1.4 Symmetric games

Symmetric games are important classes of 2 person games in which the players can use the exact same set of strategies and any payoff which player I can obtain using strategy X can be obtained by player II using the same strategy $Y = X$. Such games can be quickly identified by the rule that $A = -A^T$. Any matrix satisfying this is said to be skew symmetric.

Why is skew symmetry the correct thing? Well, if A is the payoff matrix to player I, then the entries represent the payoffs to player I and the negative of the entries, or $-A$ represent the payoffs to player II. If the players switch roles as row and column players but obtain the same payoffs then A is the payoff matrix to player I and $-A^T$ is the payoff to player II. So, if $X = Y$ we want the payoffs to player II, namely $-AY^T$ to be the same as the payoffs to player I, namely XA. That is, $-AY^T = AX = AY$, which holds if $A = -A^T$. If this is so, the matrix must be square and the diagonal elements of A, a_{ii} must be 0. We can say more.

Theorem 1.1.4 *For any skew symmetric game $v(A) = 0$ and if X^* is optimal for player I, then it is also optimal for player II.*

Proof. Let X be any strategy for I. Then

$$E(X, X) = X\ A\ X^T = -X\ A^T\ X^T = -(X\ A\ X^T)^T = -X\ A\ X^T = -E(X, X).$$

Therefore $E(X, X) = 0$.

Let X^*, Y^* be a saddle point for the game so that $E(X, Y^*) \le E(X^*, Y^*) \le E(X^*, Y)$. Then

$$E(X, Y) = X\ A\ Y^T = -X\ A^T\ Y^T = -Y\ A\ X^T = -E(Y, X).$$

Hence, from the saddle point definition,

$$-E(Y^*, X) \le -E(Y^*, X^*) \le -E(Y, X^*) \Longrightarrow E(Y^*, X) \ge E(Y^*, X^*) \ge E(Y, X^*)$$

But this says Y^* is optimal for I and X^* is optimal for II and also that $E(X^*, Y^*) = -E(Y^*, X^*) \Longrightarrow v(A) = 0.$ $\square$

Example 1.9. **General solution of** 3×3 **symmetric games** For any 3×3 symmetric game we must have

$$A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

Any of the following conditions gives a pure saddle point:

1. $a \geq 0, b \geq 0 \Longrightarrow$ saddle at $(1,1)$ position
2. $a \leq 0, c \geq 0 \Longrightarrow$ saddle at $(2,2)$ position
3. $b \leq 0, c \leq 0 \Longrightarrow$ saddle at $(3,3)$ position

Here's why. Let's assume $a \leq 0, c \geq 0$. In this case if $b \leq 0$ we get $v^- = \max\{\min\{a,b\}, 0, -c\} = 0$ and $v^+ = \min\{\max\{-a,-b\}, 0, c\} = 0$ so there is a saddle in pure strategies at $(2,2)$. All cases are treated similarly. To have a mixed strategy all three of thee must fail.

We next solve the case $a > 0, b < 0, c > 0$ so there is no pure saddle and we look for the mixed strategies.

Let player I's optimal strategy be $X^* = (x_1, x_2, x_3)$. Then

$$\begin{aligned} E(X^*, 1) &= -ax_2 - bx_3 \geq 0 \\ E(X^*, 2) &= ax_1 - cx_3 \geq 0 \\ E(X^*, 3) &= bx_1 + cx_2 \geq 0 \end{aligned}$$

Each one is nonnegative since $E(X^*, Y) \geq 0,$ for all Y. Now since $a > 0, b < 0, c > 0$ we get

$$\frac{x_3}{a} \geq \frac{x_2}{-b}, \qquad \frac{x_1}{c} \geq \frac{x_3}{a}, \qquad \frac{x_2}{-b} \geq \frac{x_1}{c}$$

so

$$\frac{x_3}{a} \geq \frac{x_2}{-b} \geq \frac{x_1}{c} \geq \frac{x_3}{a},$$

and we must have equality. Thus, $x_3 = a\,t, x_2 = -b\,t, x_1 = c\,t$ and since they must sum to one, $t = 1/(a - b + c)$. We have found the optimal strategies $X^* = Y^* = (a\,t, -b\,t, c\,t)$. The value, of course is zero.

Example 1.10. Two companies will introduce a number of new products which are essentially equivalent. They will introduce 1 or 2 products but they each must also guess how many products their opponent will introduce. If they introduce the same number of products and guess the correct number the opponent will introduce the payoff is zero. Otherwise the payoff is determined by whoever introduces more products and guesses the correct introduction of new products by the opponent. This accounts for the fact that new products result in more sales and guessing correctly results in accurate advertising, etc.. This is the payoff matrix to player I.

Player I / Player II	(1,1)	(1,2)	(2,1)	(2,2)
(1,1)	0	1	-1	-1
(1,2)	-1	0	-2	-1
(2,1)	1	2	0	1
(2,2)	1	1	-1	0

This game is symmetric. We can drop the 4th row and the 4th column by dominance and are left with the symmetric game

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & -2 \\ 1 & 2 & 0 \end{pmatrix}$$

Since $b \leq 0, c \leq 0$ we have a saddle point at $(3,3)$ and so $v = 0, X^* = (0,0,1,0) = Y^*$, each company should introduce 2 new products and guess that the opponent will introduce 1.

1.2 Matrix games and linear programming

In this section we will formulate any matrix game as a linear programming problem using the properties 1.0.2. The advantage of doing so is that there is a very efficient algorithm for solving any linear program, namely, the **simplex method**. This means that we can find the value and optimal strategies for a matrix game of any size. In many respects, this approach makes it unnecessary to know any other computational approach. The down side is that in general one needs a computer capable of running the simplex algorithm to solve a game by the method of linear programming. We will show how to set up the game in two different ways to make it amenable to the linear programming method and also the Maple commands to solve the problem.

A linear programming problem is a problem of the form (called the Primal program):

$$\text{Minimize } z = xc^T$$
$$\text{subject to } x\,A \geq b, x \geq 0$$

where $c = (c_1, \ldots, c_n), x = (x_1, \ldots, x_n), A_{n\times m}, b = (b_1, \ldots, b_m)$.

The primal problem seeks to minimize a *linear objective function*, $z(x) = c^T\ x$, over a set of constraints which are also linear. You can visualize what happens if you try to imagine a plane in 3 dimensions over a convex set. The minimum and maximum of the plane must occur on the boundary of the convex set, just as the minimum or maximum of a straight line on an interval must occur at the endpoints. The method for solving a linear program is to efficiently go through the boundary points to find the extreme points. That is essentially the simplex method (developed by George Dantzig during World War II).

For any Primal there is a related linear program called the Dual program:

$$\begin{aligned}&\text{Maximize } w = y\, b^T\\ &\text{subject to } A\, y^T \le c^T, y \ge 0\end{aligned}$$

We will now show how to formulate any matrix game as a linear program.We need the primal and the dual to find the optimal strategies for each player.

Here is the first method to do this. Let A be the game matrix. We may assume $a_{ij} > 0$ by adding a large enough constant to A if that isn't true. Adding a constant won't change the strategies and will only add a constant to the value of the game.

Hence we assume that $v(A) > 0$. Now consider the properties 1.0.2. Player I looks for a mixed strategy $X = (x_1, \ldots, x_n)$ so that

$$E(X, j) = XA_j = x_1\, a_{1j} + \cdots x_n\, a_{nj} \ge v,\ 1 \le j \le m \tag{1.2.1}$$

where $v > 0$ is as **large as possible**, and $\sum x_i = 1, x_i \ge 0$. It is player I's goal to get the largest value possible. We change variables by setting

$$p_i = \frac{x_i}{v}, 1 \le i \le n.$$

Then $\sum x_i = 1$ implies that

$$\sum_{i=1}^{n} p_i = \frac{1}{v}.$$

Thus maximizing v is the same as minimizing $1/v = \sum p_i$. This gives us our objective. For the constraints, if we divide the inequalities (1.2.1) by v and switch to the new variables, we get the set of constraints

$$\frac{x_1}{v}\, a_{1j} + \cdots \frac{x_n}{v}\, a_{nj} = p_1 a_{1j} + \cdots + p_n a_{nj} \ge 1,\ 1 \le j \le m$$

Now we write this as a linear program.

$$\text{Player I's program} = \begin{cases}\text{Minimize } z_I = \ p\, J_n^T = \sum_{i=1}^n p_i,\ J_n = (1, 1, \ldots, 1)\\ \text{subject to,} \qquad\qquad\qquad\qquad p\, A \ge J_m, p \ge 0.\end{cases}$$

Notice that the constraint of the game $\sum_i x_i = 1$ is used to get the objective function! It is not one of the constraints of the linear program. The set of constraints is $p\ A \ge J_m$ which means $p \cdot A_j \ge 1, j = 1, \ldots, m$. Also $p \ge 0$ means $p_i \ge 0, i = 1, \ldots, n$.

Once we solve player I's program we will have in our hands the optimal $p = (p_1, \ldots, p_n)$ which minimizes the objective $z_I = p\ J_n^T$. We will also know what the minimum z_I is, say z_I^*.

Unwinding the formulation we find the optimal strategy X for player I and the value of the game as follows:

$$value(A) = \frac{1}{\sum_{i=1}^{n} p_i} = \frac{1}{z_I^*} \quad \text{and} \quad x_i = p_i(value(A)).$$

Now we look at the problem for Player II. He wants to find a mixed strategy $Y = (y_1, \ldots, y_m), y_j \geq 0, \sum_j y_j = 1$, so that

$$y_1\, a_{i1} + \cdots + y_m\, a_{im} \leq u, i = 1, \ldots, n$$

with $u > 0$ as **small as possible**. Setting $q_j = \frac{y_j}{u}$ we can restate Player II's problem as the standard linear programming problem

$$\text{Player II's program} = \begin{cases} \text{Maximize } z_{II} = q\, J_m^T,\ J_m = (1, 1, \ldots, 1), m1's \\ \text{subject to,} \qquad\qquad A\, q^T \leq J_n^T, q \geq 0. \end{cases}$$

Player II's problem is the dual of player I's. At the end of solving this program we are left with the optimal maximizing vector $q = (q_1, \ldots, q_m)$ and the optimal objective value Z_{II}^*. We obtain the optimal mixed strategy for player II and the value of the game from

$$value(A) = \frac{1}{\sum_{j=1}^{m} q_j} = \frac{1}{z_{II}^*} \quad \text{and} \quad y_j = q_j(value(A)).$$

But, how do we know that the value of the game using player I's program will be the same as that given by player II's program? The important *duality theorem* gives us what we need.

Remember that if you had to add a number to the matrix to guarantee that $v > 0$ then you have to subtract that number from z_I^*, and z_{II}^*, in order to get the value of the game with the starting matrix A.

Theorem 1.2.1 *If one of the pair of linear programs (primal and dual) has a solution then so does the other. If there is at least one feasible solution (i.e., a vector which solves all the constraints so the constraint set is nonempty) then there is an optimal feasible solution for both, and their values, i.e. the objectives, are equal.*

This means that in a game we are guaranteed that $z_I^* = z_{II}^*$ and so the values given by player I's program will be the same as that given by player II's program.

Example 1.11. Find a solution of the game with matrix

$$A = \begin{pmatrix} -2 & 1 & 0 \\ 2 & -3 & -1 \\ 0 & 2 & -3 \end{pmatrix}$$

Begin by making all entries positive by adding 4 (other numbers could also be used) to everything:

$$A' = \begin{pmatrix} 2 & 5 & 4 \\ 6 & 1 & 3 \\ 4 & 6 & 1 \end{pmatrix}$$

Here is player I's program. We are looking for $X = (x_1, x_2, x_3), x_i \geq 0, \sum_{i=1}^{3} x_i = 1$, which will be found from the linear program. Setting $p_i = \frac{x_i}{v}$, where after finding the p_i's, we will set $v = \frac{1}{p_1+p_2+p_3}$ and $v - 4$ is the value of the game with matrix A, v is the value of the game with matrix A'. Then $x_i = vp_i$ will give the optimal strategy for I. Player I's problem is

$$\text{Player I's program} = \begin{cases} \text{Minimize } z_I = p_1 + p_2 + p_3 \; (= \frac{1}{v}) & \\ \text{subject to,} & \\ 2p_1 + 6p_2 + 4p_3 & \geq 1 \\ 5p_1 + p_2 + 6p_3 & \geq 1 \\ 4p_1 + 3p_2 + p_3 & \geq 1 \\ p_i \geq 0 & i = 1, 2, 3. \end{cases}$$

We next set up II's program. We are looking for $y = (y_1, y_2, y_3), y_j \geq 0, \sum_{j=1}^{3} = 1$. Setting $q_j = (y_j/v)$. Player II's problem is

$$\text{Player II's program} = \begin{cases} \text{Maximize } z_{II} = q_1 + q_2 + q_3 \; (= \frac{1}{v}) & \\ \text{subject to,} & \\ 2q_1 + 5q_2 + 4q_3 & \leq 1 \\ 6q_1 + q_2 + 3q_3 & \leq 1 \\ 4q_1 + 6q_2 + q_3 & \leq 1 \\ q_j \geq 0 & j = 1, 2, 3. \end{cases}$$

Having set up each player's linear programs we solve using the maple commands:

For player I we use

```
>with(simplex):
>cnsts:={2*x+6*y+4*z >=1,5*x+y+6*z >=1,4*x+3*y+z >=1};
>obj:=x+y+z;
>mimimize(obj,cnsts,NONNEGATIVE);
```

The minimize command incorporates the constraint that the variables be nonnegative by the use of the third argument. Maple gives the following solution to this program:

$$x = p_1 = 21/124, \; y = p_2 = 13/124, \; , z = p_3 = 1/124 \text{ and } \quad p_1+p_2+p_3 = 35/124.$$

Unwinding this to the original game, we have $35/124 = 1/v \implies v(A') = 124/35$. So, the optimal mixed strategy for player I is, using $x_i = p_i v$, $X^* = (21/35, 13/35, 1/35)$ and the value of our original game is $v(A) = 124/35 - 4 = -16/35$.

For player II we use the commands

```
>with(simplex):
>cnsts:={2*x+5*y+4*z <=1,6*x+y+3*z <=1,4*x+6*y+z <=1};
>obj:=x+y+z;
>maximize(obj,cnsts,NONNEGATIVE);
```

Maple gives the solution $x = q_1 = 13/124, y = q_2 = 10/124, z = y_q = 12/124$ so again $q_1 + q_2 + q_3 = 1/v = 35/124$, or $v(A') = 124/35$. Hence the optimal strategy for II is

$$Y^* = \frac{124}{35}(\frac{13}{124}, \frac{10}{124}, \frac{12}{124}) = (13/35, 10/35, 12/35).$$

The value of the original game is then $124/35 - 4 = -16/35$.

1.2.1 A direct formulation without transforming

It is no longer necessary to make the transformations we made in order to turn a game into a linear programming problem. In this section we give a simpler and more direct way. We will start from the beginning and recapitulate the problem.

Recall, that Player I wants to choose a mixed strategy $X^* = (x_i{}^*)$ so as to

$$\text{Maximize } v$$

subject to the constraints

$$\sum_i a_{ij} x_i^* = X^* A_j = E(X^*, j) \geq v, \ j = 1, \ldots, m,$$

and

$$\sum_{i=1}^{n} x_i^* = 1, \ x_i \geq 0, \ i = 1, \ldots, n.$$

This is a direct translation of the properties 1.0.2 which says that X^* is optimal and v is the value of the game if and only if, when X^* is played against any column for II, the expected payoff must be at least v. Thus, if we can find a solution of the problem subject to the constraints, it must give the optimal strategy for I as well as the value of the game. Similarly, Player II wants to choose a strategy $Y^* = (y_j{}^*)$ so as to

$$\text{Minimize } v$$

subject to the constraints

$$\sum_j a_{ij} y_j^* =_i A Y^{*T} = E(i, Y^*) \leq v, \ i = 1, \ldots, n,$$

and

$$\sum_{j=1}^{m} y_j^* = 1, \ y_j \geq 0, \ i = 1, \ldots, n.$$

The solution of this dual linear programming problem will give the optimal strategy for II and the same value as that for player I. We can solve these programs directly without changing to new variables. Since we don't have to divide by v in the conversion, we don't need to ensure that $v > 0$ so we can avoid having to add a constant to A. Let's work an example and give the Maple commands to solve the game.

Example 1.12. In this example we want to solve by the linear programming method the skew symmetric matrix

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

Here is the set up for solving this using Maple. We enter the matrix A. For the row player we define his strategy as $X = Vector(3, symbol = x)$ which defines a vector of size 3 and uses the symbol x for the components. We could do this with Y as well but a direct way is to use $Y =< y[1], y[2], y[3] >$. Anything to the right of the symbol # is a comment.

```
>with(LinearAlgebra):with(simplex):
>#Enter the matrix of the game here,row by row:
>A:=Matrix([[0,-1,1],[1,0,-1],[-1,1,0]]);
>#The row players Linear Programming problem:
> X:=Vector(3,symbol= x);
#Defines X as a column vector with 3 components
> B:=Transpose(X).A;
# Used to calculate the constraints, B is a vector.
>cnst:={seq(B[i] >=v,i=1..3),add(x[i],i=1..3)=1};
#The components of B must be >=v and the components of X must sum to 1.
>maximize(v,cnst,NONNEGATIVE,value);
#Player I wants v as large as possible

#Hitting enter will give X=(1/3,1/3,1/3) and v=0.


>#Column Players LP problem:
>Y:=<y[1],y[2],y[3]>;#Another way to set up the vector for Y.
>B:=A.Y;
> cnst:={seq(B[j]<=v,j=1..3),add(y[j],j=1..3)=1};
>minimize(v,cnst,NONNEGATIVE,value);

#Again, hitting enter gives Y=(1/3,1/3,1/3) and v=0.
```

Since A is skew symmetric we know ahead of time that the value of this game is 0. Maple gives the optimal strategies

$$X^* = (1/3, 1/3, 1/3) = Y^*$$

which checks with the fact that for a symmetric matrix the strategies are the same for both players.

Now let's look at a much more interesting example.

Example 1.13. **Colonel Blotto Games** This is a simplified form of a military game in which the leaders must decide how many regiments to send to attack or defend two or more targets. It is an optimal allocation of forces game. In one formulation, suppose there are 2 opponents (players) we call Red and Blue. Blue controls 4 regiments and Red controls 3. There are 2 targets of interest, say A and B. The rules of the game say that the player who sends the most regiments to a target will win 1 point for the win and 1 point for every regiment captured at that target. A tie, in which Red and Blue send the same number of regiments to a target, gives a zero payoff. The possible strategies for each player consist of the number of regiments to send to A and the number of regiments to send to B, and so they are a pair of numbers. The payoff matrix to Blue is

A=

Blue/Red	(3,0)	(0,3)	(2,1)	(1,2)
(4,0)	4	0	2	1
(0,4)	0	4	1	2
(3,1)	1	-1	3	0
(1,3)	-1	1	0	3
(2,2)	-2	-2	2	2

For example, if Blue plays $(3,1)$ against Red's play of $(2,1)$ then Blue sends 3 regiments to A while Red sends 2. So Blue will win A, which is $+1$ and then capture the 2 Red regiments for a payoff of $+3$ for A. But Blue sends 1 regiment to B and Red also sends one to B, so that is considered a tie, or standoff with a payoff to Blue of 0. So the net payoff to Blue is $+3$.

