Smooth Lyapunov Functions for Hybrid Systems—Part I: Existence Is Equivalent to Robustness

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***Abstract*—Hybrid systems are dynamical systems where the state is allowed to either evolve continuously (flow) on certain subsets of the state space or evolve discontinuously (jump) from other subsets of the state space. For a broad class of such systems, characterized by mild regularity conditions on the data, we establish the equivalence between the robustness of stability with respect to two measures and a characterization of such stability in terms of a smooth Lyapunov function. This result unifies and generalizes previous results for differential and difference inclusions with outer semicontinuous and locally bounded right-hand sides. Furthermore, we give a description of forward completeness of a hybrid system in terms of a smooth Lyapunov-like function.**

***Index Terms*—Hybrid inclusions, hybrid systems, measures, robustness, smooth Lyapunov functions, stability.**

I. INTRODUCTION

CONVERSE Lyapunov theorems relate a system's asymptotic stability properties to the existence of a function that decreases along the system's solutions. The strongest converse theorems assert the existence of *smooth* Lyapunov functions. The growing appreciation for such results is apparent as the transition from "… the so-called converse theorems are mainly of theoretical interest" [36, 1st edition, p. 165] to "… converse theorems are applied to four problems in control theory, and it is shown that converse theorems lead to elegant solutions to each of these problems" [36, 2nd edition, p. 235] displays. In this paper, we establish a link between robustness of a very general concept of stability and the existence of a smooth Lyapunov function for a broad class of hybrid systems. We hope that, as in the case of continuous-time and discrete-time systems, this result will inspire solutions to a wide range of control problems for hybrid systems.

When used for continuous-time or discrete-time systems, Lyapunov functions have played a prominent role in nonlinear control design over the last two decades, and they have been used to establish robustness of the stability induced by control to various types of practical perturbations, like small time delays, slowly varying parameters, fast unmodeled dynamics, measurement errors and observer dynamics, etc. Smooth Lyapunov functions have the potential to be just as relevant for hybrid systems. Already, the types of results reported herein have been used in [27] to develop the notion of a patchy, smooth control Lyapunov function (clf) for systems that do not admit a standard smooth clf. These patchy clf's have been used to generate new control algorithms with enhanced robustness properties. Additionally, converse Lyapunov theorems for hybrid systems have been used in [31] to establish robustness of stability induced by hybrid control to sample-and-hold and networked implementations.

A. Background

A classical problem in dynamical systems theory is to determine what asymptotic stability properties guarantee the existence of a smooth Lyapunov function. Among the early results, Kurzweil's contribution [17] for differential equations is especially notable since the system's solutions did not need to be unique. The renaissance of interest in converse theorems for continuous-time systems with nonunique solutions can be traced to [20]. Subsequently, Clarke *et al.* [7] underlined the link between robustness and smoothness of Lyapunov functions and showed that, for differential inclusions, if the right-hand side is an upper semicontinuous set-valued mapping then (strong) asymptotic stability is equivalent to the existence of a smooth Lyapunov function. The authors in [34] worked with the concept of $\mathcal{KL}$-stability with respect to two measures for differential inclusions, and they established the equivalence between robust $\mathcal{KL}$-stability and the existence of a smooth Lyapunov function. (The introduction of [34] contains an extensive overview of the literature on converse Lyapunov theorems for continuous-time systems.) For discrete-time systems given by difference inclusions, similar results were shown in [15] and [16]. We also mention [1, Theorem 2] on the equivalence between forward completeness and the existence of a smooth Lyapunov-like function (one that increases along solutions no faster than exponentially) for a class of differential inclusions.

Hybrid systems are those whose state, which can contain continuous and/or discrete variables, can evolve continuously (flow) and/or discretely (jump). Hybrid systems are ubiquitous in science and engineering [19], [35]. However, hybrid systems theory is far from being well established. Various concepts of solutions exist (see [3], [8], [12], [21], [23], [25], [33], [35]) and yield different properties of asymptotic stability. As for

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the Lyapunov characterization of asymptotic stability, many sufficient conditions have been proposed (see [4], [6], [9], [19], [22], [23], [29], [37]), and some necessary conditions (converse Lyapunov theorems) have been established (see [37]). However, to the best of the author's knowledge, no results have appeared on the existence of smooth Lyapunov functions for general hybrid systems. Moreover, a systematic approach to the robustness of stability for hybrid systems has been proposed only recently in [11] and carried out to an extent in [12], even though the capabilities of hybrid feedback have been realized before; see [14] and [26].

B. Contribution

We work with hybrid systems in the framework proposed in [11] (related to that concurrently suggested by [8]) and developed in [12]. As discussed in [11], the framework is deeply motivated by the study of the robustness of stability, and, among other benefits, makes quite general invariance principles possible; see [29]. In this framework, the sets of solutions have good sequential compactness properties and depend "uppersemicontinuously" on initial conditions. The "nice" behavior of the sets under perturbations of the system, as reported in [12], partly enables the results of this paper. Since, in [11], the solutions to hybrid systems are parameterized by both the elapsed time and the number of jumps that have occurred, we work with the concept of $\mathcal{KLL}$-stability with respect to two measures, rather than $\mathcal{KL}$-stability as in continuous-time or discrete-time systems. Details of the framework are in Section II; we refer the reader to [34] for a discussion of stability with respect to two measures.

Our main result shows the equivalence of the following.

- The existence of a smooth Lyapunov function for a hybrid system.
- The robustness, to small measurement noise, actuator error, and external disturbance, of the $\mathcal{KLL}$-stability of the hybrid system.

