THE LITTELMANN PATH MODEL

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1. Littlewood–Richardson rule

Set $\mathfrak{g} = \mathfrak{sl}_{n+1}$. Write $\lambda = \sum_i \lambda_i \varepsilon_i$ ($\lambda_1 \ge \lambda_2 \ge \cdots$) and $\mu = \sum_i \mu_i \varepsilon_i$ with $\mu_1 \ge \mu_2 \ge \cdots$. Then we have a decomposition

$$B(\lambda) \otimes B(\mu) = \bigoplus_{[j_1]\otimes\cdots[j_N]\in\mu} B(\lambda[j_1, \dots, j_N]),$$

where $\lambda[j_1, \dots, j_r]$ is obtained by adding a box at the j_rth row to $\lambda[j_1, \dots, j_{r-1}]$. This term is 0 if the result is not a Young tableau.

2. Path model

Now let $\mathfrak{g}$ be any Kac–Moody algebra. Let P be the weight lattice, $P_{\mathbb{R}} = P \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition 2.1. A **path** is a piecewise linear continuous map $\pi\colon [0,1] \to P_{\mathbb{R}}$. We say that $\pi_1 = \pi_2$ if there exists a surjective nondecreasing continuous function $p\colon [0,1] \to [0,1]$ such that $\pi_1 = \pi_2 \circ p$. Define

$$\Pi = \{\text{paths } \pi \mid \pi(0) = 0,\ \pi(1) \in P\}.$$

The **weight** of a path π is $\mathrm{wt}(\pi) = \pi(1)$.

Given a simple root α_i, let s_i be the corresponding simple reflection. Let

$$h = \min(\mathbb{Z} \cap \{\langle \pi(t), \alpha_i^\vee \rangle \mid t \in [0,1]\}).$$

If $h \ge 0$, define $\widetilde{e}_i \pi = 0$. If $h < 0$, let

$$t_1 = \min\{t \mid \langle \pi(t), \alpha_i^\vee \rangle = h\}$$
$$t_0 = \max\{t < t_1 \mid \langle \pi(t), \alpha_i^\vee \rangle = h + 1\}.$$

Define

$$\widetilde{e}_i \pi = \begin{cases} \pi(t) & t \le t_0 \\ \pi(t_0) + s_i(\pi(t) - \pi(t_0)) & t_0 \le t \le t_1 \\ \pi(t) + \alpha_i & t_1 \le t \end{cases}.$$

Define the path $\pi^\vee$ by $\pi^\vee(t) = \pi(1-t) - \pi(1)$ and set $\widetilde{f}_i \pi = (\widetilde{e}_i(\pi^\vee))^\vee$. □

Theorem 2.2. $(\Pi, \widetilde{e}, \widetilde{f}, \mathrm{wt})$ *is a (combinatorial) crystal.*

Recall the definition of the dominant weights

$$P^+ = \{\lambda \in P \mid \langle \lambda, \alpha_i^\vee \rangle \ge 0 \text{ for all } i\},$$

and the dominant chamber

$$P^+_{\mathbb{R}} = \{\lambda \in P_{\mathbb{R}} \mid \langle \lambda, \alpha_i^\vee \rangle \ge 0 \text{ for all } i\}.$$

Define Π^+ to be the set of paths that lie entirely in $P^+_{\mathbb{R}}$. For $\pi \in \Pi^+$, define

$$B_\pi = \{\widetilde{f}_{i_1} \cdots \widetilde{f}_{i_r} \pi \mid i_1, \dots, i_r \in I\}.$$

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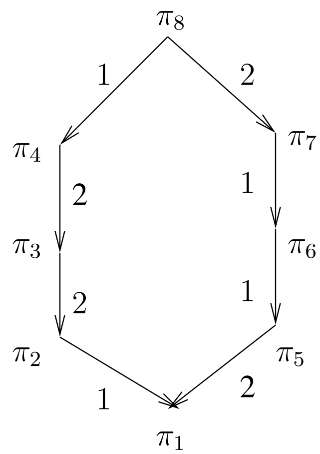

FIGURE 1. $\mathfrak{sl}_3$ adjoint representation

Theorem 2.3. (1) *For $\pi, \pi' \in \Pi^+$, $B_\pi \cong B_{\pi'}$ if and only if $\pi(1) = \pi'(1)$.*
(2) *For $\lambda \in P^+$, define $\pi_\lambda \colon [0,1] \to P^+_{\mathbb{R}}$ by $t \mapsto t\lambda$. Then $B(\lambda) \cong B_{\pi_\lambda}$.*

Example 2.4 (Adjoint representation of $\mathfrak{sl}_3$)**.** Let $\mathfrak{g} = \mathfrak{sl}_3$ and let α_1, α_2 be the simple roots. The lowest weight is $-\alpha_1 - \alpha_2$, so let π_1 be the path $t \mapsto t(-\alpha_1 - \alpha_2)$.