This game can be easily solved using Maple as a linear program, but here we will utilize the fact that the strategies have a form of symmetry which will simplify the calculations. It seems clear that the Blue strategies $(4,0)$ and $(0,4)$ should be played with the same probability. The same should be true for $(3,1)$ and $(1,3)$. Hence, we may consider that we really only have 3 numbers to determine for Blue: $X = (x_1, x_1, x_2, x_2, x_3),\ 2x_1 + 2x_2 + x_3 = 1$. Similarly, for Red we need to determine only 2 numbers:$Y = (y_1, y_1, y_2, y_2),\ 2y_1 + 2y_2 = 1$.

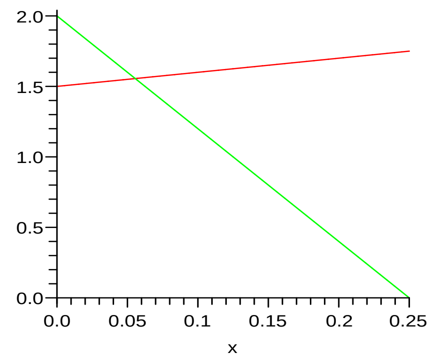
Red wants to choose $y_1 \geq 0, y_2 \geq 0$ to make $value(A) = v$ as small as possible but subject to $E(i, Y) \leq v, i = 1, 2, 3$. This says

$$4y_1 + 3y_2 \leq v,\ 3y_2 \leq v,\ \text{and}\ -4y_1 + 4y_2 \leq v.$$

Now, $4y_1 + 3y_2 \leq v$ implies $3y_2 \leq v$ so we can drop the second inequality. Substitute $y_2 = (1/2 - y_1)$ to eliminate y_2 and get

$$y_1 + 3/2 \leq v, -8y_1 + 2 \leq v.$$

The two lines $y = x + 3/2$, and $y = -8x + 2$ intersect at $x = 1/18, y = 8/18$.



and since the inequalities require that v is above the 2 lines, the smallest v is at the point of intersection. Thus, $y_1 = 1/18, y_2 = 8/18, v = 28/18$. So $Y^* = (1/18, 1/18, 8/18, 8/18)$.

Now, the middle inequality $3y_2 = 24/18 < 28/18$ and, because it is a strict inequality, a previous lemma requires that any optimal strategy for Blue would have 0 probability of using the associated row, that is, $x_2 = 0$. With that simplification we obtain the inequalities for Blue as

$$4x_1 - 2x_3 \geq v = 28/18, \; 3x_1 + 2x_3 \geq v, \; \text{and } 2x_1 + x_3 = 1.$$

The solution of these inequalities yields $X^* = (4/9, 4/9, 0, 0, 1/9)$.

Naturally, Blue, having more regiments, will come out ahead, and a rational opponent (Red) would capitulate before the game even began. Observe also, that with 2 equally valued targets, it is optimal for the superior force (Blue) to not divide it's regiments, but for the inferior force to split it's regiments, except for a small probability of doing the opposite.

Finally, if we use Maple to solve this problem we use the commands

```
>with(LinearAlgebra):
>A:=Matrix([[4,0,2,1,[0,4,1,2],[1,-1,3,0],[-1,1,0,3],[-2,-2,2,2]]);
>X:=Vector(5,symbol=x);
>B:=Transpose(X).A;
>cnst:={seq(B[i]>=v,i=1..4),add(x[i],i=1..5)=1);
>with(simplex):
>maximize(v,cnst,NONNEGATIVE,value);
```

The outcome of these commands is $X^* = (4/9, 4/9, 0, 0, 1/9)$ and $value(A) = 14/9$. Similarly, using the commands

```
>with(LinearAlgebra):
>A:=Matrix([[4,0,2,1,[0,4,1,2],[1,-1,3,0],[-1,1,0,3],[-2,-2,2,2]]);
>Y:=Vector(4,symbol=y);
>B:=A.Y;
>cnst:={seq(B[i]<=v,i=1..5),add(y[i],i=1..4)=1);
>with(simplex):
>minimize(v,cnst,NONNEGATIVE,value);
```

results in $Y^* = (7/90, 3/90, 32/90, 48/90)$ and again $v(A) = 14/9$. The optimal strategy for Red is not unique and he may play one of many optimal strategies but all resulting in the same expected outcome.

2

2 person Nonzero sum games

The previous sections considered 2 person games in which whatever one player gains, the other loses. This is far too restrictive for many games, especially games in economics or politics where both players can win something. In other words the game does not have to be a constant sum game. The generalization to zero sum games is considered in the next few sections, namely, we will drop the zero sum condition and each player may gain or lose depending on their choice of strategy. We will start with 2 person games where each player will have his own payoff matrix.

2.1 2×2 bimatrix games

In a 2 person nonzero sum game we simply assume that each player has their own payoff matrix. Suppose the payoff matrices are

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

For example, if player I plays row 1 and player II plays column 2, then the payoff to player I is a_{12} and the payoff to player II is b_{12}. In a zero sum game we always had $a_{ij} + b_{ij} = 0$, or more generally $a_{ij} + b_{ij} = k$, where k is a fixed constant. In a nonzero sum game we do not assume that.

Let $X = (x, 1-x), Y = (y, 1-y), 0 \leq x \leq 1, 0 \leq y \leq 1$ be mixed strategies of I and II, respectively. Denote by

$$E_1(X,Y) = XAY^T = xy(a_{11} - a_{12} - a_{21} + a_{22}) + (a_{12} - a_{22})x + (a_{21} - a_{22})y + a_{22},$$

$$E_2(X,Y) = XBY^T = xy(b_{11} - b_{12} - b_{21} + b_{22}) + (b_{12} - b_{22})x + (b_{21} - b_{22})y + b_{22},$$

the expected payoffs to I and II. It is the goal of each player to maximize his expected payoff *assuming the other players are doing their best to maximize their own payoffs.*

Definition 2.1.1 *A pair of mixed strategies* (X^*, Y^*) *is a Nash equilibrium if* $E_1(X, Y^*) \le E_1(X^*, Y^*)$ *for every mixed* X *and* $E_2(X^*, Y) \le E_2(X^*, Y^*)$ *for every mixed* Y.

Remark 2.1.2 *1. This says that neither player can gain any expected payoff if they choose to deviate from the choice of the equilibrium.*

2. It is not hard to see that (X^*, Y^*) *is a Nash equilibrium if*

$$E_1(i, Y^*) =_i AY^{*T} \le X^* A Y^{*T}, \quad i = 1, \ldots, m,$$
$$E_2(X^*, j) = X^* B_j \le X^* B Y^{*T}, \quad j = 1, \ldots, n.$$

3. If $B = -A$ *a bimatrix game is a zero sum 2 person game and a Nash equilibrium is the same as a saddle point.*

The questions we ask are

1. Is there a Nash equilibrium using pure strategies?
2. Is there a Nash Equilibrium using Mixed strategies?
3. How doe we compute these?

To start we consider the classic example.

Example 2.1. Prisoner's dilemma. Two criminals have just been caught after committing a crime. The police interrogate the prisoners by placing them in separate rooms so they cannot coordinate their stories and no communication between the prisoners is possible. The goal of the police is to try to get one or both of them to confess the crime. We consider the two prisoners as the players in a game in which they have two pure strategies: Confess, or Don't Confess. Their prison sentence, if any will depend on whether or not they confess and agree to testify against the other prisoner. But if they both confess, no benefit will be gained by testimony which is no longer needed. If neither confesses, there may not be enough evidence to convict either prisoner of the crime. The following matrices represent the possible payoffs remembering that they are set up to maximize the payoff.

Prisoner I/II	Confess	Don't Confess
Confess	(-5,-5)	(0,-20)
Don't Confess	(-20,0)	(-1,-1)

The individual matrices for the two prisoners are

$$A = \begin{pmatrix} -5 & 0 \\ -20 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} -5 & -20 \\ 0 & -1 \end{pmatrix}$$

The numbers are negative because they represent the number of years of the sentence and each player wants to maximize the payoff, i.e., minimize their sentence.

To see if there is a Nash equilibrium in pure strategies we are looking for a payoff pair (a, b) which simultaneously has a as the largest number in a column and b the largest number in a row. Looking at the bimatrix is the easiest way to find them. The systematic way is to put a bar over the first number which is the largest in each row and put a bar over the second number which is the largest in each column. Any pair of numbers which both have bars is a Nash equilibrium. In the prisoner's dilemma problem we have

$$\begin{pmatrix} (-\overline{5}, -\overline{5}) & (\overline{0}, -20) \\ (-20, \overline{0}) & (-1, -1) \end{pmatrix}$$

We see that there is a pure Nash and it is at (Confess, Confess), they should both confess and settle for 5 years in prison each. If either player deviates from Confess, while the other player still plays Confess, then the payoff to the player who deviates goes from -5 to -20. But clearly both players can do better if the both choose Don't Confess because then there is not enough evidence to put them in jail for more than 1 year. But there is an incentive for each player to NOT play Don't Confess. If One player chooses Don't Confess and the other chooses Confess, the payoff to the confessing player is 0, i.e., he won't go to jail at all! The players are rewarded for a betrayal of the other prisoner and so that is exactly what will happen. The payoff pair $(-1, -1)$ is unstable in the sense that a player can do better by deviating, assuming the other player does not, and $(-5, -5)$ is stable because neither player can improve their payoff. Even if they agree before they are caught to not confess, it would take extraordinary will power for both players to stick with that agreement in the face of the numbers.

Example 2.2. The Arms race. Suppose two countries have the choice of developing nuclear weapons or not. There is a cost of the development as well as a benefit. Suppose we quantify these using a bimatrix in which each player wants to maximize the payoff.

Country A \ B	Nuclear	Status Quo
Nuclear	(1,1)	(10,5)
Status Quo	(-5,10)	(1,1)

We see that we have a Nash equilibrium at the pair $(1, 1)$ corresponding to (Nuclear, Nuclear). The pair $(1, 1)$ when both countries maintain the Status Quo is NOT a Nash equilibrium because each player can improve their payoff by deviating from this. Observe too that if one country decides to go Nuclear, the other country clearly has no choice but to do likewise. The only way that the situation could change would be to make the benefits of going Nuclear much less, perhaps by third party sanctions.

Now we look for ways to find all Nash equilibria for a bimatrix game.

Theorem 2.1.3 *A necessary and sufficient condition for (x^*, y^*) associated with $X^* = (x^*, 1 - x^*), Y^* = (y^*, 1 - y^*)$ to be an equilibrium point of the game is*

$$\begin{aligned} &1)\ E_1(1, y^*) \le E_1(x^*, y^*)\\ &2)\ E_1(0, y^*) \le E_1(x^*, y^*)\\ &3)\ E_2(x^*, 1) \le E_2(x^*, y^*)\\ &4)\ E_2(x^*, 0) \le E_2(x^*, y^*)\end{aligned}$$

Proof. To see why this is true we first note that if X^*, Y^* is an equilibrium then the inequalities must hold by definition. So we need only show that they are sufficient. Let $X = (x, 1 - x)$ and $Y = (y, 1 - y)$ be any mixed strategies. By the inequalities,

$$xE_1(1, y^*) = x(1\ 0)AY^{*T} = x(a_{11}y^* + a_{12}(1 - y^*)) \le xE_1(x^*, y^*)$$

and

$$(1-x)E_1(0, y^*) = (1-x)(0\ 1)AY^{*T} = (1-x)(a_{21}y^* + a_{22}(1-y^*)) \le (1-x)E_1(x^*, y^*).$$

Add these up to get

$$\begin{aligned} xE(1, y^*)+(1-x)E(0, y^*) = x(a_{11}y^*+a_{12}(1-y^*)+(1-x)(a_{21}y^*+a_{22}(1-y^*))\\ \le E_1(x^*, y^*)\end{aligned}$$

But then, since

$$x(a_{11}y^* + a_{12}(1 - y^*)) + (1 - x)(a_{21}y^* + a_{22}(1 - y^*)) = XAY^{*T}$$

we see that

$$(x, (1 - x))AY^{*T} = XAY^{*T} = E_1(X^*, Y^*) \le E_1(X^*, Y^*).$$

This gives the first part of the definition that X^*, Y^* is a Nash equilibrium. The rest of the proof is similar. $\square$

We want to solve the inequalities (1)-(4).

Now $E_1(x, y) = XAY^T$ so the inequalities (1) (2) reduce to

$$Q(1-x)y-q(1-x) \le 0, \qquad Qxy-qx \ge 0, Q = a_{11}-a_{12}-a_{21}+a_{22}, q = a_{22}-a_{12}. \tag{2.1.1}$$

We consider cases:

Case 1: $Q = q = 0$.

Then any $x \in [0, 1], y \in [0, 1]$ solves (2.1.1).

Case 2: $Q = 0, q > 0$.

Then $-q(1 - x) \le 0,$ and $-qx \ge 0$, and so solutions are $x = 0, 0 \le y \le 1$.

Case 3: $Q = 0, q < 0$
Then $(1-x) \le 0,$ and $x \ge 0$. Solutions are $x = 1, 0 \le y \le 1$.
Case 4: $Q > 0$.
Then $Q(1-x)y - q(1-x) \le 0, \qquad Qxy - qx \ge 0$, and solutions are

$$\begin{aligned} x = 0 &\quad y \le q/Q = \alpha \\ 0 < x < 1 &\; y = q/Q = \alpha \\ x = 1 &\quad y \ge q/Q = \alpha. \end{aligned}$$

Case 5: $Q < 0$.

$$\begin{aligned} x = 0 &\quad y \ge q/Q = \alpha \\ 0 < x < 1 &\; y = q/Q = \alpha \\ x = 1 &\quad y \le q/Q = \alpha. \end{aligned}$$

Similarly for the other player let

$$R = b_{11} - b_{12} - b_{21} + b_{22}, \quad r = b_{22} - b_{21} \;\; \beta = \frac{r}{R}$$

Then the inequalities become

$$Rx(1-y) - r(1-y) \le 0, \quad Rxy - ry \ge 0.$$

Case 1: $R = 0, r = 0$. Solutions are all $0 \le x \le 1, 0 \le y \le 1$.
Case 2: $R = 0, r > 0$. Solutions are $0 \le x \le 1, y = 0$.
Case 3: $R = 0, r < 0$. Solutions are $0 \le x \le 1, y = 1$.
Case 4: $R > 0$. The solutions are

$$\begin{aligned} y = 0 &\quad x \le r/R = \beta \\ 0 < y < 1 &\; x = r/R \\ y = 1 &\quad x \ge r/R. \end{aligned}$$

Case 5: $R < 0$.

$$\begin{aligned} y = 0 &\quad x \le \beta \\ 0 < y < 1 &\; x = \beta \\ y = 1 &\quad x \ge \beta. \end{aligned}$$

Example 2.3.

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

Then $Q = 5, q = 2, \alpha = 2/5, R = 5, r = 3, \beta = 3/5$ so that we have 3 equilibria $(x, y) = (0,0), (3/5, 2/5), (1,1)$, from which we get all the Nash points (X^*, Y^*).

2.2 Interior Mixed Nash Points by Calculus

When we look for a mixed Nash point in which the rows and columns are completely mixed, we can use calculus to find the maximum of E_1, E_2. The calculus procedure for finding an interior mixed Nash equilibrium is.

1. The expected payoff to I is $E_I(X,Y) = XAY^T$, and the expected payoff to II is $E_{II}(X,Y) = XBY^T$.
2. Let $x_m = 1-(x_1+\cdots+x_{m-1})$, $y_n = 1-\sum_{j=1}^{n-1} y_j$ so each expected payoff is a function only of $x_1,\ldots,x_{m-1},y_1,\ldots,y_{n-1}$.
3. Solve the equations $\partial E_I/\partial x_i = 0,\ \partial E_{II}/\partial y_j = 0,\ i = 1,\ldots,m-1,\ j = 1,\ldots,n-1$.
4. If there is a solution with $x_i \geq 0, y_j \geq 0$ this is the mixed strategy equilibrium.

Definition 2.2.1 *The rational reaction sets for each player are defined as follows.*

$$R_I = \{(X,Y)|E_I(X,Y) = \max_p E_I(p,Y)\},$$
$$R_{II} = \{(X,Y)|E_{II}(X,Y) = \max_t E_I(X,t)\}.$$

The set of Nash equilibria is $R_I \cap R_{II}$.

In the bimatrix game case, if X^*, Y^* is an interior Nash, then

$$\sum_{j=1}^{n} y_j^*(a_{i,j} - a_{1,j}) = 0,\ \forall\, i \neq 1$$
$$\sum_{i=1}^{m} x_i^*(b_{i,j} - b_{1,j}) = 0,\ \forall\, j \neq 1$$

Example 2.4. Two partners have 2 choices for where to invest their money, say O_1, O_2 but they need to agree on joint action. We model this using the bimatrix

	O_1	O_2
O_1	(1,2)	(0,0)
O_2	(0,0)	(2,1)

If Player I chooses O_1 and II chooses O_1 the payoff to I is 1 and the payoff to II is 2 units because II prefers to invest the money into O_1 more than into O_2. If the players do not agree on how to invest then they receive 0.

To solve this game, first notice that there are 2 pure Nash points at (O_1, O_1) and (O_2, O_2), so total cooperation will be equilibrium. We want to know if there are any mixed Nash points.

Set

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $E_1(x,y) = (x\ 2(1-x)\cdot(y,1-y)^T = xy+2(1-x)(1-y) = 3xy-2x-2y+2$. Player I wants to make this as large as possible for any fixed $0 \le y \le 1$. Write

$$E_1(x,y) = x(3y-2) - 2y + 2.$$

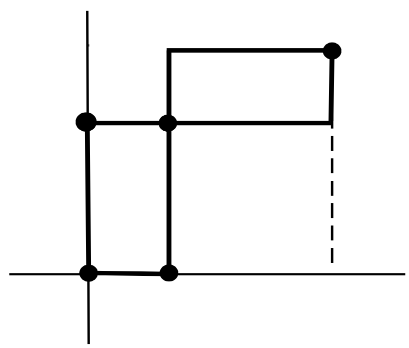
If $3y-2 \ge 0$, $E_1(x,y)$ is maximized as a function of x at $x = 1$. If $3y-2 \le 0$ the maximum will occur at $x = 0$. If $3y-2 = 0$ then $y = 2/3$ and $E_1(x, 2/3) = 2/3$. Next, we consider $E_2(x,y)$ in a similar way. Write

$$E_2(x,y) = (2x\ 1-x)\cdot(y\ 1-y)^T = 2xy+(1-x)(1-y) = 3xy-x-y+1 = y(3x-1)-x+1.$$

Player II wants to choose $y \in [0,1]$ to maximize this and that will depend on the coefficient of y namely, $3x-1$. We see as before that

$$\max_{y\in[0,1]} E_2(x,y) = \begin{cases} -x+1, & \text{if } 0 \le x < 1/3 \text{ achieved at } y = 0; \\ 2x, & \text{if } 1/3 < x \le 1 \text{ achieved at } y = 1; \\ 2/3, & \text{if } x = 1/3 \text{ achieved at any } y \in [0,1]. \end{cases}$$

Here is the graph:



The solid lines form the rational reaction sets of the players. For example, if player I decides for some reason that he will play O_1 with probability $1/2$ then player II would rationally play either $y = 1$ or $y = 2/3$. Where the zigzag lines cross are the Nash points. That is, the Nash points are at $(x,y) = (0,0), (1,1)$ and $(1/3, 2/3)$. So either they should always cooperate to the advantage of one player or the other or player I should play O_1 $1/3$ of the time and player II should play O_1 $2/3$ of the time. The associated expected payoffs are:

$$E_1(0,0) = 2, E_2(0,0) = 1,$$

$$E_1(1,1) = 1, E_2(1,1) = 2,$$

and

$$E_1(1/3, 2/3) = 2/3 = E_2(1/3, 2/3).$$

Only the mixed strategy Nash point $(X^*, Y^*) = ((1/3, 2/3), (2/3, 1/3))$ gives the same expected payoffs to the two players. This seems to be a problem. Only the mixed strategy gives the same payoff, but it will result in *less* for each player than they could get if they play the pure Nash points. So, permitting the other player the advantage results in a *bigger* payoff to both players! If one player decides to cave, they both can do better, but if both players insist that the outcome be *fair* then they both do worse.