In Section III we make these concepts precise. In Section IV we use a temperature control system to illustrate the utility of the "$\mathcal{KLL}$-stability with respect to two measures" concept, and we indicate some practical consequences of our main result for this example. The remaining sections are devoted to proving the equivalence result. The simpler implication, from the existence of a smooth Lyapunov function to robustness, is proved in Section V by using a related but more straightforward technique than the one used in continuous- and discrete-time systems. The proof of the reverse implication is outlined in Section VI, with Section VII providing the remaining technical details; we first use a classical construction technique (see [7], [16], [34]) to derive an upper semicontinuous Lyapunov function, and we then follow the smoothing technique used in discrete-time systems [16] to derive a smooth Lyapunov function[1]. The smoothing step directly relies on our result on robustness of solutions (i.e., a perturbation of a solution to a perturbed system is also a solution to a more strongly perturbed system, see Section VII-B), which is derived from the robust stability assumption. This is another main contribution in this paper, since we have not seen such a result on robustness of solutions in continuous-time or hybrid systems, though it has been used for discrete-time systems (see [16, Section 6.1.2]).

Space limitations do not allow us to give and prove sufficient conditions for robustness of $\mathcal{KLL}$-stability here. Such results, and the ensuing converse Lyapunov theorems, will be given in [5]. More specifically, [5] will show that stability of compact sets with respect to a single measure (in particular, asymptotic stability of compact attractors) is always robust. That result, combined with the main result of this paper (Theorem 3.2), will then be applied to obtain smooth Lyapunov functions for various classes of systems (for example, for asymptotically stable systems with logic variables). Some further applications of Lyapunov functions, for example in verifying robustness of asymptotic stability to varying and jumping parameters, will also be given in [5].

Finally, in part as a consequence of the techniques used in the proof of the main result, in Section VIII we show that forward completeness of a hybrid system can be equivalently characterized by a smooth Lyapunov-like function.

C. Preliminaries

- $\mathbb{R}$ denotes the real numbers, $\mathbb{Z}$ denotes the integers, $\mathbb{R}_{\geq 0}$ denotes the nonnegative real numbers, $\mathbb{Z}_{\geq 0}$ denotes the nonnegative integers.
- $\mathbb{B}$ is the open unit ball in Euclidean space (of appropriate dimension).
- The sets $\overline{\mathcal{U}}$, $\overline{\text{co}}\mathcal{U}$, and $\partial\mathcal{U}$ denote, respectively, the closure, closed convex hull, and boundary of a given set $\mathcal{U}$.
- Given two sets $\mathcal{U}_1$ and $\mathcal{U}_2$ in $\mathbb{R}^n$, their sum $\mathcal{U}_1 + \mathcal{U}_2$ is defined by $\{u_1 + u_2 : u_1 \in \mathcal{U}_1, u_2 \in \mathcal{U}_2\}$. To save on notation, we will use $x + \mathcal{U}$ in place of $\{x\} + \mathcal{U}$.
- Given a set $\mathcal{U} \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, $|x|_{\mathcal{U}} := \inf_{y\in\mathcal{U}} |x - y|$.
- Given sets $\mathcal{U} \subset \mathcal{O} \subset \mathbb{R}^n$, $\mathcal{U}$ is said to be **relatively closed** in $\mathcal{O}$ if there exists a closed set $\mathcal{U}'$ such that $\mathcal{U} = \mathcal{U}' \cap \mathcal{O}$. One can show that $\mathcal{U} \subset \mathcal{O}$ is relatively closed in $\mathcal{O}$ if and only if $\mathcal{U} = \overline{\mathcal{U}} \cap \mathcal{O}$. If $\mathcal{O}$ is open then $\mathcal{U}$ is relatively closed in $\mathcal{O}$ if and only if $\mathcal{O}\backslash\mathcal{U}$ is open.
  (In what follows, $\mathcal{O}$ denotes an open set in $\mathbb{R}^n$.)
- The **domain**, **range**, and **graph** of a set-valued mapping $M : \mathcal{O} \rightrightarrows \mathbb{R}^n$ (or $M : \mathcal{O} \rightrightarrows \mathcal{O}$) are, respectively, the sets $\text{dom}\ M := \{x \in \mathcal{O} : M(x) \neq \emptyset\}$, $\text{rge}M := \{y \in \mathbb{R}^n : \exists x \in \mathcal{O} \text{ s.t. } y \in M(x)\}$ and $\text{gph}M := \{(x,y) \in \mathcal{O} \times \mathbb{R}^n : y \in M(x)\}$.
- A set-valued mapping $M : \mathcal{O} \rightrightarrows \mathbb{R}^n$ is outer semicontinuous at $x \in \mathcal{O}$ if for all sequences $x_i \to x$ and $y_i \in M(x_i)$, if $\lim_{i\to\infty} y_i = y$ for some y, then $y \in M(x)$. The mapping is said to be **outer semicontinuous** if it is outer semicontinuous at each $x \in \mathcal{O}$. A set-valued mapping M is outer semicontinuous on $\mathcal{O}$ if and only if $\text{gph}M$ is relatively closed in $\mathcal{O} \times \mathbb{R}^n$ [28, Theorem 5.7].

[1]For continuous-time systems (see [7] and [34]), the smoothing technique is to first use the robustness assumption to embed the original upper semicontinuous differential inclusion into a larger, locally Lipschitz one, and then derive a locally Lipschitz Lyapunov function, which finally is smoothed by the standard technique. However, such an idea of using robustness to pass to a system with Lipschitz continuous dependence on initial conditions does not seem to generalize to hybrid systems.

