LECTURE 9: $U(\mathfrak{g})$ REPRESENTATIONS FROM QUIVER GRASSMANNIANS

STEVEN SAM AND PETER TINGLEY

So far we have realized $U^-(\mathfrak{g})$ geometrically in the sense that we have an injection $U^-(\mathfrak{g}) \hookrightarrow \bigoplus_v \mathcal{M}(\Lambda(v)/\mathbf{GL}(v))$ of $U^-(\mathfrak{g})$ into the ring of $\mathbf{GL}(v)$ invariant constructible functions under a geometrically defined product. We are interested in realizing the highest weight representations $V(\lambda)$. As representations of $U^-(\mathfrak{g})$, these are of the form $U^-(\mathfrak{g})/I_\lambda$ for some left ideal I_λ. We would like to understand these quotients and the actions of the raising operators E_i in terms of the geometry. The main obstacle is that, on some level, we have been working with the moduli space $\Lambda(v)/\mathbf{GL}(v)$, which is not a variety at all, but rather a "stack." The solution will be to replace $\Lambda(v)/\mathbf{GL}(v)$ by a new space which is actually a variety, and retains the information of the appropriate quotient of the ring $\bigoplus_v \mathcal{M}(\Lambda(v)/\mathbf{GL}(v))$.

This will be achieved in two ways. The first, which will be the topic of today's lecture, replaces $\Lambda(v)/\mathbf{GL}(v)$ with $\mathrm{Gr}_{\mathcal{P}}(v; M)$, the "grassmannian" of subrepresentations of a fixed representation M of the preprojective algebra. If M is injective, the result is a representation of $U(\mathfrak{g})$, and has a natural subrepresentation isomorphic an irreducible representation $V(\lambda)$.

The second approach uses Nakajima's quiver varieties [N]. The two constructions lead to isomorphic varieties, although Nakajima's construction has some advantages, since the resulting variety is naturally a Lagrangian subvariety of a larger smooth symplectic variety.

1. Naively trying to apply E_i

Recall that for $\dim V' = v + 1_i$, we defined

$$\Lambda(V'; v) = \{(x', V) \mid x' \in \Lambda(V'), V \subset V', \dim V = v\}.$$

and maps

$$\begin{tikzcd}[column sep=small] & \Lambda(v+1_i; v)/\mathbf{GL}(v+1_i) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \Lambda(v)/\mathbf{GL}(v) & & \Lambda(v+1_i)/\mathbf{GL}(v+1_i). \end{tikzcd}$$

We also defined $F_i := (\pi_1)_!(\pi_2)_* : \mathcal{M}(\Lambda(v)/\mathbf{GL}(v)) \to \mathcal{M}(\Lambda(v+1_i)/\mathbf{GL}(v+1_i))$, where $\mathcal{M}(\Lambda(v)/\mathbf{GL}(v))$ is the space of $\mathbf{GL}(V)$-invariant constructible functions on $\Lambda(V)$ for some vector space V with $\dim V = v$. E_i and F_i are supposed to be pretty symmetric, so a natural guess would be to define $E_i = (\pi_1)_!\pi_2^*$. But there are some problems, as the following example illustrates.

Example 1.1. Take $\mathfrak{g} = \mathfrak{sl}_2$ so the quiver Q is a single vertex. Consider the case $v = 0$, so that $\Lambda(v)$ and $\Lambda(v+1; v)$ are both points. Then $\Lambda(1; 0)/\mathbf{GL}(1) = \mathrm{pt}/\mathbb{C}^*$. So the fiber of π_1 over $\Lambda(0) = \mathrm{pt}$ is $\mathrm{pt}/\mathbb{C}^*$. We need to take this Euler characteristic in defining $(\pi_1)_!\pi_2^*$, so we are in some trouble. We might try to make sense of this in terms of stacks to compute Euler characteristics. In fact, stacky people say $\chi(\mathrm{pt}/\mathbb{C}^*) = \chi(\mathbb{C}P^\infty) = \infty$, which is kind of ok as we might think of this as the limiting representation as the highest weight gets large, but I'm not sure this is a meaningful answer. Things only get worse if you try to apply E to $\mathcal{M}(\Lambda(v)/\mathbf{GL}(v))$ for $v > 1$. $\square$

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2. Quiver grassmannian approach

One way to deal with the problem displayed in Example 1.1 is to work inside a fixed "universal" representation M.