3

Quadratic Programming Method for Nonzero Sum 2 person Games

In this section we present a method of finding all Nash equilibria for arbitrary 2 person nonzero sum games with any number of strategies. It was introduced by Lemke and Towson and is practically implemented with Maple, but any nonlinear programming package can solve these problems. Because it can solve any problem it can also be used to find all mixed saddle points for 2 person zero sum games because in that case we take $B = -A$.

Theorem 3.0.2 *X^*, Y^* is a Nash equilibrium iff they satisfy, along with scalars p^*, q^* the program*

$$\begin{aligned} &\max_{X,Y,p,q} XAY^T + XBY^T - p - q \\ &\qquad \textit{subject to} \\ &\qquad AY^T \leq pJ_m \\ &\qquad B^T X^T \leq qJ_n \qquad\qquad (\textit{ equivalently } XB \leq qJ_n^T) \\ &x_i \geq 0, y_j \geq 0, XJ_m = 1 = YJ_n \end{aligned}$$

where J_k is the $k \times 1$ column vector of all 1's.

Here is how the proof goes.

First we show that a strategy pair (X^*, Y^*) is a Nash equilibrium has two equivalent formulations:

$$(*) \qquad X^*AY^{*T} \geq XAY^{*T}, \forall X, \;\; X^*BY^{*T} \geq X^*BY^T, \; \forall Y.$$

and

$$(**) \qquad AY^{*T} \leq (X^*AY^{*T})J_m, \;\; X^*B \leq (X^*BY^{*T})J_n^T.$$

Now suppose that (X^*, Y^*) is a Nash point. Since every possible solution to the NLP problem must satisfy the constraints $AY^T \leq pJ_m$ and $XB \leq qJ_n^T$, multiply the first on the left by X and the second on the right by Y^T to get

$$XAY^T \leq pXJ_m = p, \qquad XBY^T \leq qJ_n^T Y^T = q.$$

Hence, any possible solution gives the objective

$$XAY^T + XBY^T - p - q \leq 0.$$

But it is easy to show that with $p^* = X^*AY^{*T}, q^* = X^*BY^{*T}$, we get that X^*, Y^*, p^*, q^* is a feasible solution of the NLP but with objective function zero. Hence this gives the maximum objective and so is a solution of the NLP.

For the opposite direction, let X_1, Y_1, p_1, q_1 be a solution of the NLP and let $X^*, Y^*, p^* = X^*AY^{*T}, q^* = X^*BY^{*T}$ be a Nash point for the game. We will show that X_1, Y_1 is another Nash point. Since X_1, Y_1 satisfy the constraints of NLP we get by multiplying the constraints appropriately

$$X_1AY_1^T \leq p_1, \qquad X_1BY_1^T \leq q_1.$$

We know, since we have a maximum objective that it is zero. This says

$$(X_1AY_1^T - p_1) + (X_1BY_1^T - q_1) = 0.$$

The terms in parentheses are nonpositive and add to zero. That means they must each be zero. Hence

$$X_1AY_1^T = p_1, \qquad X_1BY_1^T = q_1.$$

Then we write the constraints as

$$AY_1^T \leq (X_1AY_1^T)J_m, \qquad X_1B \leq (X_1BY_1^T)J_n^T.$$

This is (**) so X_1, Y_1 is a Nash. □

Using this theorem and some nonlinear programming we can numerically solve any 2 person nonzero sum game. Here are the Maple commands:

```
> with(LinearAlgebra):
>A:=Transpose(<<-1,0,0>|<2,1,0>|<0,1,1>>);
>B:=Transpose(<<1,2,2>|<1,-1,0>|<0,1,2>>);
> X:=<x[1],x[2],x[3]>;
> Y:=<y[1],y[2],y[3]>;
>Cnst:={seq((A.Y)[i]<=p,i=1..3),seq((Transpose(X).B)[i]<=q,i=1..3),
          add(x[i],i=1..3)=1,add(y[i],i=1..3)=1};
> with(Optimization);
> objective:=expand(Transpose(X).A.Y+Transpose(X).B.Y-p-q);
> expand(Transpose(X).A.Y);expand(Transpose(X).B.Y);
>QPSolve(objective,Cnst,assume=nonnegative,
                   iterationlimit=10000,maximize);
>QPSolve(objective,Cnst,assume=nonnegative,iterationlimit 10000,
                   maximize,initialpoint=({q=1,p=2}));
> NLPSolve(objective,Cnst,assume=nonnegative,maximize);
```

This gives us the result $p = 0.66, q = 0.66, x[1] = 0, x[2] = 0.66, x[3] = 0.33$ and $y[1] = 0.33, y[2] = 0, y[3] = 0.66$. By changing the initial point we also find the solution

$$X = <0,1,0> \text{ and } Y = <1,0,0>.$$

The commands indicate there are two ways Maple can solve this problem. First, recognizing the payoff objective function as a quadratic function (actually even simpler than that) we can use the command $QPSolve$ which specializes to quadratic programming problems. Second, in general for any nonlinear objective function we use $NLPSolve$.

3.1 Other forms of n person nonzero sum games

If there are n players in a game we assume that each player has its own payoff function depending on his choice of strategy and the choices of the other players. Suppose the strategies must take values in sets $Q_i, i = 1, \ldots, n$ and the payoffs are

$$u_i : Q_1 \times \cdots \times Q_n \to \mathbb{R}.$$

Here is a formal definition of a Nash point, keeping in mind that each player wants to maximize their own payoff.

Definition 3.1.1 *A collection of strategies $q^* = (q_1^*, \ldots, q_n^*)$ is a Nash equilibrium for the game with payoff functions $\{u_i(q)\}, i = 1, \ldots, n$, if for each player $i = 1, \ldots, n$, we have*

$$u_i(q_1^*, \ldots, q_{i-1}^*, q_i^*, q_{i+1}^*, \ldots, q_n^*) \geq u_i(q_1^*, \ldots, q_{i-1}^*, q_i, q_{i+1}^*, \ldots, q_n^*), \text{ for all } q_i \in Q_i.$$

The point is that no player can do better if he deviates from a Nash point.

Remark 3.1.2 *Notice that if there are 2 players and $u_1 = -u_2$ then a point (q_1^*, q_2^*) is a Nash point if*

$$u_1(q_1^*, q_2^*) \geq u_1(q_1, q_2^*), \text{ and } u_2(q_1^*, q_2^*) \geq u_2(q_1^*, q_2), \forall (q_1, q_2).$$

But then

$$-u_1(q_1^*, q_2^*) \geq -u_1(q_1^*, q_2) \Longrightarrow u_1(q_1^*, q_2^*) \leq u_1(q_1^*, q_2)$$

and putting these together says that

$$u_1(q_1, q_2^*) \leq u_1(q_1^*, q_2^*) \leq u_1(q_1^*, q_2), \forall (q_1, q_2).$$

This, of course says that (q_1^*, q_2^*) is a saddle point of the 2 person zero sum game.

In many cases, the problem of finding a Nash point can be reduced to a simple calculus problem. To do this we need to have the strategy sets Q_i to be open intervals and the payoff functions to have at least two continuous derivatives because we are going to apply the second derivative test. The steps involved are

1. Solve $\frac{\partial u_i(q_1^*,\ldots,q_n^*)}{\partial q_i^*} = 0, i = 1, 2, \ldots, n.$
2. Verify $\frac{\partial^2 u_i(q_1^*,\ldots,q_n^*)}{\partial^2 q_i^*} < 0, i = 1, 2, \ldots, n.$

Example 3.1. **Cournot duopoly** Suppose there are 2 companies producing the same gadget. Firm $i = 1, 2$ chooses to produce the quantity $q_i \geq 0$, so the total quantity produced by the 2 companies is $q = q_1 + q_2$.

We assume in this crude model that the price of a gadget is a decreasing function of the total quantity produced by the 2 firms. Let's take it to be $P(q) = \mathcal{P} - q$ when $\mathcal{P} \geq q \geq 0$ and $P(q) = 0$ when $q > \mathcal{P}$. Suppose also that it costs firm $i = 1, 2$ c_i dollars per unit gadget so the total cost to produce q_i units is $c_i q_i, i = 1, 2$. Assume that

$$\mathcal{P} > c_1 + c_2$$

because otherwise it costs more to produce the gadgets than to not produce them.

Each firm wants to maximize their profit function which, in this case is given by

$$u_1(q_1, q_2) = (\mathcal{P} - q_1 - q_2)q_1 - c_1 q_1 \text{ and } u_2(q_1, q_2) = (\mathcal{P} - q_1 - q_2)q_2 - c_2 q_2.$$

So, let's begin by taking the partials and setting to zero.

$$\frac{\partial u_1(q_1, q_2)}{\partial q_1} = 0 \Longrightarrow -2q_1 - q_2 + \mathcal{P} - c_1 = 0$$
$$\frac{\partial u_2(q_1, q_2)}{\partial q_2} = 0 \Longrightarrow -2q_2 - q_1 + \mathcal{P} - c_2 = 0$$

Notice that we take the partial of u_i with respect to q_i not the partial of each payoff function with respect to both variables. Now solving the resulting equations gives

$$q_1^* = \frac{\mathcal{P} + c_2 - 2c_1}{3} \text{ and } q_2^* = \frac{\mathcal{P} + c_1 - 2c_2}{3}.$$

At these points we have

$$\frac{\partial^2 u_1(q_1, q_2)}{\partial^2 q_1} = -2 < 0 \text{ and } \frac{\partial^2 u_2(q_1, q_2)}{\partial q_2^2} = -2 < 0.$$

The total amount the two firms should produce is

$$q^* = q_1^* + q_2^* = \frac{\mathcal{P} - c_1 - c_2}{3}$$

and $q^* > 0$ if $\mathcal{P} > c_1 + c_2$. The price function is then

$$P(q_1^* + q_2^*) = \mathcal{P} - q_1^* - q_2^* = \mathcal{P} - \frac{\mathcal{P} - c_1 - c_2}{3} = \frac{\mathcal{P} + c_1 + c_2}{3}.$$

Turn it around and suppose that the price is set at $P(q_1 + q_2) = p = \frac{\mathcal{P}+c_1+c_2}{3}$. Then the total quantity the 2 firms should produce at this price is

$$q = \mathcal{P} - p = \frac{2\mathcal{P} - c_1 - c_2}{3} = q_1^* + q_2^*.$$

So, the quantity of gadgets demanded will be exactly the amount each should produce at this price. This is called a market equilibrium and it turns out to be given by the Nash point quantity to produce.

Finally, substituting the Nash point into the profit functions gives the profit functions

$$u_1(q_1^*, q_2^*) = \frac{(\mathcal{P} + c_2 - 2c_1)^2}{9} \text{ and } u_2(q_1^*, q_2^*) = \frac{(\mathcal{P} + c_1 - 2c_2)^2}{9}.$$

Notice that the profit of each firm depends on the costs of the other firm.

Example 3.2. The Bertrand Model. We again have two companies making gadgets but the capacity of the 2 companies is fixed and they can only adjust prices, not quantities. So the quantity sold is a function of the price set, say $q = \mathcal{P} - p$. In a classic problem the model says that if the 2 firms charge the same price, they will split the market evenly, i.e., they sell exactly half of the total sold. But the company that charges a lower price will capture the entire market. We have to assume that each company has enough capacity to make the entire quantity if they capture the whole market. The cost to make gadgets is still $c_i, i = 1, 2$ dollars per unit. We first assume:

$$c_1 \neq c_2 \text{ and } \max\{c_1, c_2\} < \mathcal{P} + \min\{c_1, c_2\}$$

The profit function for firm $i = 1, 2$ assuming that firm 1 sets the price as p_1 and firm 2 sets the price at p_2 is

$$u_1(p_1, p_2) = \begin{cases} p_1(\mathcal{P} - p_1) - c_1(\mathcal{P} - p_1), & \text{if } p_1 < p_2; \\ \frac{(p - c_1)(\mathcal{P} - p)}{2}, & \text{if } p_1 = p_2 = p \geq c_1; \\ 0, & \text{if } p_1 > p_2. \end{cases}$$

This says that if firm 1 sets the price lower than firm 2, firm 1's profit will be $(price - cost) \times quantity$,; if the prices are the same firm 1's profits will be $(price - cost) \times quantity/2$,; and zero if firm 1's price is greater than firm 2's. Likewise, firm 2's profit function is

$$u_2(p_1, p_2) = \begin{cases} p_2(\mathcal{P} - p_2) - c_2(\mathcal{P} - p_2), & \text{if } p_2 < p_1; \\ \frac{(p-c_2)(\mathcal{P}-p)}{2}, & \text{if } p_1 = p_2 = p \geq c_2; \\ 0, & \text{if } p_2 > p_1. \end{cases}$$

This problem does NOT have a Nash equilibrium To see why let's suppose that there is a Nash point at (p_1^*, p_2^*). By definition,

$$u_1(p_1^*, p_2^*) \geq u_1(p_1, p_2^*) \text{ and } u_2(p_1^*, p_2^*) \geq u_2(p_1^*, p_2), \forall (p_1, p_2).$$

Let's break this down by considering cases:

Case 1. $p_1^* > p_2^*$ Then it should be true that

$$u_1(p_1^*, p_2^*) = 0 \geq u_1(p_1, p_2^*), \forall p_1.$$

But if we take any $p_1 < p_2^*$ the right side will be positive and so this cannot hold.

Case 2. $p_1^* < p_2^*$. Then it should be true that

$$u_2(p_1^*, p_2^*) = 0 \geq u_2(p_1^*, p_2), \forall p_2.$$

But if we take any $p_2 < p_1^*$ the right side will be positive and again this cannot be true. So the only case left is...

Case 3. $p_1^* = p_2^*$. But then we must have

$$u_1(p_1^*, p_2^*) = (p_1^* - c_1)(\mathcal{P} - p_1^*)/2 \geq u_1(p_1, p_2^*), \forall p_1.$$

If we take $p_1 = p_1^* - \varepsilon < p_2^*$ with really small $\varepsilon > 0$ so that

$$u_1(p_1^*, p_2^*) = \frac{(p_1^* - c_1)(\mathcal{P} - p_1^*)}{2} < u_1(p_1^* - \varepsilon, p_2^*) = (p_1^* - \varepsilon - c_1)(\mathcal{P} - p_1^* + \varepsilon).$$

The reader should check how small ε should be to see that this is true.

So, in all cases, we can find prices so that the condition for a Nash point is violated and so there is no Nash equilibrium.

What's the problem here? It is that neither player has a continuous profit function. By lowering the price just below the competitor the player with the lower price can capture the entire market.

Example 3.3. **The Stackelberg Model** In this model we will assume that there is a dominant firm, say Player I, who will announce his production publicly. Then player II will decide how much to produce.

Suppose firm 1 announces that it will produce q_1 at cost c_1 dollars per unit. It is then up to firm 2 to decide how much q_2 at cost c_2 it will produce. The price per unit will be then $p(q_1, q_2) = (\mathcal{P} - q_1 - q_2)^+$. The profit functions will be

$$u_1(q_1, q_2) = (\mathcal{P} - q_1 - q_2)q_1 - c_1 q_1 \text{ and } u_2(q_1, q_2) = (\mathcal{P} - q_1 - q_2)q_2 - c_2 q_2.$$

These are the same as before but now q_1 is fixed as given. It is not variable when firm 1 announces it. So what we are really looking for is the best response of firm 2 to the announcement q_1, that is firm 2 wants to know how to choose $q_2 = q_2(q_1)$ so as to

$$\text{Maximize } u_2(q_1, q_2(q_1)).$$

This is given by

$$q_2(q_1) = \frac{\mathcal{P} - q_1 - c_2}{2}.$$

This is the best firm 2 can do no matter what firm 1 announces. But firm 1 will know that firm 2 will do that! So, firm 1 should choose q_1 to maximize its own profit function

$$u_1(q_1, q_2(q_1)) = q_1(\mathcal{P} - q_1 - q_2(q_1)) - c_1 q_1 = q_1(\mathcal{P} - q_1 - \frac{\mathcal{P} - q_1 - c_2}{2}) - c_1 q_1.$$

Firm 1 want to choose q_1 to make this as large as possible. By calculus we find that

$$q_1^* = \frac{\mathcal{P} - 2c_1 + c_2}{2}, \text{ and } u_1(q_1^*, q_2^*) = \left(\frac{\mathcal{P} - 2c_1 + c_2}{2}\right)^2$$

and $q_2^* = q_2(q_1^*) = \frac{\mathcal{P}+2c_1-3c_2}{4}$.

In the case $c_1 = c_2 = c$ the reader should check that firm 1 produces q_1^*, the monopoly quantity if there was no firm 2, firm 2 produces less than the amount it would produce without the information announcement from firm 1. In addition, firm 1's profits will be greater than the profit without the announcement and firm 2's will be less. The total amount to be produced in Stackelberg is $\frac{3}{4}(\mathcal{P} - c) > \frac{2}{3}(\mathcal{P} - c)$, the total amount produced without information. Since the price function is decreasing in the quantity produced, this says that the Stackelberg price will be less than the price without the announcement. In other words, firm 1, by sending out the information, will make more money, firm 2 will make less, the consumer will pay less, and more will be produced. Everyone is better off except firm 2.

4

Cooperative games

There are $n > 1$ players numbered $N = \{1, 2, \dots, n\}$. We consider a game in which the players may choose to cooperate by forming coalitions. A **coalition** is any subset $S \subset N$, or numbered collection of the players. Coalitions form in order to benefit every member of the coalition so that each member might receive more than he could on his own. In this section we try to determine a fair allocation of the benefits to each member of a coalition. First we need to quantify the benefits of a coalition through the use of a real valued function, called the characteristic function, which quantifies the total reward for forming the coalition. The characteristic function of a coalition $S \subset N$ is the largest guaranteed payoff to the coalition.