- A set-valued mapping $M : \mathcal{O} \rightrightarrows \mathbb{R}^n$ is called **locally bounded** if for any compact $K \subset \mathcal{O}$ there exists a compact set $K' \subset \mathbb{R}^n$ such that $M(K) := \{y : y \in M(x),\ x \in K\} \subset K'$.
  If a set-valued mapping is outer semicontinuous and locally bounded, the mapping values are compact. For locally bounded set-valued mappings with closed values, outer semicontinuity agrees with what is often referred to as **upper semicontinuity**: for each $x \in \mathcal{O}$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that for all x' with $|x' - x| < \delta$, $M(x') \subset M(x) + \varepsilon\mathbb{B}$.
- A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to **class-$\mathcal{K}$** ($\alpha \in \mathcal{K}$) if it is continuous, zero at zero, and strictly increasing. It is said to belong to **class-$\mathcal{K}_\infty$** if, in addition, it is unbounded.
- A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to **class-$\mathcal{KL}$** ($\beta \in \mathcal{KL}$) if it satisfies: (i) for each $t \geq 0$, $\beta(\cdot, t)$ is nondecreasing and $\lim_{s \searrow 0} \beta(s,t) = 0$, and (ii) for each $s \geq 0$, $\beta(s, \cdot)$ is nonincreasing and $\lim_{t\to\infty} \beta(s,t) = 0$. We are abusing notation here since $\beta \in \mathcal{KL}$ does not necessarily imply that β is class-$\mathcal{K}$ in its first argument or class-$\mathcal{L}$ (see [13, p. 7]) in its second argument. In particular, neither continuity nor strict monotonicity are imposed. Our abuse of notation is justified by the fact that any function of the type we have defined can be upper bounded by a "true" $\mathcal{KL}$-function. See [32, Proposition 7].
- A function $\gamma : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to **class-$\mathcal{KLL}$** ($\gamma \in \mathcal{KLL}$) if, for each $r \geq 0$, $\gamma(\cdot,\cdot,r) \in \mathcal{KL}$ and $\gamma(\cdot, r, \cdot) \in \mathcal{KL}$.

II. Hybrid Systems

We consider a hybrid system, with an open state space $\mathcal{O} \subset \mathbb{R}^n$, given by

$$\mathcal{H} := \begin{cases} \dot{x} \in F(x) & \text{for } x \in C \\ x^+ \in G(x) & \text{for } x \in D \end{cases} \tag{1}$$

where the set-valued mapping $F : \mathcal{O} \rightrightarrows \mathbb{R}^n$ describes the flow, which can occur in the set $C \subset \mathcal{O}$, and the set-valued mapping $G : \mathcal{O} \rightrightarrows \mathbb{R}^n$ describes the jumps, which can occur from the set $D \subset \mathcal{O}$. The state x may include both continuous and discrete variables, the latter often representing logic modes that can be identified with integers[2]. The treatment of (1) will be based on the tools developed in [12].

We do not consider F and G to be set-valued simply for the sake of generality. Our primary motivation for using set-valued mappings is to address the situation, which is common in hybrid control systems, where the system has single-valued maps f and g that are discontinuous. In such systems, accounting for measurement noise leads to set-valued "closures" of f and g, and those fit exactly in the framework we work in here; see [30], or [11] for a general discussion. A similar motivation was given in the work [7] and [34] for continuous-time systems, and [15] and [16] for discrete-time systems. Hybrid systems with set-valued mappings F and G have also been addressed in [3], [8], [18].

[2]In the companion paper [5] we will show, among other things, how the current results can be applied to systems with logic variables. Also, see the example in Section IV.

Solutions to the hybrid inclusion (1) are functions defined on hybrid time domains, as defined in [11], [12] and [8]. As noted in these references, a hybrid time domain is essentially equivalent to the "hybrid time trajectory" in [3], [21], and [22], but gives a more prominent role to the "discrete-time variable" j for counting the number of jumps. (cf. [23] and [35].) We call a subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ a **compact hybrid time domain** if $E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq \cdots \leq t_J$. We say E is a **hybrid time domain** if for all $(T, J) \in E$, $E \cap ([0,T] \times \{0, 1, \ldots J\})$ is a compact hybrid time domain. The supremum of a hybrid time domain E, denoted $\sup E$, is defined as $(\sup_{(t,j)\in E} t,\ \sup_{(t,j)\in E} j)$.

A **hybrid arc** is a function ϕ defined on a hybrid time domain, and such that $\phi(\cdot, j)$ is locally absolutely continuous for each j. A hybrid arc can be viewed as a set-valued mapping from $\mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ whose domain is a hybrid time domain.

A hybrid arc $\phi : \operatorname{dom} \phi \mapsto \mathcal{O}$ is a **solution** to $\mathcal{H}$ if

(S1) For all $j \in \mathbb{Z}_{\geq 0}$ and almost all $t \in \mathbb{R}_{\geq 0}$ such that $(t,j) \in \operatorname{dom} \phi$

$$\phi(t,j) \in C\ , \quad \dot{\phi}(t,j) \in F(\phi(t,j))$$

(S2) For all $(t,j) \in \operatorname{dom} \phi$ such that $(t, j+1) \in \operatorname{dom} \phi$,

$$\phi(t,j) \in D\ , \quad \phi(t,j+1) \in G(\phi(t,j))\ .$$

A solution to the hybrid system is called **maximal** if it cannot be extended, and **complete** if its domain is unbounded. Complete solutions are maximal. We denote by $\mathcal{S}(x)$ the set of all maximal solutions to $\mathcal{H}$ starting from x. The hybrid system $\mathcal{H}$ is said to be **forward complete** on $\mathcal{O}$ if, for all $x \in \mathcal{O}$, each $\phi \in \mathcal{S}(x)$ is complete.

Throughout the paper, we impose the following conditions on the hybrid system we study.