$\widetilde{e}_1\pi_1 = \pi_2$ is the path $t \mapsto -t\alpha_2$. $\widetilde{e}_2\pi_2 = \pi_3$ is the path $t \mapsto -t\alpha_2$ for $0 \le t \le 1/2$ and $t \mapsto -(1-t)\alpha_2$ for $1/2 \le t \le 1$. Similarly,

$$\begin{aligned}
&\widetilde{e}_2\pi_3 = \pi_4 \colon t \mapsto t\alpha_2,\\
&\widetilde{e}_2\pi_1 = \pi_5 \colon t \mapsto -t\alpha_1,\\
&\widetilde{e}_1\pi_5 = \pi_6 \colon t \mapsto -t\alpha_1 \text{ for } 0 \le t \le 1/2,\ (t-1)\alpha_1 \text{ for } 1/2 \le t \le 1,\\
&\widetilde{e}_1\pi_6 = \pi_7 \colon t \mapsto t\alpha_1,\\
&\widetilde{e}_2\pi_7 = \widetilde{e}_1\pi_4 = \pi_8 \colon t \mapsto t(\alpha_1 + \alpha_2).
\end{aligned}$$

Thus we get the crystal for the adjoint representation of $\mathfrak{sl}_3$ as in Figure 1. □

3. Generalized Littlewood–Richardson rule

Given $\pi_1, \pi_2 \in \Pi$, define concatenation $\pi_1 * \pi_2$ by

$$(\pi_1 * \pi_2)(t) = \begin{cases} \pi_1(2t) & 0 \le t \le 1/2 \\ \pi_1(1) + \pi_2(2t-1) & 1/2 \le t \le 1 \end{cases}.$$

Theorem 3.1. *The map $\Pi \otimes \Pi \to \Pi$ given by $\pi_1 \otimes \pi_2 \mapsto \pi_1 * \pi_2$ is a morphism of crystals.*

Corollary 3.2. *Given $\pi_1, \pi_2 \in \Pi^+$, $B_{\pi_1} \otimes B_{\pi_2} = \bigoplus_\pi B_\pi$ where the sum is over all paths $\pi \in \Pi^+$ such that $\pi = \pi_1 * \eta$ for some $\eta \in B_{\pi_2}$.*

Example 3.3. We compute $B(\lambda) \otimes B(\lambda)$ for $\lambda = \alpha_1 + \alpha_2$ for $\mathfrak{g} = \mathfrak{sl}_3$ (i.e. $V(\lambda)$ is the adjoint representation.) Let $\pi(t) = (\alpha_1 + \alpha_2)t$. We see that $\pi * \eta \in \Pi^+$ for η of weights $\{\alpha_1 + \alpha_2, \alpha_1, \alpha_2, 0, 0, -\alpha_1 - \alpha_2\}$, so the decomposition is

$$B(\lambda) \otimes B(\lambda) = B(2\alpha_1 + 2\alpha_2) \oplus B(2\alpha_1 + \alpha_2) \oplus B(\alpha_1 + 2\alpha_2) \oplus B(\alpha_1 + \alpha_2) \oplus B(\alpha_1 + \alpha_2) \oplus B(0).$$

□

4. Connection to Young tableaux model

Given a semistandard Young tableau T, let $w_T = i_1 \cdots i_s$ be the word obtained by reading from bottom to top (in French notation) starting from rightmost column and then moving to the left. This gives a path $\pi_T = \pi_{\varepsilon_{i_1}} * \cdots * \pi_{\varepsilon_{i_s}}$ where $\varepsilon_1 = \omega_1$, $\varepsilon_2 = \omega_2 - \omega_1$, $\ldots$, $\varepsilon_{n-1} = \omega_{n-1} - \omega_{n-2}$, $\varepsilon_n = -\omega_{n-1}$.

Then the crystal operator on paths coincides with the crystal operator on Young tableaux.