Definition 4.0.3 *Any function $v : \mathcal{P}(N) \to \mathbb{R}$ satisfying $v(\emptyset) = 0$ and $v(N) \geq \sum_{i=1}^{n} v(i)$ is a characteristic function (of an n person cooperative game).*

In other words, the only condition placed on a characteristic function is that the benefit of the empty coalition should be 0 and the benefit of the **grand coalition** N, consisting of all the players, should be at least the sum of the benefits of the individual players if no coalitions form. That is, every one pulling together should do better than each player on his own. With that much flexibility games may have more than one characteristic function.

Example 4.1. 1.Suppose there is a factory with N workers each doing the same task. If each worker earns the same amount b dollars, then we can take the characteristic function to be $v(S) = b|S|$, where $|S|$ is the number of workers in S.

2. Suppose the owner of a car, labeled player 1, offers it for sale for S\$. There are 2 customers interested in the car. Customer C, labelled player 2, values the car at c and customer D, labelled player 3, values it at d. Assume the price is nonnegotiable. This means that if $S > c$ and $S > d$, then no deal will be made. We will assume then that $S < \min\{c, d\}$, and, for definiteness we

may assume $S < c \le d$. The possible coalitions are $123,\ 12,\ 13,\ 23,\ 1,\ 2,\ 3,\ \emptyset$. We are dropping the braces in the notation for a coalition.

It requires a seller and a buyer to reach a deal. Therefore, we may define the characteristic function as follows:

$$v(123) = d, \ \ v(1) = S, \ \ v(\emptyset) = 0$$
$$v(13) = d, \ \ v(12) = c, \ \ v(23) = 0,$$
$$v(2) = v(3) = 0.$$

Why? Well, $v(123) = d$ because the car will be sold for d, $v(1) = S$ because the car is worth S to player 1, $v(13) = d$ because 1 will sell the car to 3 for $d > S$, $v(12) = c$ because the car will be sold to 2 for $c > S$, etc..

3. Suppose a customer wants to buy a bolt and a nut for the bolt. There are 3 players but player 1 owns the bolt and players 2 and 3 each own a bolt. A bolt together with a nut is worth 5. We could define a characteristic function for this game as

$$v(123) = 5, \ v(12) = v(13) = 5, \ \ v(1) = v(2) = v(3) = 0, \ \ \text{and} \ \ v(\emptyset) = 0.$$

Example 4.2. Here is a much more complicated but systematic way to create a characteristic function given any cooperative game. The idea is to create a 2 person zero sum game in which a given coalition is played against a pure opposing coalition consisting of everybody else. The characteristic function will be the value of the game associated with each coalition S.

Precisely, suppose we have a coalition of players $S \subset N$. Consider the 2 person *zero sum game* played by S against $N - S$, i.e., S is going to play against the coalition consisting of all the other players. Any such game, as a 2 person zero sum game, has a value if the players use mixed strategies we denote by $v(S) = value(S)$ and

$$v(S) = \max_{X \in X_S} \min_{Y \in Y_{N-S}} \sum_{i \in S} E_i(X, Y) = \min_{Y \in Y_{N-S}} \max_{X \in X_S} \sum_{i \in S} E_i(X, Y).$$

Here X_S denotes the set of mixed strategies for the coalition S and Y_{N-S} is the set of mixed strategies for the opposing coalition $N - S$. If $S = \emptyset$ we define $value(S) = 0$. This way of defining $v(S)$ uses the expected sum over all of the members in the coalition S of the expected payoffs to player $i \in S$. The example following will show exactly how to calculate that.

It is not hard to check that $value(S)$ as we let S range over all possible coalitions satisfies the conditions to be a characteristic function. For each coalition S, $value(S) = v(S)$ represents the least amount the coalition S can get no matter what the opposing coalition $N - S$ does.

Let's work out a specific example using a 3 player game.

Suppose we have a 3 player game with the following matrices:

$\boxed{3}$ *plays A*		*Player* $\boxed{2}$	
		A	B
Player $\boxed{1}$	A	$(1,1,0)$	$(4,-2,2)$
	B	$(1,2,-1)$	$(3,1,-1)$

$\boxed{3}$ *plays B*		*Player* $\boxed{2}$	
		A	B
Player $\boxed{1}$	A	$(-3,1,2)$	$(0,1,1)$
	B	$(2,0,-1)$	$(2,1,-1)$

Each player has the two pure strategies A and B. Because there are 3 players, in matrix form this could be represented in 3 dimensions (a cube $3 \times 3 \times 3$ matrix). That is a little hard to write down so instead we have broken this into 2 2×2 matrices. Each matrix assumes that player 3 plays a fixed strategy. So, now, we want to find the characteristic function of this game. We need to consider all the possible 2 player coalitions $\{12\}, \{13\}, \{23\}$ versus the single player coalitions.

1. **Play** $S = \{12\}$ **versus** $\{3\}$**.** So, players 1 and 2 team up against player 3. We first write down the associated matrix game.

$\boxed{12}$ *vs* $\boxed{3}$		*Player* $\boxed{3}$	
		A	B
Player $\boxed{12}$	AA	2	-2
	AB	2	1
	BA	3	2
	BB	4	3

For example, if 1 plays A and 2 plays A and 3 plays B, the payoffs are $(-3, 1, 2)$ and so the payoff to player 12 is -3+1=-2, the sum of the payoff to player 1 and player 2 which is our coalition. Now we calculate the value of the zero sum 2 person game with this matrix to get the value(12 vs 3)=3 and we write $v(12) = 3$. Observe there is a saddle point at BB vs B. Consequently, in the game $\{3\}$ vs $\{12\}$ we would get $v(3) = -1$.

2. **Play** $S = \{13\}$ **versus** $\{2\}$. The game matrix is

$\boxed{13}$ *vs* $\boxed{2}$		*Player* $\boxed{2}$	
		A	B
Player $\boxed{13}$	AA	1	6
	AB	-1	1
	BA	0	2
	BB	1	1

We see that the value of this game is 1 so that $v(13) = 1$. In the game $\{2\}$ versus $\{13\}$ we have $\boxed{2}$ as the row player and the matrix

$\boxed{2}$ vs $\boxed{13}$		$\boxed{13}$		
	AA	AB	BA	BB
$\boxed{2}$ A	1	1	2	0
B	-2	1	1	1

The value of this game is $1/4$ and so $v(2) = 1/4$.

Continuing in this way we summarize that the characteristic function for this 3 person game is

$$v(1) = 1, v(2) = 1/4, v(3) = -1$$
$$v(12) = 3, v(13) = -1, v(23) = 1$$
$$v(123) = 4$$

The value $v(123) = 4$ is obtained by taking the largest sum of the payoffs which they would achieve if they all cooperated.

All characteristic functions, including the one defined in the preceding example, must satisfy specific properties which are listed next.

Properties of characteristic functions:

1. $v(\emptyset) = 0$.
2. $v(S \cup T) \geq v(S) + v(T), \quad \forall S, T \subset N, S \cap T = \emptyset$. This is called superadditivity. It says that the benefits of the larger coalition $S \cup T$ must be at least the total benefits of the individual coalitions S and T.
3. Any game with $v(S \cup T) = v(S) + v(T), \quad \forall S, T \subset I, S \cap T = \emptyset$, is called an **inessential** game.
4. A game is inessential if and only if $v(N) = \sum_{i=1}^{n} v(i)$. An **essential** game therefore is one with $v(N) > \sum_{i=1}^{n} v(i)$.

The word *inessential* implies that these games are not important. That turns out to be true. The reason is that they turn out to be easy to analyze.

We need a basic definition. Recall that $v(N)$ represents the reward available if all players cooperate.

Definition 4.0.4 *Let x_i be the share of the value of $v(N)$ received by player $i = 1, 2, \ldots, n$. A vector $\mathbf{x} = (x_1, \ldots, x_n)$ is an imputation if*

- $x_i \geq v(i)$ *(individual rationality)*
- $\sum_{i=1}^{n} x_i = v(N)$ *(group rationality)*

The imputation $\mathbf{x}$ is also called a payoff vector or an allocation.

Remark 4.0.5 *1. It is possible for x_i to be a negative number!*

2. Individual rationality means that the share received by player i should be at least what he could get on his own. Each player is rational.

3. Group rationality means that the total rewards to all of the individuals should equal the total rewards available by cooperation.

4. Any inessential game (i.e., $v(N) = \sum_{i=1}^{n} v(i)$) has one and only one imputation and it is $\mathbf{x} = (v(1), \ldots, v(n))$. These games are uninteresting because there is no incentive for any of the players to form any sort of coalition.

The main objective in cooperative game theory is to determine the imputation which results in a *fair* allocation of the total rewards. Of course this will depend on the definition of *fair*. That word is not at all precise. If you change the meaning of *fair* you will change the imputation.

We begin by presenting a way to transform a given characteristic function for a cooperative game to one which is frequently easier to work with. It is called the (0,1) normalization of the original game. This is not strictly necessary but it does simplify the computations in many problems. The normalized game will result in a characteristic function with $v(i) = 0, v(N) = 1$. The proof of the lemma will show how to make the conversion.

Lemma 4.0.6 *Any essential game with characteristic function v' has a $(0,1)$ normalization with characteristic function v.*

Proof. Consider the $n+1$ system of equations for constants $c, a_i, 1 \le i \le n$, given by

$$\begin{aligned} v(i) &= cv'(i) + a_i = 0, 1 \le i \le n, \\ v(N) &= cv'(N) + \sum_{i=1}^{n} a_i = 1. \end{aligned}$$

Solving this system we get

$$c = \frac{1}{v'(N) - \sum v'(i)} > 0, \qquad a_i = -\frac{v'(i)}{v'(N) - \sum_i v'(i)}.$$

Then, for any coalition $S \subset N$ define the characteristic function

$$v(S) = cv'(S) + \sum_{i \in S} a_i, \quad \text{for all } S \subset N.$$

The reader can verify that $v(N) = 1$ and $v(i) = 0, i = 1, 2, \ldots, n.$ $\square$

Example 4.3. In the 3 person nonzero sum game considered above we found the (unnormalized) characteristic function to be

$$\begin{aligned} v(1) = 1, v(2) = 1/4, v(3) = -1 \\ v(12) = 3, v(13) = -1, v(23) = 1 \\ v(123) = 4 \end{aligned}$$

To normalize this game we compute

$$c = \frac{1}{v(N) - \sum_{i=1}^{3} v(i)} = \frac{1}{1 - 1/4} = \frac{4}{15}, \text{ and } a_i = -\frac{4}{15}v(i).$$

Let's denote the normalized characteristic function by $\bar{v}$. We get

$$\begin{aligned}
\bar{v}(i) &= 4/15v(i) + a_i = 0 \\
\bar{v}(12) &= 4/15v(12) + a_1 + a_2 = 7/15 \\
\bar{v}(13) &= 4/15v(13) + a_1 + a_3 = 4/15 \\
\bar{v}(23) &= 4/15v(23) + a_2 + a_3 = 7/15 \\
\bar{v}(123) &= 4/15v(123) + a_1 + a_2 + a_3 = 1.
\end{aligned}$$

Remark 4.0.7 *If we have an imputation for an unnormalized game what does it become for the normalized game? If* $\mathbf{x}' = (x'_1, \ldots, x'_n)$ *is an imputation for* v' *then the imputation for* v *becomes* $\mathbf{x} = (x_1, \ldots, x_n)$ *where* $x_i = cx'_i + a_i, i = 1, 2, \ldots, n$ *and the constants* c *and* a_i *are the same as in the lemma.*

The set of imputations for the original game is

$$X' = \{\mathbf{x}' = (x'_1, \ldots, x'_n) \mid x'_i \geq v'(i), \sum_{i=1}^{n} x'_i = v'(N)\}.$$

For the normalized game, indeed for any game with $v(i) = 0, v(N) = 1$, *it becomes*

$$X = \{\mathbf{x} = (x_1, \ldots, x_n) \mid x_i \geq 0, \sum_{i=1}^{n} x_i = 1\}.$$

In the rest of this section we let X denote the set of imputations $\mathbf{x}$. We look for an $\mathbf{x} \in X$ as a "solution" to the game. The problem is the definition of the word "solution". It is as vague as the word "fair". We are seeking an imputation which, in some sense, is fair and allocates a fair share of the payoff to each player. To get a handle on the idea of *fair* we introduce the following subset of X.

Definition 4.0.8 *The* **reasonable set** $R \subset X$ *is the set*

$$R = \{\mathbf{x} \in X \mid x_i \leq \max_{T \in \Pi^i} \{v(T) - v(T - i)\}, i = 1, 2, \ldots, n\},$$

where $\mathcal{P}^i$ *is the set of all coalitions for which player* i *is a member. So, if* $T \in \mathcal{P}^i$ *then* $i \in T \subset N$, *and* $T - i$ *denotes the coalition* T *without the player* i.

In other words, the reasonable set is the set of imputations so that the amount allocated to each player is no greater than the maximum benefit that the player brings to any coalition. The difference $v(T)-v(T-i)$ is the measure of the rewards for coalition T due to player i. The reasonable set gives us a first way to reduce the size of X and try to focus in on a solution.

If the reasonable set has only one element, which is extremely unlikely for most games, then that is our solution. If there are many elements in R we need to cut it down further. In fact we need to cut it down to the **core** imputations. Here is the definition.

Definition 4.0.9 *Let $S \subset N$ be a coalition and let $\mathbf{x} \in X$. The* **excess** *of coalition $S \subset N$ for imputation $\mathbf{x} \in R$ is defined by*

$$e(S,x) = v(S) - \sum_{i \in S} x_i.$$

The **core of the game** *is*

$$C(0) = \{\mathbf{x} \in X \mid e(S,\mathbf{x}) \leq 0, \ \forall S \subset N\} = \{\mathbf{x} \in X \mid v(S) \leq \sum_{i \in S} x_i, \ \forall S \subset N\}.$$

*The $\varepsilon-$***core***, for $-\infty < \varepsilon < +\infty$, is*

$$C(\varepsilon) = \{\mathbf{x} \in X \mid e(S,x) \leq \varepsilon, \ \forall S \subset N, S \neq N, S \neq \emptyset\}$$

Let $\varepsilon^1 \in (-\infty,\infty)$ be the smallest ε for which $C(\varepsilon) \neq \emptyset$. The **least core** *$LC = X^1$ is $C(\varepsilon^1)$. It is possible for ε^1 to be positive, negative, or zero.*

Remark 4.0.10 *1. The idea behind the definition of the core is that an imputation $\mathbf{x}$ is a member of the core if no matter which coalition, S, is formed the total payoff given to the members of S, namely $\sum_{i \in S} x_i$, must be at least as large as $v(S)$, the maximum possible benefit of forming the coalition S. If $e(S,\mathbf{x}) > 0$ this would say that the maximum possible benefits of joining the coalition S is greater than the total allocation to the members of S using the imputation $\mathbf{x}$. That would mean that the members of S would not be very happy with $\mathbf{x}$ and they would move to improve their allocation by moving to a different imputation. In that sense, if $\mathbf{x} \in C(0)$ then $e(S,\mathbf{x}) \leq 0$ for every coalition S and so there would be no incentive for any coalition to try to use a different imputation. Likewise, if $\mathbf{x} \in C(\varepsilon)$, then the measure of dissatisfaction of a coalition with $\mathbf{x}$ is limited to ε.*

2. It is possible for the core of the game, $C(0)$, to be empty, but there will always be some $\varepsilon \in (-\infty,\infty)$ so that $C(\varepsilon) \neq \emptyset$. The least core uses the smallest such ε.

3. It should be clear, since $C(\varepsilon)$ is just a set of inequalities, that as ε increases, $C(\varepsilon)$ gets bigger, and as ε decreases, $C(\varepsilon)$ gets smaller. So, the idea is that we should shrink (or expand if necessary) $C(\varepsilon)$ by adjusting ε until we get one and only one imputation in it, if possible.

4. We will see shortly that $C(0) \subset R$, every allocation in the core is always in the reasonable set.

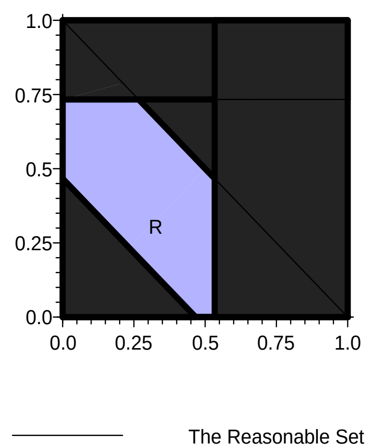
Example 4.4. We have the characteristic function in a 3 player game $v'(1) = 1, v'(2) = 1/4, v'(3) = -1, v'(12) = 3, v'(13) = 1, v'(23) = 1, v'(123) = 4$. This is an essential game so we normalize to get the characteristic function we work with.

$$v(i) = 0, v(123) = 1, v(12) = 7/15, v(13) = 4/15, v(23) = 7/15.$$

The reasonable set is easy to find and it is

$$R = \{(x_1, x_2, 1 - x_1, x_2) | x_1 \leq 8/15, x_2 \leq 11/15, 7/15 \leq x_1 + x_2 \leq 1.\}$$

Here is a graph of R :



Now let's calculate the $\varepsilon-$core for any $\varepsilon \in (-\infty, \infty)$. The $\varepsilon-$core is

$$\begin{aligned} C(\varepsilon) &= \{\mathbf{x} \in X \mid e(S, x) \leq \varepsilon,\ \forall S \subset N, S \neq N, S \neq \emptyset\} \\ &= \{x_1 \geq 0, x_2 \geq 0, 7/15 - x_1 - x_2 \leq \varepsilon, -11/15 + x_2 \leq \varepsilon, -8/15 + x_1 \leq \varepsilon, \\ &\qquad -x_1 \leq \varepsilon, -x_2 \leq \varepsilon, -1 + x_1 + x_2 \leq \varepsilon\}. \end{aligned}$$

We have used the fact that $x_1 + x_2 + x_3 = 1$.

By working with the inequalities in $C(\varepsilon)$ we can verify that the smallest ε so that $C(\varepsilon) \neq \emptyset$ is $\varepsilon_1 = -8/30$. That means that the least core is

$$C(\varepsilon_1) = X^1 = \{(x_1, x_2, 1 - x_1 - x_2) \mid 4/15 \geq x_1 \geq 0, 7/15 \geq x_2 \geq 0, x_1 + x_2 = 11/15\}.$$

It seems that the core $C(0)$ would be the good imputations and so would be considered the solution of our game if in fact $C(0)$ contained exactly one point. Unfortunately, the core may contain many points, or may even be empty. Here is an example of a game with an empty core.

Example 4.5. Suppose the characteristic function of a 3 player game is given by

$$v(123) = 1 = v(12) = v(13) = v(23), \quad \text{and} \quad v(1) = v(2) = v(3) = 0.$$

The set of imputations is then (this is a zero-one game)

$$X = \{\mathbf{x} = (x_1, x_2, x_3) \mid x_i \geq 0, \sum_{i=1}^{3} x_i = 1\}.$$

To calculate the reasonable set R we need to find

$$x_i \leq \max_{T \in \Pi^i} \{v(T) - v(T - i)\}, i = 1, 2, 3.$$

Starting with $\mathcal{P}^1 = \{1, 12, 13, 123\}$ we calculate

$$v(1) - v(1 - 1) = 0,\ v(12) - v(2) = 1, v(13) - v(3) = 1, v(123) - v(23) = 0$$

so $x_1 \leq \max\{0, 1, 1, 0\} = 1$. This is true for x_2 as well as x_3. So all we get from this is $R = X$, all the imputations are reasonable.