Standing Assumption 2.1: (Hybrid Basic Conditions)

The open state space $\mathcal{O} \subset \mathbb{R}^n$ and the data (F, G, C, D) of the system $\mathcal{H}$ satisfy:

(SA1) the sets $C \subset \mathcal{O}$ and $D \subset \mathcal{O}$ are relatively closed in $\mathcal{O}$;

(SA2) the (set-valued) mapping $F : \mathcal{O} \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded, and $F(x)$ is nonempty and convex for all $x \in C$; (equivalently, F is upper semicontinuous with compact values that, on C, are nonempty and convex);

(SA3) the (set-valued) mapping $G : \mathcal{O} \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded, $G(x)$ is nonempty and $G(x) \subset \mathcal{O}$ for all $x \in D$; (equivalently, G is upper semicontinuous with compact values that, on D, are nonempty and contained in $\mathcal{O}$).

General statements about the existence and structural properties of the solutions under the hybrid basic conditions can be found in [3], [8], [11], [12]. We will use the following result:

Proposition 2.2: The following statements are equivalent.

1) For each $x \in \mathcal{O}$, the set $\mathcal{S}(x)$ is nonempty and, for each $\phi \in \mathcal{S}(x)$, either $\operatorname{dom} \phi$ is unbounded (i.e., the solution is complete) or, as $(t,j) \to \sup \operatorname{dom} \phi$, either $|\phi(t,j)| \to \infty$ or $\phi(t,j) \to \partial\mathcal{O}$.
2) $C \cup D = \mathcal{O}$.

III. Main Results

In this paper, we are interested in the existence of smooth Lyapunov functions. In particular, given the state-space $\mathcal{O}$, the hybrid system data (F, G, C, D), and two continuous functions $\omega_i : \mathcal{O} \to \mathbb{R}_{\geq 0}, i = 1, 2$, we are interested in whether there exist a smooth function $V : \mathcal{O} \to \mathbb{R}_{\geq 0}$ and two class-$\mathcal{K}_\infty$ functions α_i, $i = 1, 2$ such that

$$\alpha_1(\omega_1(x)) \leq V(x) \leq \alpha_2(\omega_2(x)) \quad \forall x \in \mathcal{O} \tag{2}$$

$$\max_{f \in F(x)} \langle \nabla V(x), f \rangle \leq -V(x) \quad \forall x \in C \tag{3}$$

$$\max_{g \in G(x)} V(g) \leq e^{-1} V(x) \quad \forall x \in D. \tag{4}$$

Such a function V will be called a **smooth Lyapunov function** for $(\mathcal{O}, F, G, C, D, \omega_1, \omega_2)$.

We will show that the existence of such a function is guaranteed when (appropriately small) perturbations of the hybrid system retain stability properties of the original system. For a function $\sigma : \mathcal{O} \to \mathbb{R}_{\geq 0}$ we define the σ-perturbation of $\mathcal{H}$, denoted $\mathcal{H}_\sigma$, as

$$F_\sigma(x) := \overline{\text{co}} F((x + \sigma(x)\overline{\mathbb{B}}) \cap C) + \sigma(x)\overline{\mathbb{B}} \quad \forall x \in \mathcal{O} \tag{5}$$

$$\begin{aligned} G_\sigma(x) := \{ & v \in \mathcal{O} : v \in g + \sigma(g)\overline{\mathbb{B}}, \\ & g \in G((x + \sigma(x)\overline{\mathbb{B}}) \cap D)\} \quad \forall x \in \mathcal{O} \end{aligned} \tag{6}$$

$$C_\sigma := \{x \in \mathcal{O} : (x + \sigma(x)\overline{\mathbb{B}}) \cap C \neq \emptyset\} \tag{7}$$

$$D_\sigma := \{x \in \mathcal{O} : (x + \sigma(x)\overline{\mathbb{B}}) \cap D \neq \emptyset\} \tag{8}$$

$$\mathcal{H}_\sigma := \begin{cases} \dot{x} \in F_\sigma(x) & \text{for } x \in C_\sigma \\ x^+ \in G_\sigma(x) & \text{for } x \in D_\sigma. \end{cases} \tag{9}$$

The perturbation (5) resembles that in [7] and [34] while (6) resembles that in [16], considered in the analysis of robustness of stability for, respectively, differential and difference inclusions.

In what follows, **admissible perturbation radius** will denote any continuous $\sigma : \mathcal{O} \to \mathbb{R}_{\geq 0}$ such that $\{x\} + \sigma(x)\overline{\mathbb{B}} \subset \mathcal{O}$ for all $x \in \mathcal{O}$. The proposition below implies that several properties of sets of solutions to the hybrid systems, guaranteed by the hybrid basic conditions as described in [12], are present not just for $\mathcal{H}$ but also for $\mathcal{H}_\sigma$. The proof is in Appendix A.

Proposition 3.1: If σ is an admissible perturbation radius, then the system $\mathcal{H}_\sigma$ satisfies the hybrid basic conditions.

The existence of a smooth Lyapunov function for $(\mathcal{O}, F, G, C, D, \omega_1, \omega_2)$ will be guaranteed by the existence of an admissible perturbation radius σ with the properties listed below. (We write $\mathcal{S}_\sigma(x)$ for the set of maximal solutions to $\mathcal{H}_\sigma$ starting at x.)

1) $\mathcal{H}_\sigma$ is forward complete on $\mathcal{O}$ and there exists $\gamma \in \mathcal{KLL}$ such that, for each $x \in \mathcal{O}$ and each $\phi \in \mathcal{S}_\sigma(x)$, we have

$$\omega_1(\phi(t, j)) \leq \gamma(\omega_2(x), t, j) \quad \forall (t, j) \in \text{dom } \phi \tag{10}$$

2) $\mathcal{A}_0 = \mathcal{A}_\sigma$, where

$$\mathcal{A}_0 := \left\{ \xi \in \mathcal{O} : \sup_{\substack{\phi \in \mathcal{S}(\xi) \\ (t,j) \in \text{dom } \phi}} \omega_1(\phi(t, j)) = 0 \right\} \tag{11}$$

and

$$\mathcal{A}_\sigma := \left\{ \xi \in \mathcal{O} : \sup_{\substack{\phi \in \mathcal{S}_\sigma(\xi) \\ (t,j) \in \text{dom } \phi}} \omega_1(\phi(t, j)) = 0 \right\}. \tag{12}$$

3) $\sigma(x) > 0$ for all $x \in \mathcal{O} \setminus \mathcal{A}_\sigma$.

