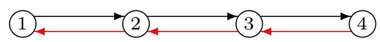
LUSZTIG'S NILPOTENT VARIETY AND $B(\infty)$

STEVEN SAM AND PETER TINGLEY

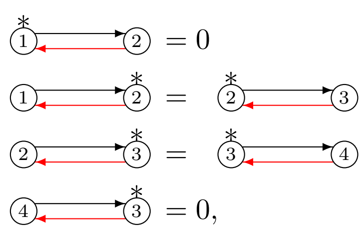
Last week we defined Lusztig's nilpotent variety [L], and discussed how it is used to give a geometric realization of $U^-(\mathfrak{g})$ (precisely, an embedding of $U^-(\mathfrak{g})$ into a geometrically defined algebra). This week, we will give a similar construction of the crystal $B(\infty)$. Here the vertices of the crystal will be irreducible components of the varieties. Note that we have not defined a realization of $U_q(\mathfrak{g})$, so we can't really talk of this as a "crystal basis." Instead we will use the recognition theorems from lecture 4 to see that we obtain $B(\infty)$.

1. Review from last week

We defined Q to be the doubled quiver of some graph $\Gamma = (I, E)$, with a fixed orientation (i.e. chosen direction for each edge). For example, if Γ is the A_4 Dynkin diagram,

$Q =$  .

where the red edges are the negatively oriented edges. If we choose a different orientation, we will end up with an isomorphic variety below, so this choice is of minimal importance. The preprojective algebra $\mathcal{P}$ is the quotient of the path algebra $\mathbb{C}Q$ by the moment map condition, which in this case consists of the relations



where each diagram represents a path of length two starting at the stared vertex. Lusztig's nilpotenet variety $\Lambda(V)$ is the variety of representations of the completion of $\mathcal{P}$ on an I graded vector space $V = V_1 \oplus \cdots \oplus V_n$, subject to the condition $\pi_i V = V_i$. Here π_i is the projection corresponding to the trivial path at vertex i. In more general cases we need to take a completion of the preprojective algebra, but that is unnecessary in finite type.

Up to isomorphism, $\Lambda(V)$ only depends on the dimension vector v of V. Assuming we are working with $\mathbf{GL}(V) = \prod_I \mathbf{GL}(V_i)$ invariant constructions, we can safely denote it by $\Lambda(v)$. We constructed a product $*$ on the space $\bigoplus_v \mathfrak{M}(\Lambda(v)/\mathbf{GL}(v))$ of $\mathbf{GL}(V)$-invariant constructible functions on all $\Lambda(v)$:

$$*\colon \mathfrak{M}(\Lambda(v)/\mathbf{GL}(v)) \otimes \mathfrak{M}(\Lambda(v')/\mathbf{GL}(v')) \to \mathfrak{M}(\Lambda(v+v')/\mathbf{GL}(v+v')).$$

There is an embedding

$$U^-(\mathfrak{g}) \hookrightarrow \bigoplus_v \mathfrak{M}(\Lambda(v)/\mathbf{GL}(v))$$

which takes F_i to the function "1" on $\Lambda(1_i)$ (which is a point).

Date: April 8, 2011.

2. Crystals from $\Lambda(V)$

The following construction was originally give in [KS]. We wish to show that there is a realization of $B(\infty)$ where the vertices are the irreducible components of $\coprod_v \Lambda(v)$. Call this latter set $B^{\mathcal{P}}$. In order to make sense of this claim, we need to define $e_i, f_i\colon B^{\mathcal{P}} \to B^{\mathcal{P}} \cup \{\emptyset\}$, $\mathrm{wt}, \varphi, \varepsilon\colon B^{\mathcal{P}} \to P$.

Fix $x = (x_a)_{a\colon i\to j} \in \Lambda(v)$. Define

$$x_i := \bigoplus_{a\colon i\to j} x_a \colon V_i \to \bigoplus_{a\colon i\to j} V_j \quad \text{and} \quad {}_ix := \bigoplus_{a\colon j\to i} \epsilon(a)x_a : \bigoplus_{a\colon j\to i} V_j \to V_i, \tag{2.1}$$

where $\epsilon(a) = 1$ is a is black, and -1 if a is red. Note that the moment map condition becomes ${}_ix \circ x_i = 0$ for all i, or equivalently

$$\mathrm{im} x_i \subset \ker {}_ix. \tag{2.2}$$

Fix $Z \in \mathrm{Irr}\,\Lambda(v)$. Let

$$Z_i^0 = \{T = (x, v) \in Z \mid \dim \mathrm{im}(x_i) \text{ is maximal, and } \dim \mathrm{im}({}_ix) \text{ is maximal}\}. \tag{2.3}$$

Note that, for all i, Z_i^0 is an open dense subset of Z.