Next we get $C(0) = \{\mathbf{x} \in X \mid v(S) \leq \sum_{i \in S} x_i,\ \forall S \subset N\}$. If $\mathbf{x} \in C(0)$ we calculate

$$e(i, \mathbf{x}) = v(i) - x_i = -x_i \leq 0,\ e(12, \mathbf{x}) = 1 - (x_1 + x_2) \leq 0$$

and, in likewise fashion,

$$e(13, \mathbf{x}) = 1 - (x_1 + x_3) \leq 0,\ e(23, \mathbf{x}) = 1 - (x_2 + x_3) \leq 0,$$

The set of inequalities we have to solve are

$$x_1 + x_2 \geq 1, x_1 + x_3 \geq 1, x_2 + x_3 \geq 1, x_1 + x_2 + x_3 = 1, x_i \geq 0.$$

But there is no $\mathbf{x} \in X$ which satisfy these, so $C(0) = \emptyset$.

The excess function $e(S, \mathbf{x})$ is a measure of dissatisfaction (especially if $e(S, \mathbf{x}) > 0$), of S with the imputation $\mathbf{x}$. So it makes sense that the best imputation would minimize the largest dissatisfaction over all the coalitions. This leads us to one possible definition of a solution.

Definition 4.0.11 *An allocation $\mathbf{x}^* \in X$ is a solution to the cooperative game if*

$$\min_{\mathbf{x} \in X} \max_S e(S, x) = \max_S e(S, x^*)$$

so that $\mathbf{x}^*$ minimizes the maximum excess for any coalition S.

Remark 4.0.12 *Maple can give us a simple way of determining if the core is empty. Consider the linear program:*

$$\textit{Minimize } z = x_1 + \cdots + x_n$$
$$\textit{Subject to } v(S) \leq \sum_{i \in S} x_i, \textit{ for every } S \subset N.$$

It is not hard to check that $C(0)$ is not empty if and only if the linear program has a minimum, say z^*, and $z^* \leq v(N)$. If the game is normalized, then we need $z^* \leq 1$.

Let's begin looking for the solution by showing that the core must be a subset of the reasonable set.

Lemma 4.0.13 $C(0) \subset R$.

Proof. We may assume the game is in normalized form because we can always transform it to one that is and then work with that one. So $v(N) = 1, v(i) = 0, i = 1, \ldots, n$. Let $\mathbf{x} \in C(0)$. If $\mathbf{x} \notin R$ there is j such that

$$x_j > \max_{T \in \Pi^j} v(T) - v(T - j).$$

This means that for every $T \subset N$ with $j \in T$, $x_j > v(T) - v(T - j)$ and so the amount allocated to player j is larger than the amount of his benefit to any coalition. Take $T = N$. Then

$$x_j > v(N) - v(N - j) = 1 - v(N - j).$$

But then, $v(N - j) > 1 - x_j = \sum_{i \neq j} x_i$ and so $e(N - j, \mathbf{x}) > 0$ which means $x \notin C(0)$. $\square$

The next theorem formalizes the idea above that when $e(S, \mathbf{x}) \leq 0$ for all coalitions, then the player should be happy with the imputation $\mathbf{x}$ and would not want to switch to another one.

Theorem 4.0.14 *The core of a game is the set of all undominated imputations for the game. That is,*

$$C(0) = \{\mathbf{x} \in X | \textit{there is no } \mathbf{y} \in X \ \& \ S \subset N, \\ \textit{such that } y_i > x_i, \forall\, i \in S, \& \sum_{i \in S} y_i \leq v(S)\}.$$

Proof. Call the right hand side the set B. We have to show $C(0) \subset B$ and vice versa.

Let $\mathbf{x} \in C(0)$ and suppose $\mathbf{x} \notin B$. Then there is $\mathbf{y} \in X$ and $S \subset N$ such that $y_i > x_i$ for all $i \in S$ and $v(S) \geq \sum_{i \in S} y_i$. Summing on $i \in S$ this shows

$$v(S) \geq \sum_{i \in S} y_i > \sum_{i \in S} x_i \Longrightarrow e(S, \mathbf{x}) > 0,$$

contradicting the fact that $\mathbf{x} \in C(0)$.

Now let $\mathbf{x} \in B$. If $\mathbf{x} \notin C(0)$ there is $S \subset N$ so that $v(S) > \sum_{i \in S} x_i$. Let $\varepsilon = v(S) - \sum_{i \in S} x_i > 0$. Let $\alpha = 1 - v(S) \geq 0$. Let $s = |S|$, the number of points in S, and

$$z_i = \begin{cases} x_i + \frac{\varepsilon}{s}, & \text{if } i \in S; \\ \frac{\alpha}{n-s}, & \text{if } i \notin S. \end{cases}$$

We will show that $\mathbf{z} = (z_1, \ldots, z_n)$ is an imputation and z dominates x, i.e., $\mathbf{z}$ is a better allocation for the members of S than is $\mathbf{x}$.

First $z_i \geq 0$ and

$$\sum_{i=1}^{n} z_i = \sum_{S} x_i + \sum_{S} \frac{\varepsilon}{s} + \sum_{N-S} \frac{\alpha}{n-s} = \sum_{S} x_i + \varepsilon + \alpha = v(S) + 1 - v(S) = 1.$$

Therefore $\mathbf{z}$ is an imputation.

Next we show $\mathbf{z}$ is a better imputation than is $\mathbf{x}$ for the coalition S. If $i \in S$ $z_i = x_i + \frac{\varepsilon}{s} > x_i$ and $\sum_{i \in S} z_i = \sum_{i \in S} x_i + \varepsilon = v(S)$. Therefore z dominates x. But this says $x \notin B$ and that is a contradiction. Hence $B \subset C(0)$. $\square$

Example 4.6. This example will show what we can do when the game has an empty core $C(0)$. We will see that when we calculate the least core $X^1 = C(\varepsilon)$, where ε is the smallest value for which $C(\varepsilon) \neq \emptyset$, we will obtain a reasonable fair allocation (and only one).

Suppose Bill has 150 sinks to give away to the whoever shows up to take them away. Amy(1), Agnes(2), and Agatha(3) simultaneously show up to take as many of the sinks as their trucks can haul. Amy can haul 45, Agnes 60 and Agatha 75, for a total of 180, 30 more than the maximum of 150. The wrinkle in this problem is that the sinks are too heavy for any one person to load onto the trucks so they must cooperate in loading the sinks. The question is: how many sinks should be allocated to each person?

Define the characteristic function $v'(S)$ as the number of sinks the coalition $S \subset N = 1, 2, 3$ can load. We have $v'(i) = 0, i = 1, 2, 3$, since they must cooperate to receive any sinks at all, and

$$v'(12) = 105,\ v'(13) = 120,\ v'(23) = 135,\ v'(123) = 150.$$

It will be easier to not normalize this problem. In that case we calculate

$$\begin{aligned} C(\varepsilon) &= \{x \in X | e(S, x) \leq \varepsilon, \forall S \subsetneq N\} \\ &= \{x \in X | v(S) - \textstyle\sum_{i \in S} x_i \leq \varepsilon\} \\ &= \{x | 105 \leq x_1 + x_2 + \varepsilon, 120 \leq x_1 + x_3 + \varepsilon, \\ &\qquad\qquad 135 \leq x_2 + x_3 + \varepsilon, -x_i \leq \varepsilon\} \end{aligned}$$

We know that $x_1 + x_2 + x_3 = 150$ so we obtain as conditions on ε that

$$120 \leq 150 - x_2 + \varepsilon,\ 135 \leq 150 - x_1 + \varepsilon,\ 105 \leq x_1 + x_2 + \varepsilon.$$

We see that $45 \geq x_1 + x_2 - 2\varepsilon \geq 105 - 3\varepsilon$ implying that $\varepsilon \geq 20$. This is in fact the smallest $\varepsilon = 20$ making $C(\varepsilon) \neq \emptyset$. Of course that means that the core $C(0)$ would be empty. Using $\varepsilon = 20$ we calculate $C(20) = \{x_1 = 35, x_2 = 50, x_3 = 65\}$. Hence the fair allocation is to let Amy have 35 sinks, Agnes 50, and Agatha 65 sinks and they all cooperate. Notice that $C(0) = \emptyset$.

We conclude that our fair allocation of sinks is as follows

Player	Truck Capacity	Allocation
Amy	45	35
Agnes	60	50
Agatha	75	65
Total	180	150

Observe that each player in the fair allocation gets 10 less than the capacity of the truck. It seems that this is certainly a reasonably fair way to allocate the sinks. That is, there is an undersupply of 30 sinks so each player will receive $30/3 = 10$ less than their truck can haul.

In the next example we will determine a necessary and sufficient condition for any cooperative game with 3 players to have a nonempty core.

Example 4.7. We take $N = \{1, 2, 3\}$ and a characteristic function in normalized form

$$v(i) = v(\emptyset) = 0, i = 1, 2, 3, \quad v(123) = 1, \quad v(12) = a_{12}, v(13) = a_{13}, v(23) = a_{23}.$$

Of course we have $0 \le a_{ij} \le 1$. We can state then the proposition.

Proposition 4.0.15 *For the 3 person cooperative game with normalized characteristic function v we have $C(0) \neq \emptyset$ if and only if*

$$a_{12} + a_{13} + a_{23} \le 2.$$

Proof. We have

$$C(0) = \{(x_1, x_2, 1 - x_1 - x_2) \mid a_{12} \le x_1 + x_2,$$
$$a_{13} \le x_1 + (1 - x_1 - x_2) = 1 - x_2, a_{23} \le 1 - x_1\}.$$

So, $x_1 + x_2 \ge a_{12}$, $x_2 \le 1 - a_{13}$, and $x_1 \le 1 - a_{23}$. Adding the last two says $x_1 + x_2 \le 2 - a_{12} - a_{13}$ so that $a_{12} \le 2 - a_{12} - a_{13}$ so that if $C(0) \neq \emptyset$ it must be true that $a_{12} + a_{13} + a_{23} \le 2$.

For the other side, if $a_{12} + a_{13} + a_{23} \le 2$ then we can find numbers $c_1 \ge 0, c_2 \ge 0, c_3 \ge 0$ so that

$$(a_{12} + c_1) + (a_{13} + c_2) + (a_{23} + c_3) = 2.$$

Define the imputation

$$\mathbf{x} = (1 - (a_{13} + c_2), 1 - (a_{23} + c_3), 1 - (a_{12} + c_1)) = (x_1, x_2, x_3).$$

Then $x_1 + x_2 + x_3 = 1$. Furthermore,

$$x_1 + x_2 = 2 - [(a_{13} + c_2) + (a_{23} + c_3) = a_{12} + c_1 \ge a_{12}.$$

Similarly, $x_2 \le 1 - a_{13}$ and $x_1 \le 1 - a_{23}$. Hence $\mathbf{x} \in C(0)$ and so $C(0) \neq \emptyset$. □

Remark 4.0.16 *This remark summarizes the ideas behind the use of the least core.*

1. Remember $e(S, \mathbf{x})$ is the level of dissatisfaction of the coalition S with the allocation $\mathbf{x}$. The larger $e(S, \mathbf{x})$ is the less $\mathbf{x}$ will be acceptable to S.

2. For a given grand allocation $\mathbf{x}$, the coalition which least objects to $\mathbf{x}$ is the coalition satisfying

$$e(S_0, \mathbf{x}) = \max_{S \subsetneq N} e(S, \mathbf{x}).$$

For each fixed coalition S, the allocation giving the minimum dissatisfaction is

$$e(S, \mathbf{x}_0) = \min_{\mathbf{x} \in X} e(S, \mathbf{x}).$$

3. The value of ε giving the least $\varepsilon-$core is

$$\varepsilon_1 := \min_{\mathbf{x} \in X} \max_{S \subsetneq N} e(S, \mathbf{x}).$$

4. The point of calculating the $\varepsilon-$core is that the core is not a sufficient set to ultimately solve the problem in the case when the core $C(0)$ is (i)empty, or (ii) consists of more than one point. In case (ii) the question of course is which point should be chosen as the fair allocation. The $\varepsilon-$core seeks to answer this question by shrinking the core at the same rate from each side of the feasible set until we reach a single point. We can use Maple to do this and here are the commands:

```
> with(simplex):
> obj:=z
> cnsts:={7/15-x-y<=z,y-11/15<=z,x-8/15<=z,
                          -x<=z,-y<=z,x+y<=1+z};
> minimize(obj,cnsts);
> with(plots):subs(z=0,cnsts);
> fcnsts := ;
> #This is a plot of the core:

> inequal(fcnsts,x=0..1,y=0..1,optionsfeasible=(color=red),
>     optionsopen=(color=blue,thickness=2),
>     optionsclosed=(color=green, thickness=3),
>     optionsexcluded=(color=yellow));

> #The next command animates the shrinking of the core to a point.
>animate(inequal,[7/15-x-y<=z,y-11/15<=z,x-8/15<=z,
                   -x<=z,-y<=z,x+y<=1+z\},
                   x=0..1,y=0..1,
optionsfeasible=(color=red),
optionsopen=(color=blue,thickness=2),
optionsclosed=(color=green, thickness=3),
optionsexcluded=(color=yellow)],z=-1..0,frames=50);
```

The minimum value $z = \varepsilon$ is the smallest ε so that the constraint set is nonempty. Hence $C(z)$ is the least ε-core.

4.1 The Nucleolus

The core $C(0)$ might be empty but we can find an ε so that $C(\varepsilon)$ is not empty. We can fix the *empty* problem. Even if $C(0)$ is not empty, it may contain more than one point and again we can use $C(\varepsilon)$ to *maybe* shrink the core down to one point. The problem is what happens when the least core $C(\varepsilon)$ itself has too many points.

In this section we will answer the question of what to do when the least core $C(\varepsilon)$ contains more than one point. Remember that $e(S, \mathbf{x}) = v(S) - \sum_{i \in S} x_i$ and the larger the excess, the more unhappy the coalition S is with the allocation $\mathbf{x}$. So, no matter what, we want the excess to be as small as possible for all coalitions and we want the imputation which achieves that.

Let's begin by working through an example.

Example 4.8. Let us take the normalized characteristic function for the 3 player game

$$v(12) = 4/5, v(13) = 2/5, v(23) = 1/5 \text{ and } v(123) = 1, v(i) = 0, i = 1, 2, 3.$$

Step 1. Calculate the least core.

$$\begin{aligned} C(\varepsilon) \quad &= \quad \{(x_1, x_2, x_3) \in X \mid e(S, x) \leq \varepsilon, \forall S \subsetneq N\} \\ &= \quad \{(x_1, x_2, 1 - x_1 - x_2) | -\varepsilon \leq x_1 \leq 4/5 + \varepsilon, \\ &\qquad -\varepsilon \leq x_2 \leq 3/5 + \varepsilon, 4/5 - \varepsilon \leq x_1 + x_2 \leq 1 + \varepsilon\} \end{aligned}$$

We calculate that the smallest ε for which $C(\varepsilon) \neq \emptyset$ is $\varepsilon_1 = -1/10$ and then

$$\begin{aligned} C(\varepsilon_1 = -1/10) = \{(x_1, x_2, 1 - x_1 - x_2) | x_1 \in [2/5, 7/10], \\ x_2 \in [1/5, 1/2], x_1 + x_2 = 9/10\}. \end{aligned}$$

This is a line segment in the $x_1 - x_2$ plane. So we have the problem that the least core does not have only one imputation which we would be able to call our solution. What is the fair allocation now? Here is what to do.

2. Calculate the next least core

The idea is, restricted to the allocations in the first least core, minimize the maximum excesses over all the allocations in the least core.

If we take any allocation $\mathbf{x} \in C(\varepsilon_1)$ we want to calculate the excesses for each coalition:

$$\begin{array}{ll} e(1,x) = -x_1 & e(2,x) = -x_2 \\ e(13,x) = x_2 - 3/5 & e(23,x) = x_1 - 4/5 \\ e(12,x) = -1/10 & e(3,x) = -1/10 \end{array}$$

Since $\mathbf{x} \in C(\varepsilon_1)$ we know that these are all $\leq -1/10$. Observe that the excesses $e(12,\mathbf{x}) = e(3,\mathbf{x}) = -1/10$ do not depend on the allocation $\mathbf{x}$ as long as it is in $C(\varepsilon_1)$.

Now we set

$$\Sigma^1 := \{S \subsetneq N | e(S,\mathbf{x}) < \varepsilon_1, \text{ for some } \mathbf{x} \in C(\varepsilon_1)\}.$$

This is a set of coalitions with excesses for some imputation smaller than ε_1. So these coalitions can use some imputation which gives a better allocation for them, as long as the allocations used are also in $C(-1/10)$. In our example

$$\Sigma^1 = \{1, 2, 13, 23\}$$

Now define

$$\varphi^1(\mathbf{x}) = \max_{S \in \Sigma^1} e(S,\mathbf{x}), \text{ and } \varepsilon_2 := \min_{\mathbf{x} \in X} \varphi^1(\mathbf{x}) = \min_{\mathbf{x} \in X} \max_{S \in \Sigma^1} e(S,\mathbf{x})$$

and

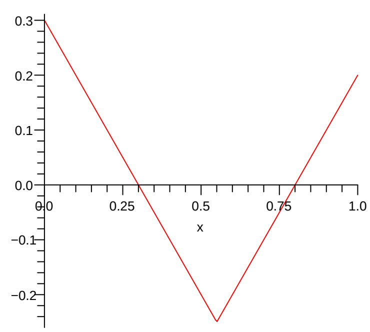
$$X^2 := \{\mathbf{x} \in X^1 = C(\varepsilon_1) | \varphi^1(\mathbf{x}) = \varepsilon_2\}.$$

The set X^2 is the subset of allocations from X^1 which are preferred by the coalitions in Σ^1. If X^2 contains exactly one imputation, then that point is our fair allocation, i.e., the solution.

In our example we get, subject to $\mathbf{x} \in X^1$ so that $x_1 + x_2 = 9/10, 2/5 \leq x_1 \leq 7/10$,

$$\begin{aligned} \varphi^1(\mathbf{x}) &= \max\{e(1,\mathbf{x}), e(2,\mathbf{x}), e(13,\mathbf{x}), e(23,\mathbf{x})\} \\ &= \max\{-x_1, -x_2, x_2 - 3/5, x_1 - 4/5\} \\ &= \max\{-x_1, x_1 - 9/10, x_1 - 4/5, 3/10 - x_1\} \end{aligned}$$

Here is a plot of the function $\varphi_1(\mathbf{x})$.



The minimum of this function is then $\varepsilon_2 = -1/4$ and it is achieved where the two lines cross, namely $x_1 = 11/20$. Then, since $x_1 + x_2 = 9/10$ we must have $x_2 = 7/20$ and, finally, $x_3 = 2/20$. We have found $X^2 = \{(11/20, 7/20, 2/20)\}$ containing exactly one point and so is our solution to the problem.