Example 2.1. Continuing with the $\mathfrak{sl}_2$ example: Fix a vector space M of dimension m, thought of as a representation of the quiver with one node and no arrows. Instead of working with $\Lambda(v)/\mathbf{GL}(v)$, we work with the variety of subrepresentations of M, which is just the grassmannian $\mathrm{Gr}(v, M)$. Furthermore, we replace $\Lambda(v+1, v)$ with the 2-step flag variety $\mathrm{Fl}(v+1, v; M)$. Thus

$$\begin{tikzcd} & \Lambda(v';v)/\mathbf{GL}(v') \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \Lambda(v)/\mathbf{GL}(v) & & \Lambda(v')/\mathbf{GL}(v') \end{tikzcd}$$

becomes

$$\begin{tikzcd} & \mathrm{Fl}(v+1,v;M) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathrm{Gr}(v;M) & & \mathrm{Gr}(v+1;M). \end{tikzcd}$$

The fiber of π_1 over any point is $\mathbb{P}^{m-v+1}$ and the fiber of π_2 is $\mathbb{P}^v$. One can then calculate that

$$(\pi_2)_! \pi_1^* \mathbf{1}_{\mathrm{Gr}(v;M)} = v \mathbf{1}_{\mathrm{Gr}(v+1;M)} \quad \text{and} \quad (\pi_1)_! \pi_2^* \mathbf{1}_{\mathrm{Gr}(v+1;M)} = (u-v)\mathbf{1}_{\mathrm{Gr}(v;M)},$$

where $\mathbf{1}_{\mathrm{Gr}(v;M)}$ denotes the constant function 1 on $\mathrm{Gr}(v;M)$. It follows that $\mathrm{span}\{\mathbf{1}_{\mathrm{Gr}(v;M)} : 0 \leq v \leq u\}$ is the standard $m+1$ dimensional representation of $U(\mathrm{sl}_2)$, where we define the action of the generators by $F_i \to (\pi_2)_! \pi_1^*$ and $E_i \to (\pi_1)_! \pi_2^*$. □

So we have realized the finite dimensional irreps of $U(\mathrm{sl}_2)$ geometrically, achieving our goal in the simplest case. Now consider the double quiver of a general symmetric Cartan matrix (not necessarily of finite type). Fix a representation M of the completed preprojective algebra $\overline{\mathcal{P}}$. For each dimension vectors $v' > v$, let $\mathrm{Gr}_{\overline{\mathcal{P}}}(v, M)$ be the variety of v-dimensional subrepresentations of M, let $\mathrm{Fl}_{\overline{\mathcal{P}}}(v', v; M)$ be the variety of two step flags of subrepresentations of dimensions v' and v, and let $\mathcal{M}(\mathrm{Gr}_{\overline{\mathcal{P}}}(v, M))$ be the space of constructible functions on $\mathrm{Gr}_{\overline{\mathcal{P}}}(v, M)$. Then we can consider the correspondence

$$\begin{tikzcd} & \mathrm{Fl}_{\overline{\mathcal{P}}}(v+1_i,v;M) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathrm{Gr}_{\overline{\mathcal{P}}}(v;M) & & \mathrm{Gr}_{\overline{\mathcal{P}}}(v+1;M), \end{tikzcd}$$

and attempt to define an action of $U(\mathfrak{g})$ on $\oplus_v \mathrm{Gr}_{\overline{\mathcal{P}}}(v;M)$ by

$$F_i \to (\pi_2)_! \pi_1^* \quad \text{and} \quad E_i \to (\pi_1)_! \pi_2^*. \tag{2.2}$$

It is not always true that this defines an action of $U(\mathfrak{g})$ on $\oplus_v \mathrm{Gr}_{\overline{\mathcal{P}}}(v;M)$. However, if $M = q$ is injective, it does. In fact, we can make the following somewhat more precise claim:

Theorem 2.3. *Let S_i denote the simple module for $\overline{\mathcal{P}}$ with dimension vector 1_i (i.e. 1 over node i, and zero over all other nodes). Fix a highest weight $\lambda = \sum_i a_i \omega_i$, where ω_i in the i^{th} fundamental weight. Let q be the injective hull of $S := \oplus_i S_i^{\oplus a_i}$. Then Equation (2.2) defines an action of $U(\mathfrak{g})$ on $\oplus_v Gr_{\overline{\mathcal{P}}}(v;q)$.*

Remark 2.4. If the underlying Dynkin diagram is not of finite type, then the injective module q is infinite dimensional. However, one can show that, for any finite v, $\mathrm{Gr}(v;q)$ is a finite dimensional variety, and the theory still works in this case.

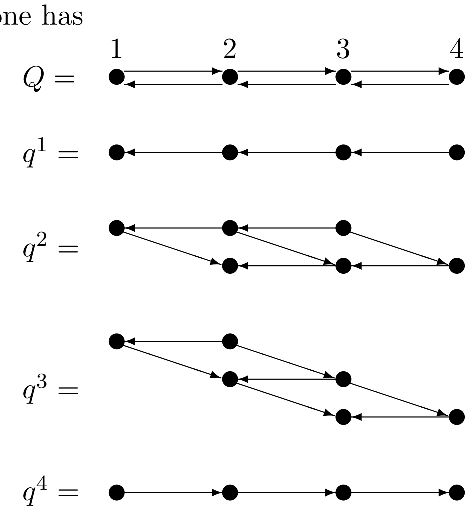
I believe theorem 2.3 can be proven by directly checking all the defining relations of $U(\mathfrak{g})$, but proofs that exist in the literature all work by showing that $\mathrm{Gr}_{\overline{\mathcal{P}}}(v;q)$ is isomorphic to a variety of Nakajima [N], and appealing to known results in that case. See [L, S] for the proof of this isomorphism in a slightly different content (they use projectives instead of injectives and also need to impose some conditions on the representations in the grassmannians) and [ST] for the isomorphism in the present setting.

Remark 2.5. In fact, the operators F_i make sense on $\oplus_v \mathcal{M}(\mathrm{Gr}_{\overline{\mathcal{P}}}(v, M))$ for any M, and define an action of $U(\mathfrak{g})$ on this space. The submodule generated by the function 1 on the point $\mathrm{Gr}(0, M)$ is always contained in the space of functions which have the property that they are equal on subrepresentations which are isomorphic as representations (i.e. isomorphic, but where the isomorphism is not require to extend to an automorphism of M). In general, the operators E_i do not respect this subvariety, essentially because subrepresentations being isomorphic is not equivalent to quotients being isomorphic. However, if $M = q$ is injective, the E_i do preserve this space.

Remark 2.6. It is instructive to prove that Equation (2.2) does not define an action of $U(\mathfrak{g})$ on $\oplus_{\overline{\mathcal{P}}} \mathcal{M}(\mathrm{Gr}(v, M))$ in a particular case. For instance, this fails for $M = S_1 \oplus S_1 \oplus S_2$ in the A_2 case.

Remark 2.7. Morally, the point of the injective assumption in Theorem 2.3 is that, if $V \subset q$ is a subrepresentation of q, then all non-trivial extensions of V are isomorphic to subrepresentations of q. Thus $\mathrm{Gr}_{\overline{\mathcal{P}}}(v, q)$ "sees" enough information about the category of representations of $\overline{\mathcal{P}}$.

Example 2.8. The injective module q from Theorem 2.3 is actually the direct sum of a copy of q^i for each copy of S_i in its socle, where q^i is the injective hull of S_i. Often these q^i are very simple. For instance, in type A_4, one has



Here each node represents a basis vector in V_i, where i is the node of the quiver directly above it, and each arrow represents a matrix element of 1 for the corresponding arrow of the quiver. All other matrix elements are 0. $\square$

Next time we will discuss Nakajima's construction [N], and how the two constructions are related.

References

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