Definition 2.4. For $Z \in \mathrm{Irr}\,\Lambda(v)$, let

(i) $e_i(Z)$ be the closure of $\{T \in \Lambda(v - 1_i) \mid T$ is isomorphic to a submodule of some $T' \in Z_i^0\}$.
(ii) $f_i(Z)$ be the closure of $\{T \in \Lambda(v + 1_i) \mid T$ has a submodule in $Z_i^0\}$.
(iii) $\varepsilon_i(Z) := \dim \mathrm{im} x_i - \dim \ker {}_ix$ for some (equivalently any) $x \in Z_i^0$.
(iv) $\varepsilon(Z) := \sum \varepsilon_i(Z)\omega_i$.
(v) $\mathrm{wt}(Z) := -\sum_I v_i\alpha_i$.
(vi) $\varphi(Z) := \mathrm{wt}(Z) + \varepsilon(Z)$.

□

To see that $e_i(Z), f_i(Z)$ are indeed single irreducible components (or $\emptyset$), one shows that they are all closures of vector bundles over an open subset of $e_i^{\varepsilon_i(Z)}(Z)$, and that this also holds for Z itself. Since Z is irreducible, this implies that $e_i^{\varepsilon_i(Z)}(Z)$ is irreducible, from which it follows that each of the vector bundles corresponding to $e_i^k(Z)$ and $f_i^k(Z)$ are irreducible.

Definition 2.5 (Alternative definition of f_i)**.** Take $T \in Z$ generic and a generic extension

$$0 \to T \to T' \to S_i \to 0.$$

Then T' will be in a unique $Z' \in \Lambda(v + 1_i)$ and we set $f_i(Z) = Z'$. □

Recall the definition of the stupid crystal $B^{(i)}$:

$$\cdots b^{(i)}(-1) \leftarrow b^{(i)}(0) \leftarrow b^{(i)}(1) \leftarrow \cdots$$

where $\mathrm{wt} = 0$, $\varepsilon = 0$, $\varphi = 0$ at $b^{(i)}(0)$ and the arrows are given by f_i.

Theorem 2.6 (Kashiwara–Saito)**.** *Let B be a combinatorial highest weight crystal with an involution $*$. Define $e_i^* = * \circ e_i \circ *$ and define $\Phi_i\colon B \to B \otimes B^{(i)}$ by $b \mapsto (e_i^*)^{\varepsilon_i^*(b)}(b) \otimes b(-\varepsilon_i^*(b))$. If Φ_i is a morphism for all i, then $B \cong B(\infty)$.*

Proof. As discussed in Lecture 4 (and proven in [KS]), $B(\infty)$ has this property, where $*$ is Kashiwara's involution inherited from the algebra anti-automorphism of $U_q^-(\mathfrak{g})$ fixing all F_i. Thus it is enough to see that the conditions of the theorem uniquely characterize B. Choose a sequence of $i \in I$ so that each element appears infinitely many times. Then the conditions imply that B is isomorphic to the crystal generated by $\cdots \otimes b^{(i_3)}(0) \otimes b^{(i_2)}(0) \otimes b^{(i_1)}(0) \subset \cdots \otimes B^{(i_3)} \otimes B^{(i_2)} \otimes B^{(i_1)}$. □

Now define $*\colon \Lambda(V) \to \Lambda(V^*)$ by $(V, x) \mapsto (V^*, *x)$, where $*x_a = x^*_{\overline{a}}$. Choosing an I-graded isomorphism of vector spaces $V \cong V^*$. This gives us an involution

$$*\colon \operatorname{Irr}(\Lambda(V)) \to \operatorname{Irr}(\Lambda(V^*)) \cong \operatorname{Irr}(\Lambda(V))$$

which is independent of the choice of isomorphism by **GL**-equivariance.

To apply the theorem, we need to show that

$$\varepsilon_i^*(f_i(Z)) = \begin{cases} \varepsilon_i^*(Z) & \text{if } \varphi_i((e_i^*)^{\varepsilon_i^*(Z)}(z)) > \varepsilon_i^*(Z) \\ \varepsilon_i^*(Z) + 1 & \text{otherwise} \end{cases}.$$

Using the explicit definitions of $*$ and ε, we see that $\varepsilon^*(Z) := \varepsilon(*Z)$ is given by $\dim \ker x_i$ for a generic x in Z.

Now, fix Z and $x \in Z$ generic. Using definition 2.5, we see that $\varepsilon_i^*(f_i(Z)) = \varepsilon_i^*(Z)$ if and only if $\dim \mathrm{im} x_i < \dim \ker x_i$. Thus we need to show that

$$\dim \mathrm{im} x_i < \dim \ker x_i \iff \varphi_i((e_i^*)^{\varepsilon_i^*(Z)}(z)) > \varepsilon_i^*(Z). \tag{2.7}$$

This is an elementary (although slightly tricky) exercise, which we leave to the reader. It can also be found in [KS].

References

[L] G. Lusztig. Quivers, Perverse sheaves and quantized enveloping algebras. *Journal of the american mathematical society* **4** No 2, April 1991.

[KS] Kashiwara, Masaki; Saito, Yoshihisa. Geometric construction of crystal bases. Duke Math. J. 89 (1997), no. 1, 936.