Notice that for this allocation

$$\begin{aligned}
&e(13, \mathbf{x}) = x_2 - 3/5 = 7/20 - 12/20 = -1/4 \\
&e(23, \mathbf{x}) = x_1 - 4/5 = 11/20 - 4/5 = -1/4 \\
&e(1, \mathbf{x}) = -11/20, e(2, x) = -7/20
\end{aligned}$$

and each of these is smaller than $-1/10$ so these coalitions are happy with this imputation in the sense that if they don't use it, some other coalition will be unhappy with the allocation.

In general, we would need to continue this procedure if X^2 also contained more than one point. Here are the sequence of steps to take until we get down to one point:

Step $0. X^0 := X, \Sigma^0 := \{S \subsetneq N, S \neq \emptyset\}$

Step $k \geq 1. {:}\varepsilon^k := \min_{\mathbf{x} \in X^{k-1}} \max_{S \in \Sigma^{k-1}} e(S, \mathbf{x})$

$X^k := \{\mathbf{x} \in X | \mathbf{x} \in X^{k-1}, \varepsilon^k = \min_{\mathbf{x} \in X^{k-1}} \max_{S \in \Sigma^{k-1}} e(S, x) = \max_{S \in \Sigma^{k-1}} e(S, \mathbf{x})\}$

$\Sigma_k = \{S \in \Sigma^{k-1} | e(S, x) = \varepsilon^k, \forall\, \mathbf{x} \in X^k\}$

$\Sigma^k := \Sigma^{k-1} \setminus \Sigma_k$

If $\Sigma^k = \emptyset$ STOP : ELSE $k = k + 1$ GO TO Step k

When this algorithm stops at, say $k = m$, then X^m is the **nucleolus** of the core and will satisfy the relationships

$$X^m \subset X^{m-1} \subset \cdots \subset X^1 \subset X^0$$

Also, $\Sigma^0 \supset \Sigma^1 \supset \Sigma^2 \cdots \Sigma^{m-1} \supset \Sigma^m$.

The nucleolus is guaranteed to contain only one allocation $\mathbf{x}$ and this is the solution of the game.

The procedure to find the nucleolus can be formulated as a sequence of linear programs which can be solved using Maple. Here is the set up.

To begin, set $k = 1$. Then

$$\varepsilon_1 = \min_{\mathbf{x} \in X^0} \max_{S \in \Sigma^0} e(S, \mathbf{x}).$$

Set $\alpha = \max_{S \in \Sigma^0} e(S, \mathbf{x})$. Then $\alpha \geq e(S, \mathbf{x}) = v(S) - \sum_{i \in S} x_i, \forall S \in \Sigma^0$. We see that

$$\alpha + \sum_{i \in S} x_i \geq v(S), \forall S \in \Sigma^0.$$

So, our first linear programming problem which will yield e_1, X^1, Σ^1 is

$$\begin{array}{l} \text{Minimize } \alpha \\ \text{subject to } \alpha + \sum_{i \in S} x_i \geq v(S),\ \mathbf{x} \in X^0 = X. \end{array}$$

The minimum such α is ε_1 and

$$\Sigma^1 = \{S \in \Sigma^0 | e(S, \mathbf{x}) = \varepsilon^1\}.$$

The next linear programming problem can now be formulated. It is

$$\begin{array}{l} \text{Minimize } \alpha \\ \text{subject to } \alpha + \sum_{i \in S} x_i \geq v(S),\ \mathbf{x} \in X^1, S \in \Sigma^1 = \Sigma^0 - \Sigma_1. \end{array}$$

The minimum such α is ε_2 and

$$\Sigma^2 = \{S \in \Sigma^1 | e(S, \mathbf{x}) = \varepsilon^2\}.$$

Continue until we end up with exactly one allocation.

Example 4.9. Let's look at a game with 3 players. The characteristic function is $v(i) = 0, i = 1, 2, 3,\ v(12) = 1/3, v(13) = 1/6, v(23) = 5/6, v(123) = 1$, and this is in normalized form. We see that $1/3 + 1/6 + 5/6 < 2$ and so the core of the game $C(0)$ is not empty. We need to find the allocation within the core which solves our problem and that is done by finding the nucleolus.

1. First linear programming problem: We start with the full set of possible coalitions excluding the grand coalition $\Sigma^0 = \{1, 2, 3, 12, 13, 23\}$. In addition with this characteristic function we get

$$\begin{array}{l} e(1, \mathbf{x}) = -x_1, e(2, x) = -x_2, e(3, \mathbf{x}) = -x_3 \\ e(12, \mathbf{x}) = 1/3 - x_1 - x_2, e(13, \mathbf{x}) = 1/6 - x_1 - x_3 \\ e(23, \mathbf{x}) = 5/6 - x_2 - x_3 \end{array}$$

The first linear program is

$$\begin{array}{l} \text{Minimize } \alpha = \varepsilon^1 \\ \text{subject to } \alpha + \sum_{i \in S} x_i \geq v(S),\ \mathbf{x} \in X^0 = X, S \in \Sigma^0. \end{array}$$

The constraints are explicitly

$$\begin{array}{l} \alpha + x_1 \geq 0, \alpha + x_2 \geq 0, \alpha + x_3 \geq 0 \\ \alpha + x_1 + x_2 \geq 1/3, \alpha + x_1 + x_3 \geq 1/6, \alpha + x_2 + x_3 \geq 5/6 \\ x_1 + x_2 + x_3 = 1, x_i \geq 0, i = 1, 2, 3. \end{array}$$

The Maple commands that give the solution are

```
>with(simplex):
>obj:=a:
>cnsts:={a+x>=0,a+y>=0,a+z>=0,a+x+y>=1/3,a+x+z>=1/6,a+y+z>=5/6,
x+y+z=1,x>=0,y>=0,z>=0};
>minimize(a,cnsts);
```

Maple gives the solution $\varepsilon^1 = \alpha = a = -1/12, x_1 = x = 1/12, x_2 = y = 4/12, x_3 = z = 7/12$. So this gives the allocation $\mathbf{x} = (1/12, 4/12, 7/12)$. But this is not necessarily the unique allocation and therefore the solution to our game. To see if there are more, substitute $\alpha = 1/12$ in the constraint set to see that

$$cnsts := \{1/12 \leq x, 1/12 \leq y, 1/12 \leq z, 5/12 \leq x+y, 1/4 \leq x+z, \\ 11/12 \leq y+z, x+y+z = 1, 0 \leq x, 0 \leq y, 0 \leq z\}. \quad (4.1.1)$$

This substitution is easily performed with the Maple command

```
>subs(a=-12,cnsts);
```

Now, to see if there are other solutions we need to solve the system of inequalities in *cnsts*. To do that we have to first convert all the equalities in *cnsts* to inequalities. So we work with the modified constraints:

$$fcnsts := \{0 <= x - 1/12, 0 \leq y - 1/12, 0 \leq z - 1/12, 5/12 \leq x+y, \\ 1/4 \leq x+z, 11/12 \leq y+z, x+y+z \leq 1, 1 <= x+y+z\}; \quad (4.1.2)$$

So we want to solve the system $fcnsts$ for x, y, z. Maple does that as follows

```
>with(SolveTools:-Inequality):
>LinearMultivariateSystem(fcnsts,[x,y,z]);
```

Maple gives the following output

```
{[{1/12 <= x, x <= 1/12}, {5/12-x <= y, y <= 2/3+x},
{1-x-y <= z, z <= 1-x-y}]}
```

We see that $x = x_1 = 1/12$ but there are more solutions for $y = x_2$ and $z = x_3$. To find them we now substitute $x = 1/12$ in $fcnsts$ and clean it up to work with

```
hcnsts := {0 <= -1/12+y, 0 <= -1/12+z, 0 <= -11/12+y+z,
0 <= -1/3+y, y+z <= 11/12, 0 <=-1/6+z}
```

Then we use once again

```
LinearMultivariateSystem(hcnsts,[y,z]);
```

with output

```
{[{y <= 3/4, 3/4 <= y}, {z <= 11/12-y, 1/6 <= z}],
[{1/3 <= y, y < 3/4}, {11/12-y <= z, z<= 11/12-y}]}.
```

It is the second set of solutions we want, namely $1/3 \le y < 3/4$ and $y + z = 11/12$.

Summarizing the results of this procedure we have

$$\varepsilon^1 = -1/12 \text{ and}$$
$$X^1 = C(\varepsilon^1) = \{x_1 = 1/2, 4/12 \le x_2 < 9/12, x_3 \le 7/12, x_2 + x_3 = 11/12\}. \tag{4.1.3}$$

The least core contains a line segment and we will need to proceed to the second linear program.

2. From the first step we get

$$\varepsilon^1 = -1/12 \text{ and}$$
$$X^1 = C(\varepsilon^1) = \{x_1 = 1/2, 4/12 \le x_2 < 9/12, x_3 \le 7/12, x_2 + x_3 = 11/12\}. \tag{4.1.4}$$

and

$$\Sigma_1 = \{S \in \Sigma^0 | e(S, \mathbf{x}) = \varepsilon^1\} = \{1, 23\} \text{ and } \Sigma^1 = \Sigma^0 - \Sigma_1 = \{2, 3, 12, 13\}.$$

Our next LP problem can now be formulated:

$$\begin{array}{l} \text{Minimize } \alpha = \varepsilon^2 \\ \text{subject to } \alpha + \sum_{i \in S} x_i \ge v(S),\ \mathbf{x} \in X^1, S \in \Sigma^1. \end{array}$$

The constraints are explicitly

$$\begin{array}{l} \alpha + x_2 \ge 0, \alpha + x_3 \ge 0 \\ \alpha + x_1 + x_2 \ge 1/3, \alpha + x_1 + x_3 \ge 1/6, \\ x_2 + x_3 = 11/12, x_2 \le 9/12, x_3 \le 7/12. \end{array}$$

Using Maple we again obtain

$$\varepsilon^2 = -7/24, X^2 = \{x_1 = 1/12, x_2 = 13/24, x_3 = 9/24\}$$

and

$$\Sigma_2 = \{12, 23\}, \qquad \Sigma^2 = \Sigma^1 - \Sigma_2 = \{2, 3\}.$$

This is very promising because X^2 has only one point but since $\Sigma^2 \ne \emptyset$ we have to calculate again.

3. The third LP problem gives

$$\alpha = \varepsilon^3 = -9/24, X^3 = X^2, \Sigma_3 = \{3\}, \text{ and } \Sigma^3 = \Sigma^2 - \Sigma_3 = \{2\}.$$

So we do it again since $\Sigma^3 \neq \emptyset$ even though we are not changing the allocation X^3.

4. The fourth LP problem results in

$$\alpha = \varepsilon^4 = -13/24, X^4 = X^3 = X^2, \Sigma_4 = \{2\}, \text{ and } \Sigma^4 = \Sigma^3 - \Sigma_4 = \emptyset.$$

So we are finally done and we conclude that

$$\text{Nucleolus} = X^4 = \{(1/12, 13/24, 9/24)\}.$$

4.2 The Shapley Value

In an entirely different approach to deciding a fair allocation we change the definition of *fair* from minimizing the maximum dissatisfaction to allocating an amount proportional to the benefit each coalition derives from having a specific player as a member. The fair allocation would be the one recognizing the amount each member adds to a coalition. Players who add nothing, should receive nothing and players who are indispensable should be allocated a lot. The question is how do we figure out how much benefit each player adds to a coalition. Shapley came up with a way.

Definition 4.2.1 *An allocation* $\mathbf{x} = (x_1, \ldots, x_n)$ *is called the Shapley value if*

$$x_i = \sum_{\{S \in \Pi^i\}} [v(S) - v(S - i)] \frac{(|S| - 1)!(|N| - |S|)!}{|N|!}, i = 1, 2, \ldots, n,$$

where Π^i *is the set of all coalitions* $S \subset N$ *so that* $i \in S$*.*

To see where this definition comes from fix a player, say i, and consider the random variable Z_i which takes its values in the set of all possible coalitions $\mathcal{P}(N)$. Z_i is the coalition in which i is the jth player to join the grand coalition and $n - j$ players join after player i. We assume that Z_i has the probability distribution

$$P(Z_i = S) = \frac{(|S| - 1)!(n - |S|)!}{n!}$$

We choose this distribution because $|S| - 1$ players have joined before player i, and this can happen in $(|S| - 1)!$ ways; and $n - |S|$ players join after player i, and this can happen in $(n - |S|)!$ ways. The denominator is the total number of coalitions that can form among n players. This distribution assumes that they are all equally likely. One could debate this choice of distribution but this one certainly seems reasonable.

Therefore, for the fixed player i, if we compute

$$\begin{aligned} x_i \equiv E[v(Z_i) - v(Z_i - i)] &= \sum_{\{S \in \Pi_i\}} [v(S) - v(S-i)] P(Z_i = S) \\ &= \sum_{\{S \in \Pi^i\}} [v(S) - v(S-i)] \frac{(|S|-1)!(n-|S|)!}{n!} \end{aligned}$$

this is the expected benefit player i brings to the grand coalition. The Shapley value (or vector) is then the allocation $\mathbf{x} = (x_1, \ldots, x_n)$.

Example 4.10. 2 players have to divide \$M and they each get zero if they can't agree. What is the fair allocation? Obviously, without regard to the benefit derived from the money the allocation should be $M/2$ to each player. Define $v(1) = v(2) = 0, v(12) = M$. Then

$$x_1 = [v(1) - v(\emptyset)] \frac{0!}{1!} 2! + [v(12) - v(2)] \frac{1!0!}{2!} = \frac{M}{2}.$$

Note that if we solve this problem using the least core approach we get

$$\begin{aligned} C^+(\varepsilon) &= \{(x_1, x_2) | e(S, x) \le \varepsilon, \forall S \subsetneq N\} \\ &= \{(x_1, x_2) | -x_1 \le \varepsilon, -x_2 \le \varepsilon, M - x_1 - x_2 \le \varepsilon\} \\ &= \{(x_1, x_2) | x_1 = x_2 = M/2\} \end{aligned}$$

Example 4.11. Let's go back to the sink allocation problem with Amy, Agnes, and Agatha. Using the core concept we obtained

Player	Truck Capacity	Allocation
Amy	45	35
Agnes	60	50
Agatha	75	65
Total	180	150

Let's see what we get for the Shapley value. Recall that the characteristic function was $v(i) = 0, v(13) = 120, v(12) = 105, v(23) = 135, v(123) = 150$. In this case $n = 3, n! = 6$ and for player $i = 1$, $\Pi^1 = \{1, 12, 13, 123\}$, so

$$
\begin{aligned}
x_1 &= \sum_{\{S \in \Pi^1\}} [v(S) - v(S-1)] \frac{(|S|-1)!(3-|S|)!}{6} \\
&= [v(1) - v(\emptyset)] P(Z_1 = 1) + [v(12) - v(2)] P(Z_1 = 12) \\
&\qquad\qquad + [v(13) - v(3)] P(Z_1 = 13) + [v(123) - v(23)] P(Z_1 = 123) \\
&= 0 + 105 \frac{1!}{1!6} + 120 \frac{1}{6} + [150 - 135] \frac{2}{6} \\
&= 42.5
\end{aligned}
$$

Similarly

$$
\begin{aligned}
x_2 &= \sum_{\{S \in \Pi^2\}} [v(S) - v(S-2)] \frac{(|S|-1)!(3-|S|)!}{6} \\
&= [v(2) - v(\emptyset)] P(Z_1 = 2) + [v(12) - v(1)] P(Z_1 = 12) \\
&\qquad\qquad + [v(23) - v(3)] P(Z_1 = 23) + [v(123) - v(13)] P(Z_1 = 123) \\
&= 0 + 105 \frac{1!}{1!6} + 135 \frac{1}{6} + [150 - 135] \frac{2}{6} \\
&= 50
\end{aligned}
$$

$$
\begin{aligned}
x_3 &= \sum_{\{S \in \Pi^3\}} [v(S) - v(S-3)] \frac{(|S|-1)!(3-|S|)!}{6} \\
&= [v(3) - v(\emptyset)] P(Z_3 = 3) + [v(13) - v(1)] P(Z_3 = 13) \\
&\qquad\qquad + [v(23) - v(2)] P(Z_3 = 23) + [v(123) - v(12)] P(Z_3 = 123) \\
&= 0 + 120 \frac{1!}{1!6} + 135 \frac{1}{6} + [150 - 105] \frac{2}{6} \\
&= 57.5
\end{aligned}
$$

Consequently, the Shapley vector is $\mathbf{x} = (42.5, 50, 57.5)$, or, since we can't split sinks $\mathbf{x} = (43, 5057)$, quite a different allocation from the nucleolus solution.

Example 4.12. A typical and interesting problem involves a debtor who owes money to more than one creditor. The problem is that the debtor does not have enough money to pay off all the creditors so he must negotiate the amount each creditor will receive of the money that he has to pay back. To be specific, suppose that debtor D has exactly \$ 100,000 to pay off 3 creditors A, B, C. D owes A \$50,000; D owes B \$65,000, and D owes C \$10,000. Now it is possible for D to split up the 100K on the basis of percentages, i.e., the total owed is

\$ 145,000 and the amount to A is about 35% of that, to B is about 45%, and to C about 20%. We want to know what the Shapley allocation would be.

Let's take the characteristic function as follows: The 3 players are A, B, C and (with amounts in thousands of dollars)

$$v(A) = 25,\ v(B) = 40,\ v(C) = 0, \tag{4.2.1}$$

$$v(AB) = 90,\ v(AC) = 35,\ v(BC) = 50,\ v(ABC) = 100. \tag{4.2.2}$$

For example, if we consider the coalition AC they look at the fact that in the worst case B gets paid his 65K and they have 35K left as the value of their coalition. This is a little pessimistic since it is also possible to consider that AC would be paid 75K and then $v(AC) = 75$. That is, other characteristic functions are possible.

Now we compute the Shapley values.

$$\begin{aligned} x_A &= [v(A) - v(\emptyset)]\frac{1}{3} + [v(AB) - v(B)]\frac{1}{6} + [v(AC) - v(C)]\frac{1}{6} + [v(ABC) - v(BC)]\frac{1}{3} \\ &= 25/3 + 50/6 + 35/6 + 50/3 = 235/6 = 39.17K \end{aligned}$$

$$\begin{aligned} x_B &= [v(B) - v(\emptyset)]\frac{1}{3} + [v(AB) - v(A)]\frac{1}{6} + [v(BC) - v(C)]\frac{1}{6} + [v(ABC) - v(AC)]\frac{1}{3} \\ &= 40\frac{1}{3} + 65\frac{1}{6} + 50\frac{1}{6} + 65\frac{1}{3} = 325/6 = 54.17K \end{aligned}$$

$$\begin{aligned} x_C &= [v(C) - v(\emptyset)]\frac{1}{3} + [v(BC) - v(B)]\frac{1}{6} + [v(AC) - v(A)]\frac{1}{6} + [v(ABC) - v(AB)]\frac{1}{3} \\ &= 0\frac{1}{3} + 10\frac{1}{6} + 10\frac{1}{6} + 10\frac{1}{3} = 40/6 = 6.67K \end{aligned}$$

The Shapley allocation is $\mathbf{x} = (39.17, 54.17, 6.67)$ compared to the allocation by percentages of $(35, 45, 20)$. Players A, B will receive more at the expense of C who is owed the least.