When such an admissible perturbation radius σ exists, we will say that the hybrid system $\mathcal{H}$ is **robustly $\mathcal{KLL}$-stable** with respect to (ω_1, ω_2) on $\mathcal{O}$.

We note that it is possible for the sets $\mathcal{A}_0$ and $\mathcal{A}_\sigma$ to be empty in the characterization of robust $\mathcal{KLL}$-stability. Also, in the special case when $\omega_1 \equiv \omega_2 =: \omega$, item 2 above is guaranteed by item 1, and item 3 reduces to $\sigma(x) > 0$ for all $x \in \mathcal{O} \setminus \{z : \omega(z) = 0\}$.

We now come to the main result of this paper:

Theorem 3.2: Let $\omega_i : \mathcal{O} \to \mathbb{R}_{\geq 0}$, $i = 1, 2$, be continuous. The following statements are equivalent:

(A) The hybrid system $\mathcal{H}$ is forward complete on $\mathcal{O}$ and there exists a smooth Lyapunov function V for $(\mathcal{O}, F, G, C, D, \omega_1, \omega_2)$ that satisfies $\{x : V(x) = 0\} = \mathcal{A}_0$.

(B) $C \cup D = \mathcal{O}$ and $\mathcal{H}$ is robustly $\mathcal{KLL}$-stable with respect to (ω_1, ω_2) on $\mathcal{O}$.

The somewhat simpler implication from (A) to (B) is shown in Section V. The outline of the reverse implication is in Section VI, and Section VII provides the missing details.

IV. Example: Temperature Control System

In this section, we discuss an example of a hybrid temperature control system. We establish the type of robust-$\mathcal{KLL}$ stability considered in this paper and then use our main result (the existence of a smooth Lyapunov function) to indicate how additional robustness properties accrue. These include robustness to perturbations that are often encountered in temperature control systems, perturbations like small measurement noise, slowly-varying parameters, small time-delays, etc.

Consider a control system that uses a heater to maintain the temperature of a plant in a desired temperature band. The heater has its own internal temperature and is required to shut off when its internal temperature is too high, in order to avoid overheating. Because of this, the desired temperature band for the plant is not a forward invariant set: if the plant temperature is at the lower end of the desired band and the heater is too hot to operate, the temperature in the plant will drop below the lower limit of the desired band. On the other hand, eventually the heater will cool down, and in its normal mode of operation it will be able to maintain the temperature of the plant in the desired band. Because of these scenarios, it is not the case that the system is $\mathcal{KLL}$-stable with respect to a single measure that is zero if the plant temperature is in the desired band and positive otherwise. On the other hand, the system will be $\mathcal{KLL}$-stable with respect to two measures, the first one of the type just described and the second one chosen appropriately. We will explain briefly why this $\mathcal{KLL}$-stability with respect to two measures is robust and thus the system admits a smooth Lyapunov function. Then we will touch upon how the existence of such a Lyapunov function can be used to generate *semiglobal practical robustness to persistent perturbations* that