Shapley vectors can also quickly analyze the winning coalitions in games where that is all we care about: who do we team up with to win. Here are the definitions.

Definition 4.2.2 *Suppose we are given a normalized characteristic function $v(S)$ which satisfies that for every $S \subset N$, either $v(S) = 0$ or $v(S) = 1$. This is called a simple game. If $v(S) = 1$, the coalition S is said to be a winning coalition. Let $W^i = \{S \in \Pi^i | v(S) = 1, v(S - i) = 0\}$, the set of coalitions who win with player i and lose without player i.*

In the case of a simple game for player i we need only consider coalitions $S \in \Pi^i$ for which S is a winning coalition, but $S - i$, that is, S without i is a losing coalition. This is because $v(S) - v(S - i) = 1$ only when $v(S) = 1$, and $v(S - i) = 0$. In all other cases $v(S) - v(S - i) = 0$. Hence, the Shapley value for a simple game is

$$x_i = \sum_{\{S \in \Pi^i\}} [v(S) - v(S - i)] \frac{(|S| - 1)!(n - |S|)!}{n!}$$
$$= \sum_{\{S \in W^i\}} \frac{(|S| - 1)!(n - |S|)!}{n!}$$

The Shapley allocation for player i represents the power that player holds in a coalition.

Example 4.13. A corporation has 4 stockholders who each vote their shares on any major decision. The majority of shares voted decides an issue. A majority consists of more than 50 shares. Suppose the holdings of each stockholder are as follows:

Player	1	2	3	4
Shares	10	20	30	40

The winning coalitions, i.e., with $v(S) = 1$ are

$$W = \{24, 34, 123, 124, 234, 1234\}$$

We find the Shapley allocation. For x_1, $W^1 = \{123\}$ because $S = \{123\}$ is winning but $S - 1 = \{23\}$ is losing. Hence,

$$x_1 = \frac{(4 - 3)!(3 - 1)!}{4!} = \frac{1}{12}.$$

Similarly, $W^2 = \{24, 123, 234\}$ and so

$$x_2 = \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{1}{4}.$$

Also, $x_3 = 1/4$ and $x_4 = 5/12$. Hence the Shapley allocation is $\mathbf{x} = (1/12, 3/12, 3/12, 5/12)$. Notice that player 1 has the least power, but players 2 and 3 have the same power even though player 3 controls 10 more shares than does player 2. Player 5 has the most power but a coalition is still necessary to constitute a majority.

Continuing this example but changing the shares to

Player	1	2	3	4
Shares	10	30	30	40

We compute the Shapley value as $x_1 = 0, x_2 = x_3 = x_4 = 1/3$ and we see that player 1 is completely marginalized as he doesn't contribute anything to any coalition. He has no power. In addition, player 4's additional shares over players 2 and 3 provide no advantage over those players since a coalition is essential to carry a majority in any case.

Example 4.14. In this example we show how cooperative game theory can determine a fair allocation of taxes to a community. For simplicity assume that there are only 4 households and that the community requires expenditures of$ 10,000. The question is how to allocate the cost of the $ 10,000 among the 4 families. So, as in most communities we consider the wealth of the family as represented by the value of their property. Suppose the wealth of household i is w_i, with specific values $w_1 = 50K, w_2 = 75K, w_3 = 100K, w_4 = 200K$. In addition, suppose that there is a cap on the amount each household will have to pay (based on age or income or some other extraneous factors) which is independent of the value of their property. In our case we take the maximum amount each of the 4 households will be willing to pay as $u_1 = 5K, u_2 = 4K, u_3 = 6K, u_4 = 8K$. What should each family fairly have to pay?

Let us consider the general problem:

T	total cost
u_i	maximum amount i will have to pay
w_i	net worth of player i
z_i	amount player i will have to pay
$u_i - z_i$	surplus of the cap over the assessment

We will assume that

$$\sum_{i=1}^{n} w_i > T, \ \sum_{i=1}^{n} u_i > T, \text{ and } \sum_{i=1}^{n} z_i = T.$$

Here is the characteristic function we will use:

$$v(S) = \begin{cases} \left(\sum_{i \in S} u_i - T \right)^+, & \text{if } \sum_{i \in S} w_i \geq T; \\ 0, & \text{if } \sum_{i \in S} w_i < T. \end{cases}$$

In other words, if the total wealth in coalition S is less than the total cost, or if the sum of the caps over the coalition S is less than T, then the benefit of that coalition is 0. Otherwise, the benefit of a coalition is the surplus over the cost of the total sum of the caps of the coalition. Under our previous assumption we can simplify to

$$v(S) = \sum_{i \in S} u_i - T.$$

Next we compute for a given coalition S and $j \in S$, assuming that $v(S) > 0$ and $v(S-j) > 0$,

$$v(S) - v(S-j) = \sum_{i \in S} u_i - T - \left(\sum_{i \in S, i \neq j} u_i - T \right) = u_j.$$

Assuming $v(S) > 0, v(S-j) = 0$,

$$v(S) - v(S-j) = \sum_{i \in S} u_i - T.$$

Summarizing we have

$$v(S) - v(S-j) = \begin{cases} u_j, & \text{if } v(S) > 0, v(S-j) > 0; \\ \sum_{i \in S} u_i - T, & \text{if } v(S) > 0, v(S-j) = 0; \\ 0, & \text{if } v(S) = v(S-j) = 0. \end{cases}$$

We are ready to compute the Shapley allocation. We have, for player $j = 1, \ldots, n$,

$$\begin{aligned} x_j = & \sum_{\{S | j \in S, v(S-j) > 0\}} u_j \frac{|S|!(n-|S|)!}{n!} \\ & + \sum_{\{S | j \in S, v(S) > 0, v(S-j) = 0\}} \left(\sum_{k \in S} u_k - T \right) \frac{|S|!(n-|S|)!}{n!} \end{aligned} \tag{4.2.3}$$

Observe that x_j represents the amount *of the surplus of the amount j is willing to pay* to be allocated to player j. Consequently, the amount player j will be billed is actually $z_j = u_j - x_j$.

For the 4 person problem data above we have $T = \$10,000, \sum w_i = 425 > 10, \sum_i u_i = 23 > 10$. so all our assumptions are verified. Then we have

$$\begin{aligned} & v(i) = 0, v(12) = v(23) = 0, v(13) = 1000, v(14) = 3000, \\ & v(24) = 2000, v(34) = 4000, \\ & v(123) = 5000, v(134) = 9000, v(234) = 8000, \\ & v(124) = 7000, v(1234) = 13000. \end{aligned}$$

For example $v(123) = (u_1 + u_2 + u_3 - 10)^+ = 15 - 10 = 5$. We compute

$$\begin{aligned} x_1 = & \frac{1!}{2!} 4! \left[(v(13) - v(3)) + (v(14) - v(4)) \right] + \frac{2!1!}{4!} [v(123) - v(23)] \\ & + \frac{2!1!}{4!} [v(134) - v(34) + v(124) - v(24)] \\ & + \frac{3!0!}{4!} [v(1234) - v(234)] \\ = & \frac{34000}{12}. \end{aligned}$$

Then, the amount player 1 will be billed is $z_1 = u_1 - x_1 = 5000 - \frac{34000}{12} = 2166.67$. In a similar way we calculate

$$x_2 = 2166.67 \quad x_3 = 3333.33, \quad \text{and } x_4 = 4666.67$$

so that

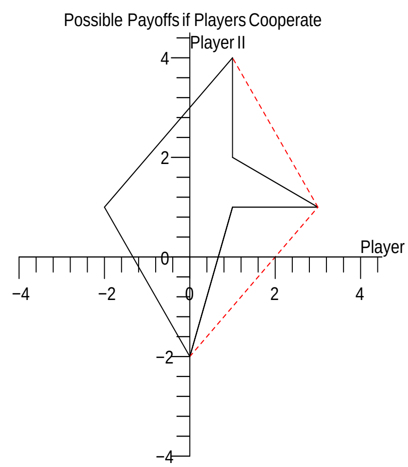
$$z_2 = 4000 - 2166.67 = 1833.33, \quad z_3 = 6000 - 3333.33 = 2666.67, \\ \text{and} \quad z_4 = 8000 - 4666.67 = 3333.33.$$

4.3 Bargaining

In this section we will introduce a new type of cooperative game in which the players bargain to improve both of their payoffs. Let us start with a simple example to illustrate the benefits of bargaining and cooperation. Consider the 2 player nonzero sum game with bimatrix

	II_1	II_2	II_3
I_1	$(1,4)$	$(-2,1)$	$(1,2)$
I_2	$(0,-2)$	$(3,1)$	$(1,1)$

We will draw the points of this game on a graph and connect the payoffs with straight lines.



The vertices of the polygon are the payoffs from the matrix. The solid lines connect the payoffs. If the players play mixed strategies then any payoff in the region bounded by the solid lines are achievable. The dotted lines extend the region of achievable payoffs. Now, if the players do not cooperate, they will achieve one of two possibilities. (1) The vertices of the figure, if they play pure strategies, or (2) Any point along the solid lines in the figure, if they play mixed strategies. However, if the players agree to cooperate, then any point on the boundary of the extended figure, including the dotted lines, are achievable payoffs.

For example, suppose that player I always chooses row 2, I_2 and player II plays the mixed strategy $Y = (y_1, y_2, y_3)$, where $y_i \geq 0, y_1 + y_2 + y_3 = 1$. The expected payoff to I is then

$$E_I(I_2, Y) = 0\ y_1 + 3y_2 + 1\ y_3,$$

and the expected payoff to II is

$$E_{II}(I_2, Y) = -2y_1 + 1y_2 + 1y_3.$$

Hence,

$$(E_I, E_{II}) = y_1(0, -2) + y_2(3, 1) + y_3(1, 1) = (1, 1) + y_1(-1, -3) + y_2(2, 0),$$

which, as a linear combination of the 3 points is in the convex hull of the 3 points. The *convex hull* of a set of points is the smallest convex set containing all the points. So any point in the convex hull of the payoff points is achievable if the players agree to cooperate. This defines the feasible set:

Definition 4.3.1 *The* **feasible set** *is the set of the convex hull of all the payoff points corresponding to pure strategies of the players.*

The objective of Player I is to obtain a payoff as far to the right as possible in the figure and the objective of player II is to obtain a payoff as far to the north as possible in the figure. Player I's ideal payoff is at the point $(3, 1)$, but that is attainable only if II agrees to play II_2. Why would he do that? Similarly, II would do best at $(1, 4)$, which is only going to happen if I plays I_1, and why would he do that? So, there is an incentive for the players to reach a compromise agreement in which they would agree to play in such a way so as to obtain a payoff along the line connecting $(1, 4)$ and $(3, 1)$.

In any bargaining problem there is always the possibility that negotiations will fail. Hence, each player must know what their payoff will be if there was no bargaining.

Definition 4.3.2 *The status quo payoff point, or safety point, in a 2 person game is the pair of payoffs (u^*, v^*) which each player can achieve if there is no cooperation.*

Example 4.15. In the example above we will determine the status quo point for each player.

Consider the payoff matrix for player I:

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 3 & 1 \end{pmatrix}$$

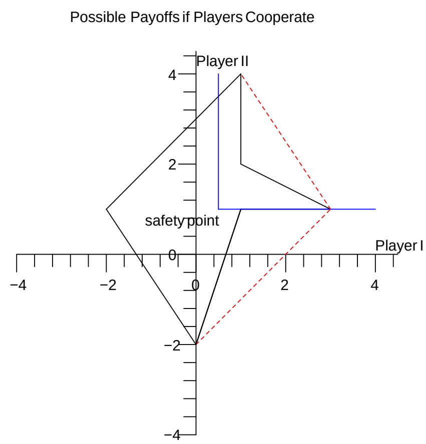
We want the value of the game with matrix A. By the methods of chapter one we find that $v(A) = 1/2$ and the optimal strategies are $Y = (0, 1/6, 5/6)$ for player II and $X = (1/2, 1/2)$ for player I.

Next we consider the payoff matrix for player II. We call this matrix B but since we want to find the value of the game from player II's point of view we actually need to work with B^T since it is always the row player who is the maximizer (and II is trying to maximize his payoff). So,

$$B^T = \begin{pmatrix} 4 & -2 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

For this matrix $v(B^T) = 1$ and we have a saddle point at row 3 column 2.

We conclude that the status quo point for this game is $(I, II) = (1/2, 1)$ since that is the guaranteed payoff to each player without cooperation or negotiation. This means that any bargaining must begin with the guaranteed payoff pair $(1/2, 1)$. This cuts off the feasible set as in the figure.



The new feasible set consists of the points above and to the right of the blue lines emanating from the safety point. It is like moving the origin to the new safety point. The question now is to find the cooperative, negotiated best payoff for each player.

How does cooperation help? Well, suppose, for example, the players agree as follows: I will play I_1 1/2 the time and I_2 1/2 the time as long as II plays II_1 1/2 the time and II_2 1/2 the time. This is not optimal for player II in terms of his safety level. But, if they agree to play this way, they will get

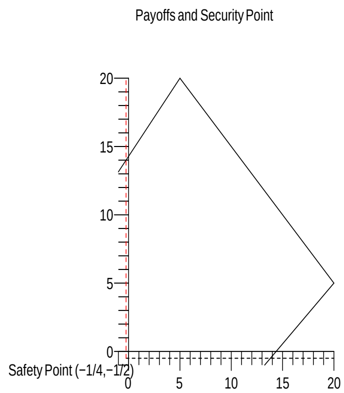
$$\frac{1}{2}(1, 4) + \frac{1}{2}(3, 1) = (2, 5/2).$$

So I gets $2 > 1/2$ and II gets $5/2 > 1$, a big improvement for each player over their safety levels. So, they definitely have an incentive to cooperate.

Example 4.16. Here is another example. The bimatrix is

	II_1	II_2
I_1	$(5, 20)$	$(-7, -19)$
I_2	$(-16, -4)$	$(20, 5)$

The reader can calculate the safety level $(I, II) = (-1/4, -1/2)$, so negotiations start from that point. The figure has the picture for this problem.



It appears that a negotiated set of payoffs will benefit both players and will be on the line farthest to the right. Player I would love to get $(20, 5)$ while player II would love to get $(5, 20)$. That probably won't happen but they could negotiate a point along the line connecting these two points and compromise on obtaining, say the midpoint

$$\frac{1}{2}(5, 20) + \frac{1}{2}(20, 5) = (12.5, 12.5).$$

Now suppose that player II threatens player I by saying that he will always play strategy II_1 unless I cooperates. $II's$ goal is to get the 20 if and when I plays I_1, so I would receive 5. Of course I does not have to play I_1 but if he doesn't, then I will get -16, and II will get -4. So, if I does not cooperate and II carries out his threat they will both lose but I will lose much more than II. Therefore, II is in a much stronger position than I in this game and he can essentially *force* I to cooperate. This implies that the safety level of $(-1/4, -1/2)$ loses it's effect here because II has a credible threat he can use to force I to cooperate. This also seems to imply that maybe II should expect to get more than 12.5 to reflect his stronger bargaining position from the start.

The preceding example indicates that there may be a more realistic choice for a safety level than the values of the associated games, taking into account various threat possibilities. We will see that this is indeed the case.

4.3.1 The Nash model with security point

We start with a security status quo point (u^*, v^*) for a 2 player cooperative game with matrices A, B. This leads to a feasible set of possible negotiated outcomes depending on the point we start from (u^*, v^*). For any given such point and feasible set S, we are looking for a negotiated outcome, call it $(\overline{u}, \overline{v})$. This point will depend on (u^*, v^*) and the set S, so we may write

$$(\overline{u}, \overline{v}) = f(S, u^*, v^*).$$

The question is how to determine the point $(\overline{u}, \overline{v})$? John Nash proposed the following requirements for the point to be a negotiated solution:

- Axiom 1. We must have $\overline{u} \geq u^*$ and $\overline{v} \geq v^*$. Each player must get at least the security point.
- Axiom 2. The point $(\overline{u}, \overline{v}) \in S$, i.e., it must be a feasible point.
- Axiom 3. If (u, v) is any point in S, so that $u \geq \overline{u}$ and $v \geq \overline{v}$, then it must be the case that $u = \overline{u}, v = \overline{v}$. In other words, there is not other point in S, where *both* players receive more. This is *Pareto optimality.*
- Axiom 4. If $(\overline{u}, \overline{v}) \in T \subset S$ and $(\overline{u}, \overline{v})$ is the solution to the bargaining problem with feasible set T, then for the larger feasible set S, either $(\overline{u}, \overline{v})$ is the bargaining solution for S, or the actual bargaining solution for S is in $S - T$. We are assuming that the security point is the same for T and S. So, if we have more alternatives, the new negotiated position can't be one of the old possibilities.
- If T is an affine transformation of $S, T = \varphi(S)$ and $(\overline{u}, \overline{v})$ is the bargaining solution of S with security point (u^*, v^*), then $\varphi(\overline{u}, \overline{v})$ is the bargaining solution associated with T and security point $\varphi(u^*, v^*)$.
- Axiom 6. If the game is symmetric with respect to the players, then so is the bargaining solution. That is, if $(\overline{u}, \overline{v}) = f(S, u^*, v^*)$ and (i) $u^* = v^*$, (ii) $(u, v) \in S \Rightarrow (v, u) \in S$, then $\overline{u} = \overline{v}$.

The amazing thing is that Nash proved that if we assume these axioms, there is one and only one solution of the bargaining problem. In addition, the theorem gives a constructive way of finding the bargaining solution.

Theorem 4.3.3 *Consider the nonlinear programming problem*

$$\textit{Maximize } g(u, v) := (u - u^*)(v - v^*)$$
$$\textit{Subject to } (u, v) \in S, u \geq u^*, v \geq v^*.$$

Assume there is at least one point $(u, v) \in S$ with $u > u^*, v > v^*$. There exists one and only one point $(\overline{u}, \overline{v}) \in S$ which solves this problem and this point is the unique solution of the bargaining problem $(\overline{u}, \overline{v}) = f(S, u^*, v^*)$.

Proof. 1. *Existence.* The set

$$S^* = \{(u, v) \in S | u \geq u^*, v \geq v^*\},$$

is convex, closed, and bounded. Since $g : S^* \to \mathbb{R}$ is continuous, a theorem of analysis (any continuous function on a closed and bounded set achieves a maximum and a minimum on the set) guarantees that g has a maximum at some point $(\overline{u}, \overline{v}) \in S^*$. By assumption there is at least one feasible point with $u > u^*, v > v^*$. For this point $g(u, v) > 0$ and so the maximum of g over S^* must be > 0 and so does not occur at $u = u^*$ or $v = v^*$.

2. *Uniqueness.* Suppose the maximum $0 < M = g(u', v') = g(u'', v'')$. If $u' = u''$ then $(u' - u^*)(v' - v^*) = (u'' - u^*)(v'' - v^*)$ so that $v' = v''$ as well. So we may assume that $u' < u''$ and that implies $v' > v''$ because $(u' - u^*)(v' - v^*) = (u'' - u^*)(v'' - v^*) = M$.

Set $(u, v) = \frac{1}{2}(u', v') + \frac{1}{2}(u'', v'')$. Since S is convex, $(u, v) \in S$ and $u > u^*, v > v^*$. So $(u, v) \in S^*$. Some simple algebra shows that

$$g(u, v) = M + \frac{(u' - u'')(v'' - v')}{4} > M.$$

This contradicts the fact that (u', v') provides a maximum for g over S^*.

3. Consider the straight line passing through $(\overline{u}, \overline{v})$ and the security point (u^*, v^*). It has equation

$$v = m\, u + (\overline{v} - m\, \overline{u})$$

where $m = \frac{\overline{v} - v^*}{\overline{u} - u^*} u$ is the slope. Now consider the new line with slope $-m$ and passing through the point $(\overline{u}, \overline{v})$. This line has equation

$$v - \overline{v} = -m(u - \overline{u}). \tag{4.3.1}$$

We claim that every point in $(u, v) \in S$ lies on or below the line and so the line is a support line for the convex set S. We verify this claim later.

4. It is not hard to show that the unique point $(\overline{u}, \overline{v}) := f(S, u^*, v^*)$ maximizing g over S^* satisfies the axioms and we leave that for the reader as an exercise. We will show it is the unique solution which satisfies them.

Set $h(u, v) = (\overline{v} - v^*)u + (\overline{u} - u^*)v$ and the closed half plane

$$H := \{(u, v) | h(u, v) \leq h(\overline{u}, \overline{v})\}$$

We know that $S \subset H$ since the line is a support line. Consider the linear transformation $\varphi : (u, v) \in H \to (u', v') \in H'$ given by

$$u' = \frac{u - u^*}{\overline{u} - u^*}, \quad v' = \frac{v - v^*}{\overline{v} - v^*}.$$

One can check that $H' = \{(u', v') | u' + v' \leq 2\}$. Notice that

$$\varphi(u^*, v^*) = (0, 0) \text{ and } \varphi(\overline{u}, \overline{v}) = (1, 1).$$

Now we look at the *symmetric* bargaining problem with $S = H'$ and security point $(0, 0)$. Let $(u', v') = f(H', 0, 0)$ be the solution. By symmetry $u' = v'$ so

$u' + v' = 2 \Rightarrow u' = v' = 1$and since $(1,1)$ is a solution, by Pareto optimality it follows that $(1,1)$ is the one and only solution so $(1,1) = f(H', 0, 0)$. Now we apply the inverse of the transformation φ to conclude that $(\overline{u}, \overline{v}) = \varphi^{-1}(1,1)$ is the unique solution of (H, u^*, v^*). Since $(\overline{u}, \overline{v}) \in S \subset H$ it must be true that $(\overline{u}, \overline{v})$ is the unique solution of (S, u^*, v^*). $\square$

Example 4.17. For the problem with security point $(u^*, v^*) = (-1/4, -1/2)$ we have the nonlinear programming problem

$$\begin{gathered}\text{Maximize } (u + 1/4)(v - 1/2)\\ \text{Subject to } u \geq -1/4, v \geq 1/2, v \leq -u + 25,\\ v \geq 24/27u - 12.77778,\ v \leq 24/21 + 14.2857\end{gathered}$$

The maple commands to solve this are:

```
> with(Optimization):
> NLPSolve(u+1/4)*(v-1/2),{u>=-1/4,v>=1/2,v<=-u+25,
v>=24/27*u-12.777778,v<=24/21*u+14.2857},maximize);
```

This gives the solution $g(\overline{u}, \overline{v}) = 153.14, \overline{u} = 12.125, \overline{v} = 12.875$. We do NOT get the point we thought, namely $(12.5, 12.5)$. That is due to the fact that the security point is not symmetric.

4.3.2 Threats

Negotiations of the type considered in the previous section do not take into account the relative strength of the positions of the players in the negotiations. For instance, a simplified version of strike negotiations might go as follows.

Labor may work for minimum subsistence level wages and no benefits or choose to strike. Management may choose the strategy of not acceding to labors demands and watching labor strike, or deciding to pay a decent wage with decent benefits. This could be a reasonable representation of the payoffs: The bimatrix with I =labor and II =management is

	$II_1(Pay)$	$II_2(Stand)$
$I_1(Work)$	$(10, 5)$	$(1, 20)$
$I_2(Strike)$	$(15, 15)$	$(-100, -4)$

Player I's payoff matrix is

$$A = \begin{pmatrix} 10 & 1 \\ 15 & -100 \end{pmatrix}$$

and for II

$$B = \begin{pmatrix} 5 & 20 \\ -4 & 0 \end{pmatrix} \text{ so we look at } B^T = \begin{pmatrix} 5 & -4 \\ 20 & 0 \end{pmatrix}$$

It is left to the reader to verify that $value(A) = 1, value(B^T) = 0$ so the security point is $(u^*, v^*) = (1, 0)$.

In a threat game we replace the security levels (u^*, v^*), which in the preceding sections was the value of the associated games $u^* = value(A), v^* = value(B^T)$, with the expected payoffs to each player if the threat strategies are used.

Suppose that in the bimatrix game player I has a threat strategy X_t and player II has a threat strategy Y_t. The new status quo point will be the expected payoffs to the players if they both use their threat strategies:

$$u^* = {X_t}^T A Y_t, \text{ and } v^* = {X_t}^T B^T Y_t.$$

Then we return to the cooperative game and apply the same procedure as before but with the new threat security point. That is, we seek to

$$\begin{gathered}\text{Maximize } g(u,v) := (u - {X_t}^T A Y_t)(v - {X_t}^T B^T Y_t)\\ \text{Subject to } (u,v) \in S, u \geq {X_t}^T A Y_t, v \geq {X_t}^T B^T Y_t.\end{gathered}$$

The question is how to pick the threat strategies? For the time being we put that question aside and assume we have a security point (u^*, v^*). We look at the example to see how to solve this problem.

Example 4.18. Start with the bimatrix

	II_1	II_2
I_1	$(-3, 0)$	$(-1, -2)$
I_2	$(2, 1)$	$(1, 3)$

To begin we find the values of the associated matrices

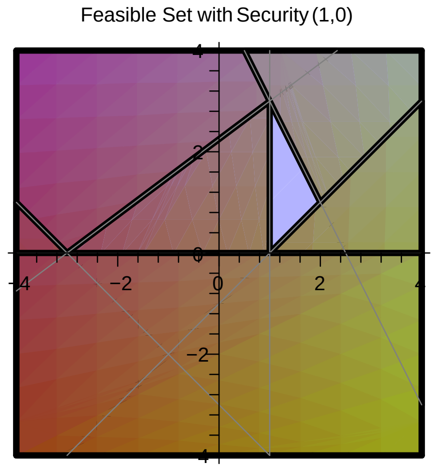
$$A = \begin{pmatrix} -3 & -1 \\ 2 & 1 \end{pmatrix}, \text{ and } B^T = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$$

Then, $value(A) = 1$ with a saddle at $X = (0, 1), Y = (0, 1)$ and $value(B^T) = 0$ with a saddle at $X = (1, 0) = Y$. Hence the security point is $(u^*, v^*) = (1, 0)$. The feasible set taking into account the security point is

$$S^* = \{(u,v) | u \geq 1, v \geq 0, v \leq -2u + 5, v \geq u - 1, v \geq -u - 3, v \leq \frac{3}{4}u + \frac{9}{4}\}$$

The nonlinear programming problem is then

$$\begin{gathered}\text{Maximize } g(u,v) := (u - 1)v\\ \text{Subject to } (u,v) \in S^*.\end{gathered}$$



We get the solution $\overline{u} = 1.75, \overline{v} = 1.5$. This is on the line $v = -2u + 5$, and this line is the *Pareto optimal boundary* for this problem. In the figure, notice that if you take any point on this line, *both* players cannot do strictly better at the same time and still stay in S^*. The line through $(1.75, 1.5)$ and $(1, 0)$ has slope $1.5/.75 = 2$ and the line of the Pareto optimal boundary is -2. The function g has increasing contour curves which will look like hyperbolas moving to the right and upward and the last contour curve which touches S^* must touch at the Pareto optimal boundary at $\overline{u}, \overline{v}$.

The conclusion is that with this security point, I receives the negotiated solution $\overline{u} = 1.75$ and II the amount $\overline{v} = 1.5$.

How should they cooperate in order to achieve these solutions? To find out, the only points in the bimatrix which are of interest are the endpoints of the Pareto optimal boundary, namely, $(1, 3)$ and $(2, 1)$. So the cooperation must be a linear combination of the strategies yielding these payoffs. Solve

$$(1.75, 1.5) = \lambda(1, 3) + (1 - \lambda)(2, 1),$$

to get $\lambda = 1/4$. This says that (I, II) must play $(row2, col2)$ with probability $1/4$ and $(row2, col1)$ with probability $3/4$.

Threat security point:

The Pareto optimal boundary for this problem is the line segment $v = -2u + 5, 1 \leq u \leq 2$. This line has slope $m = -2$. Consider a line with slope $-m = 2$ through any possible threat security point (u^*, v^*) then the line will intersect the Pareto optimal boundary line at the possible negotiated solution $(\overline{u}, \overline{v})$. The equation of the line through (u^*, v^*), whatever the point is, with slope $-m$ has equation

$$v - v^* = -m(u - u^*).$$

The equation of the Pareto optimal boundary line is $v = -2u + 5$ so the intersection point of the two lines will be at the coordinates

$$\overline{u} = \frac{m\ u^*+v^*-5}{m-2} = \frac{-2u^*+v^*-5}{-4}, \text{ and}$$
$$\overline{v} = \frac{5*m-2(m\ u^*+v^*)}{m-2} = \frac{-10-2(-2u^*+v^*)}{-4}.$$

Now, remember, we are trying to find the best threat strategies to use but fundamentally, the players want to maximize their payoffs $\overline{u}, \overline{v}$. But that tells us exactly what to do for the threats. Player I will maximize $\overline{u}$ if he chooses threat strategies to maximize $m\ u^* + v^* = -2 * u^* + v^*$ and player II will maximize $\overline{v}$ if he chooses threat strategies to *minimize* the same quantity $m\ u^* + v^*$ because there is a minus sign in front of this quantity for $\overline{v}$. So here is the procedure for finding u^*, v^* and the optimal threat strategies:

1. Identify the Pareto optimal boundary and find the slope of that line, call it m.
2. Construct the new matrix for a zero sum game
$$-m * u^* - v^* = -m(X_t A Y_t^T - X_t B Y_t^T) = X_t(-mA - B)Y_t^T$$
with matrix $-mA - B$.
3. Find the optimal strategies X_t, Y_t for that game and compute $u^* = X_t A Y_t^T$ and $v^* = X_t B Y_t^T$. This (u^*, v^*) is the threat status quo point to be used to solve the bargaining problem.
4. Once we know (u^*, v^*) we may use the formulas above for $(\overline{u}, \overline{v})$.

Carrying out these steps for our example, we find

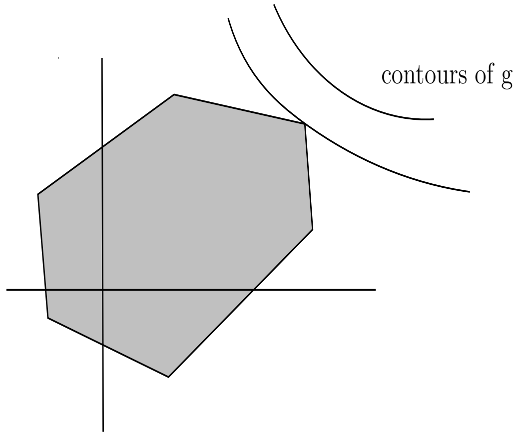
$$2A - B = \begin{pmatrix} -6 & 0 \\ 3 & -1 \end{pmatrix}$$

We find $value(2A-B) = -60$ and optimal threat strategies $X_t = (.4, .6), Y_t = (.1, .9)$. Then $u^* = X_t A Y_t^T = .18$, and $v^* = X_t B Y_t^T = .96$. Once we know that we can use the formulas above to get

$$\overline{u} = \frac{-2(.18)+.96-5}{-4} = .11, \text{ and}$$
$$\overline{v} = \frac{-10-2(-2(.18)+.96)}{-4} = .28$$

The example gives us a general procedure to solve for the threat strategies. Notice however that several things can make this procedure more complicated. First, the determination of the Pareto optimal boundary of S is of critical importance. In the example it consisted of only one line segment, but in practice there may be many such line segments and we have to work separately with each segment. That is because we need the slopes of the segments. This means that the threat strategies and the threat point u^*, v^* could change from segment to segment. An example will illustrate this.

Another problem is that the Pareto optimal boundary could be a point of intersection of two segments so there is no slope for the point. Then what do we do?



g is maximized at an intersection point

Summary approach for Bargaining with Threat Strategies.

1. Identify the Pareto optimal boundary of S and find the slope of that line, call it $m < 0$. Suppose the end point (u_B, v_B) is on the line.
2. Construct the new matrix for a zero sum game $-mA - B$ associated by

$$-m * u^* - v^* = -m(X_t A Y_t^T - X_t B Y_t^T) = X_t(-mA - B)Y_t^T.$$

3. Find the optimal strategies X_t, Y_t for that game with matrix $-mA - B$ and then compute $u^* = X_t A Y_t^T$ and $v^* = X_t B Y_t^T$. This (u^*, v^*) is the threat status quo point to be used to solve the bargaining problem.
4. Once we know (u^*, v^*) we may use the formulas for $(\overline{u}, \overline{v})$ which is the point of intersection of the line through (u^*, v^*) with slope $-m$ and the line through (u_B, v_B) with slope m on the Pareto optimal boundary. This formula is

$$\overline{u} = \frac{v_B - v^* - m(u_B + u^*)}{-2m}, \text{ and } \overline{v} = \frac{v_b + v^* - m(u_B - u^*)}{2}.$$

Alternatively we may simply solve the nonlinear programming problem using Maple.

Example 4.19. Consider the bimatrix game

	II_1	II_2	II_3
I_1	(1,4)	(-2,0)	(4,1)
I_2	(0,-1)	(3,3)	(-1,4)

1. We find the security status quo point first. The matrices for the players are

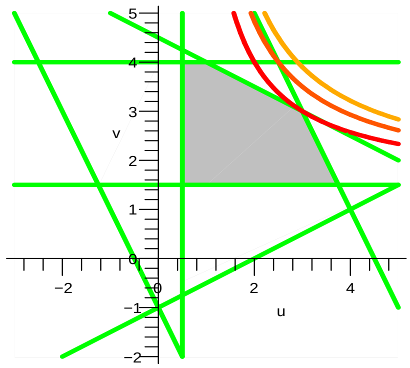
$$A = \begin{pmatrix} 1 & -2 & 4 \\ 0 & 3 & -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 4 & 0 & 1 \\ -1 & 3 & 4 \end{pmatrix}$$

Then, $value(A) = 1/2, value(B^T) = 3/2$ so the status quo security level is $(u^*, v^*) = (1/2, 3/2)$.

The set of constraints with this security level is

$$\begin{aligned} S^* := \{(u,v) | u \geq 1/2, v \geq 3/2, v \leq 4, v-4 \leq (-1/2)(u-1), \\ v-3 \leq (-2)(u-3), v-1 \geq (2/4)(u-4), v+1 \geq (-2)u\} \end{aligned}$$

Applying the nonlinear programming method to the objective function $g(u,v) = (u-1/2)(v-3/2)$ subject to the constraints $(u,v) \in S^*$, we get that the maximum of g is achieved at $(\overline{u}, \overline{v}) = (3,3)$ and the maximum is $g(\overline{u}, \overline{v}) = 3.75$. One can see from the figure that the contours of g just touch at the point $(3,3)$ before they leave the set S^*.



2. Now we look for optimal threat strategies.

Example 4.20. Consider the cooperative game with bimatrix

	II_1	II_2
I_1	(-1,1)	(1,1)
I_2	(2,-2)	(-2,2)

So the individual matrices are

$$A = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \text{ and } B = \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}$$

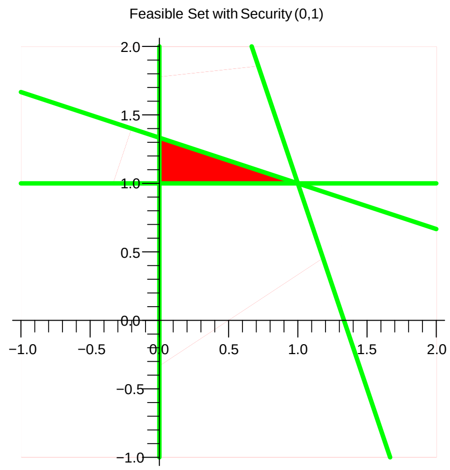
It is easy to calculate that $value(A) = 0, value(B^T) = 1$ and so the status quo security point for this game is at $(u^*, v^*) = (0, 1)$. The problem we then need to solve is

$$\text{Maximize } u(v-1), \text{ Subject to } (u,v) \in S^*,$$

where

$$S^* = \{(u,v) | v \leq (-1/3)u + 4/3, v \leq -3u + 4, u \geq 0, v \geq 1\}$$

The solution of this problem is at the unique point $(\overline{u}, \overline{v}) = (1/2, 7/6)$.



Next, to find the threat strategies we note that $m = -3$ so we look for the value of the game with matrix $3A - B$. This is

$$3A - B = \begin{pmatrix} -2 & 2 \\ 8 & -8 \end{pmatrix}$$

Then $value(3A - B) = 0$, and the optimal threat strategies are $X_t = (1/2, 1/2) = Y_t$. Then the status quo threat points are

$$u^* = X_t A {Y_t}^T = 0 \text{ and } v^* = X_t B {Y_t}^T = 0.$$

This means that each player threatens to use (X_t, Y_t) and receive 0 rather than cooperate and receive more.

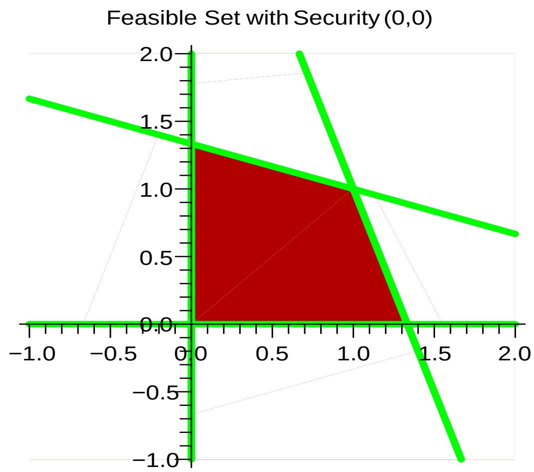
Now the maximization problem becomes

$$\text{Maximize } uv, \text{ Subject to } (u,v) \in S^*,$$

where

$$S^* = \{(u,v) | v \leq (-1/3)u + 4/3, v \leq -3u + 4, u \geq 0, v \geq 0\}$$

The solution of this problem is at the unique point $(\overline{u}, \overline{v}) = (1, 1)$.


Feasible Set with Security (0,0)
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0