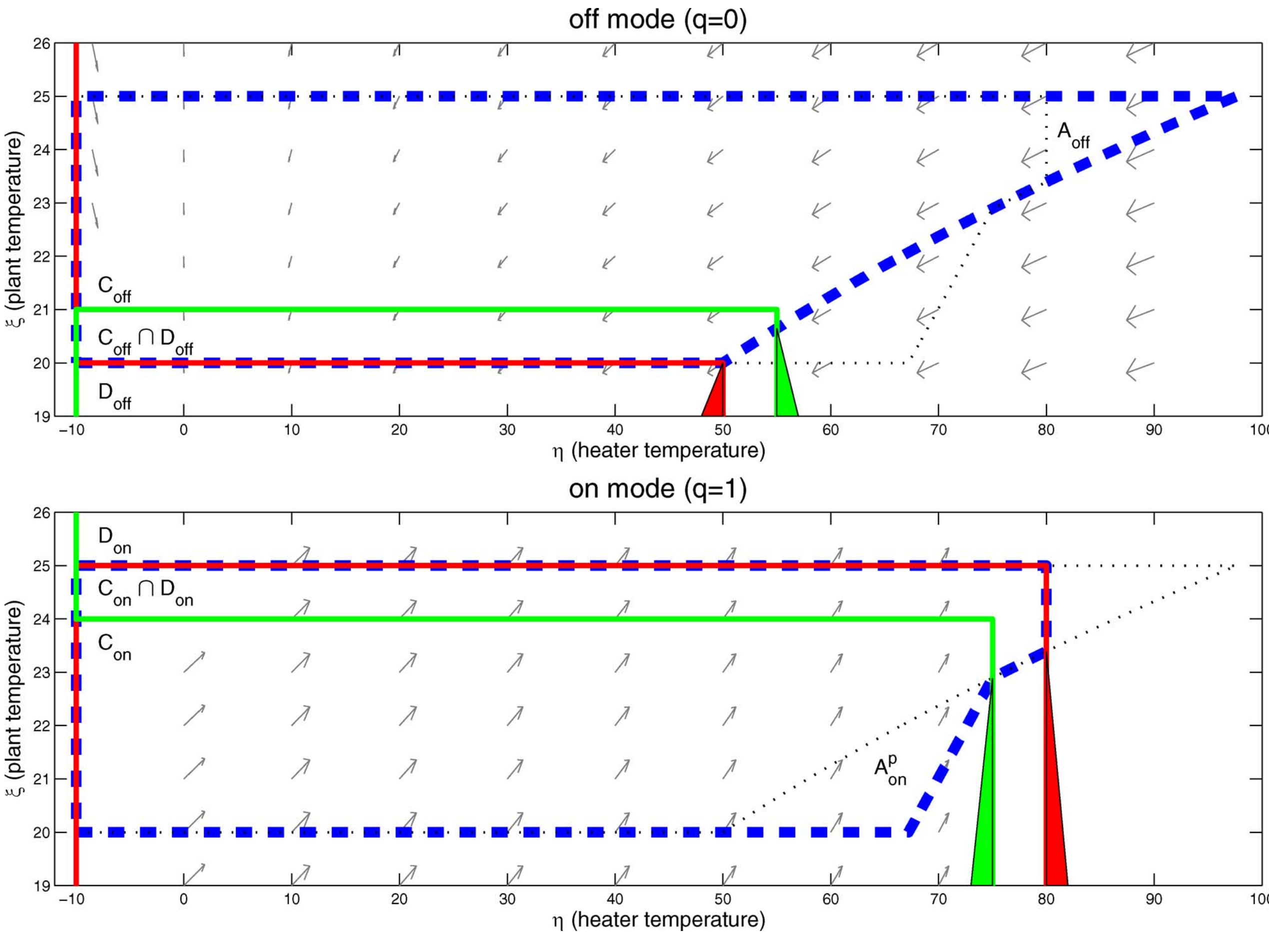

Fig. 1. The temperature control system with $p = 100$ (The arrows represent vector field for flow dynamics. The sets $A_{\rm off}$ and $A_{\rm on}^p$ are contoured with dash lines. The filled areas represent the added flow/jump sets under perturbation (note that the size of perturbation decreases to 0 as the state approaches $\mathcal{A}_0$).

cover small measurement noise, slowly varying parameters, small time delays, and "average dwell-time" perturbations.

A. The Hybrid Model

The hybrid model is based on the one given in [24]. The state of the closed-loop temperature control system consists of the plant temperature ξ, the heater's internal temperature η, a logic variable q used to indicate the off/on mode of the heater, and a parameter p that characterizes the effectiveness of the heater. We take the state-space to be $\mathcal{O} = \mathbb{R}^4$, although q takes values in the set $Q := \{0, 1\}$, corresponding to "off" and "on" respectively, p takes values in the compact interval [80, 120], and the heater temperature is constrained to the temperature range $[-10{,}150]$.

The desired temperature band for the plant is chosen as the interval [20, 25]. To keep the plant temperature in the desired band and avoid overheating, the heater is governed by the rules:

- The heater must turn off when $\xi > 25$ or $\eta > 80$; it may turn off when $\xi \geq 24$ or $\eta \geq 75$.
- The heater must turn on when $\xi < 20$ and $\eta < 50$; it may turn on when $\xi \leq 21$ and $\eta \leq 55$.

We observe that the behavior of the heater may not be unique and that there is some hysteresis to keep the heater from chattering between the on and off modes.

Define the state $x := [\xi,\ \eta,\ q,\ p]^\top$, and define the flow map as follows

$$F(x) := [qp - \xi,\ q(300 + \eta) - 3\eta,\ 0,\ 0]^\top .$$

where $\top$ denotes the vector transpose. Flows are enabled (meaning that the logic variable does not need to change) when either

- $q = 0$ and either $\xi > 20$ or $\eta > 50$ (this set is indicated as $C_{\rm off}$ in Fig. 1), or
- $q = 1$ and both $\xi < 25$ and $\eta < 80$ (this set is indicated as $C_{\rm on}$ in Fig. 1).

The closure of this set, intersected with the heater temperature constraint interval $[-10{,}150]$, is the flow set C.

The jump map simply toggles the logic variable. Jumps are enabled (the logic variable is allowed to change) when either

- $q = 0$ and both $\xi < 21$ and $\eta < 55$ (this set is indicated as $D_{\rm off}$ in Fig. 1), or
- $q = 1$ and either $\xi > 24$ or $\eta > 75$ (this set is indicated as $D_{\rm on}$ in Fig. 1).

The closure of this set, intersected with the heater temperature constraint set $[-10{,}150]$, is called $\widetilde{D}$. The jump set will be expanded later, for the purposes of analysis, to make the union of the jump set and the flow set covers $\mathbb{R}^4$.

B. (Robust) $\mathcal{KLL}$-Stability With Respect to Two Measures

While the plant temperature band [20, 25] is not forward invariant for the system described above, it is the case that the trajectories with initial conditions in $C \cup \widetilde{D}$ eventually reach and remain in this band for all time. In fact, the convergence rate to this temperature band is uniform over each compact set of initial conditions. For this reason, it can be shown that the set

$$\mathcal{A}_0 := \bigcup_{80 \leq p \leq 120} (A_{\text{off}} \times \{0\} \times \{p\}) \cup (A_{\text{on}}^p \times \{1\} \times \{p\})$$

is forward invariant and uniformly attractive, where the sets A_{off} and A_{on}^p can be computed analytically and are plotted and contoured by dashed lines in Fig. 1. Hence, taking $\omega_1(x) = |\xi|_{[20,25]}$ (which denotes the distance of the plant temperature to the band [20, 25]) and $\omega_2(x) = |x|_{\mathcal{A}_0}$, it can be shown that the system is $\mathcal{KLL}$-stable with respect to (ω_1, ω_2), where the $\mathcal{KLL}$ estimate γ is defined by

$$\gamma(r,t,j) := \sup\{\omega_1(\phi(t,j)) : \omega_2(x) \leq r,\ \phi \in \mathcal{S}(x)\ (\tau,k) \in \operatorname{dom}\phi,\ \tau + k \geq t + j\}.$$

Now, the obstruction to applying our main theorem is that we have not yet established robustness, and we do not yet have $\mathcal{O} = C \cup D$. The latter problem is easily remedied by taking $D = \mathbb{R}^4$ and the jump map to be set-valued, with the values in $\widetilde{D}$ being the union the previous jump map and any compact subset of $\mathcal{A}_0$, while the values being that same subset when outside of $\widetilde{D}$. This augmentation preserves the hybrid basic conditions, and while it will certainly introduce extra solutions (consisting of jumps to $\mathcal{A}_0$), the extra solutions will still obey the previous $\mathcal{KLL}$-stability with respect to two measures property. To verify robustness of the $\mathcal{KLL}$-stability, one can graphically argue the existence of an admissible perturbation radius σ (see Fig. 1 the filled area representing the added flow/jump sets under perturbation) and then follow similar stability arguments to the ones we used above for the original system, or one can treat $\mathcal{A}_0$ as an asymptotically stable set and then use the result that asymptotic stability of compact sets is robust in [5].

With robustness, it follows from Theorem 3.2 that there exists a smooth Lyapunov function V for $(\mathbb{R}^4, F, \widetilde{G}, C, \widetilde{D}, \omega_1, \omega_2)$. Moreover, we can use this V to show that, given $0 < \ell_1 < \ell_2 < \infty$, there exists $\delta > 0$ such that the inequalities in (13) hold (for example, by using similar arguments as those in the proof of Claim 5.1), where F_δ, $\widetilde{G}_\delta$, C_δ, and $\widetilde{D}_\delta$ are defined as in (5)–(8), respectively, by letting $\sigma(\cdot) \equiv \delta$. The key here is the smoothness of V, which, thanks to compactness of the set $\left\{x \in C \cup \widetilde{D} : V(x) \leq \ell\right\}$ for any $\ell \geq 0$, yields uniform continuity of V and ∇V on $\left\{x \in C \cup \widetilde{D} : V(x) \leq \ell\right\}$ and on the other sets appearing in (13) shown at the bottom of the page.

Finally, using terminology from parameterized differential equations, we can combine (2) and (13) to conclude that the temperature control system is *semiglobally practically $\mathcal{KLL}$-stable with respect to two measures in the parameter* δ, i.e., for each compact set $K \subset C \cup \widetilde{D}$ and each $\varepsilon > 0$, there exists $\delta^* > 0$ such that, for each $\delta \in [0, \delta^*]$, there exists a class-$\mathcal{KLL}$ function $\widetilde{\gamma}$ such that, each solution ϕ to $\widetilde{\mathcal{H}}_\delta := (F_\delta, \widetilde{G}_\delta, C_\delta, \widetilde{D}_\delta)$ with $\phi(0,0) \in K$ satisfies

$$\omega_1(\phi(t,j)) \leq \max\{\varepsilon,\ \widetilde{\gamma}(\omega_2(\phi(0,0)), t, j)\}$$

for all $(t,j) \in \operatorname{dom}\phi$. Furthermore, we can use these ideas to establish extra robustness properties that are different from the robustness used to get the existence of V, such as robustness to

1) small measurement noise;
2) small time delays at the measurements of the plant temperature;
3) slow time variations and small jumps in the parameter p; and
4) insertion of jumps according to an "average dwell-time" rule;

since the effects caused by these perturbations will always be overcome by (13) if the magnitudes of the perturbations are reasonably small. Space constraints preclude us from going into details (cf. [5, Section V]).

V. From (A) to (B) in Theorem 3.2

Throughout this section, we assume that $\mathcal{H}$ is forward complete and V is a smooth Lyapunov function as described in (A) of Theorem 3.2. We will show that $\mathcal{H}$ is robustly $\mathcal{KLL}$-stable.

A. Overview of the Proof

First, note that the forward completeness of $\mathcal{H}$ entails $C \cup D = \mathcal{O}$. Indeed, for any $x \notin C \cup D$ there exist (maximal and not complete) trivial solutions $\phi \in \mathcal{S}(x)$ with $\operatorname{dom}\phi = (0,0)$. The rest of the proof, broken into three steps, will do the following.

1) Show that, because $\mathcal{H}$ is forward complete on $\mathcal{O}$, there is an admissible perturbation radius σ_c that is positive everywhere on $\mathcal{O}$ such that $\mathcal{H}_{\sigma_c}$ is forward complete on $\mathcal{O}$.
2) Show that, because V is a smooth Lyapunov function for $(\mathcal{O}, F, G, C, D, \omega_1, \omega_2)$, there is an admissible perturbation radius $\sigma(\cdot) \leq \sigma_c(\cdot)$ that is positive where V is positive

$$\begin{aligned}
\max_{f \in F_\delta(x)} \langle \nabla V(x), f\rangle &\leq -\frac{1}{2}V(x) \forall x \in C_\delta \cap \left\{x \in C \cup \widetilde{D} : \ell_1 \leq V(x) \leq \ell_2\right\} \\
\max_{g \in \widetilde{G}_\delta(x)} V(g) &\leq e^{-1/2}V(x) \forall x \in \widetilde{D}_\delta \cap \left\{x \in C \cup \widetilde{D} : \ell_1 \leq V(x) \leq \ell_2\right\} \\
\max_{g \in \widetilde{G}_\delta(x)} V(g) &\leq \ell_1\ \forall x \in \widetilde{D}_\delta \cap \left\{x \in C \cup \widetilde{D} : V(x) \leq \ell_1\right\}.
\end{aligned} \tag{13}$$

and

$$\begin{aligned} \max_{f\in F_\sigma(x)} \langle \nabla V(x), f\rangle &\le -\frac{1}{2}V(x) \quad \forall x\in C_\sigma \\ \max_{g\in G_\sigma(x)} V(g) &\le e^{-1/2}V(x) \quad \forall x\in D_\sigma. \end{aligned} \tag{14}$$

3) Show that, because $\mathcal{H}_\sigma$ is forward complete (since σ is bounded by σ_c, each solution of $\mathcal{H}_\sigma$ is a solution of $\mathcal{H}_{\sigma_c}$ which is forward complete), (14) holds, and $\{x : V(x) = 0\} = \mathcal{A}_0$, it follows that the system $\mathcal{H}$ is robustly $\mathcal{KLL}$-stable with respect to (ω_1, ω_2) on $\mathcal{O}$, as certified by using the admissible perturbation radius σ.

B. Details

For the first step, note that in light of Propositions 2.2 and 3.1, given any admissible perturbation radius σ, forward completeness of $\mathcal{H}_\sigma$ fails only if there exists a maximal but not complete solution ϕ to $\mathcal{H}_\sigma$ with either $|\phi(t,j)| \to \infty$ or $\phi(t,j) \to \partial\mathcal{O}$ as $(t,j) \to \sup \mathrm{dom}\,\phi$. This is only possible if there exists a (maximal but not complete) solution ψ to the differential inclusion $\dot\psi(t) \in F_\sigma(\psi(t))$ subject to $\psi(t) \in C_\sigma$. Hence, we only need to show the existence of σ for which the existence of such ψ is excluded. For the case of $C = \mathcal{O}$, this has been done in [34, Lemma 7]. However, the proof carries over essentially without change for the case of a nontrivial C. The prerequisites – results that, for the constrained differential inclusion $\dot\psi(t) \in F(\psi(t))$, $\psi(t) \in C$, reachable sets from compact subsets of $\mathcal{O}$ are compact and that for small perturbations σ, reachable sets for the perturbed inclusion are close to those for the unperturbed one – follow from the more general results for hybrid systems in [12, Corollaries 4.7 and 5.5] but can be easily shown directly.

For the second step, assume that σ is an admissible perturbation radius that is positive everywhere on $\mathcal{O}$. In particular $x + \sigma(x)\overline{\mathbb{B}} \subset \mathcal{O}$ for all $x \in \mathcal{O}$, and also for each $\delta \in (0,1]$, the function $\delta\sigma$ is also an admissible perturbation radius.

Claim 5.1: For each compact $K \subset \mathcal{O}\backslash\mathcal{A}_0$ there exists $\delta \in (0,1]$ such that

$$\begin{aligned} \max_{f\in F_{\delta\sigma}(x)} \langle \nabla V(x), f\rangle &\le -\frac{1}{2}V(x) \quad \forall x\in C_{\delta\sigma}\cap K \\ \max_{g\in G_{\delta\sigma}(x)} V(g) &\le e^{-1/2}V(x) \quad \forall x\in D_{\delta\sigma}\cap K. \end{aligned} \tag{15}$$

Proof: Suppose the claim is false. Then, for each $i = 1, 2, \ldots$, either there exist $x_i \in C_{i^{-1}\sigma} \cap K$ and $f_i \in F_{i^{-1}\sigma}(x_i)$ such that

$$\langle \nabla V(x_i), f_i\rangle > -\frac{1}{2}V(x_i) \tag{16}$$

or there exist $x_i \in D_{i^{-1}\sigma} \cap K$ and $g_i \in G_{i^{-1}\sigma}(x_i)$ such that

$$V(g_i) > e^{-1/2}V(x_i)\,. \tag{17}$$

Since $C_{i^{-1}\sigma}$, $D_{i^{-1}\sigma}$ converge as sets to C and D, while the (locally bounded, uniformly in i) mappings $F_{i^{-1}\sigma}, G_{i^{-1}\sigma}$ converge graphically to F and G as $i \to \infty$ (see [12, Lemma 5.4]), and by compactness of K, without loss of generality we can assume that either x_i converge to $x \in C\cap K$ and f_i converge to $f \in F(x)$, or x_i converge to $x \in D\cap K$ and g_i converge to $g \in G(x)$. In the former case, using the continuity of ∇V and V, and (15) and (16) we have

$$\begin{aligned} -\frac{1}{2}V(x) &= \lim_{i\to\infty} -\frac{1}{2}V(x_i) \\ &\le \lim_{i\to\infty} \langle \nabla V(x_i), f_i\rangle \\ &= \langle \nabla V(x), f\rangle \\ &\le -V(x) \end{aligned}$$

which is impossible since $x \in \mathcal{O}\backslash\mathcal{A}_0$, i.e., $V(x) > 0$. In the latter case, using the continuity of V and (15) and (17), we have

$$\begin{aligned} e^{-1/2}V(x) &= \lim_{i\to\infty} e^{-1/2}V(x_i) \\ &\le \lim_{i\to\infty} V(g_i) \\ &= V(g) \\ &\le e^{-1}V(x) \end{aligned}$$

which again is impossible since $V(x) > 0$. ■

Since V is continuous and $\{x : V(x) = 0\} = \mathcal{A}_0$, the set $\mathcal{O}\backslash\mathcal{A}_0$ is open. Let $\{\mathcal{U}_i\}_{i=1}^\infty$ be a locally finite open cover of $\mathcal{O}\backslash\mathcal{A}_0$ with $\overline{\mathcal{U}}_i$ a compact subset of $\mathcal{O}\backslash\mathcal{A}_0$ and let $\{\kappa_i\}_{i=1}^\infty$ be a smooth partition of unity on $\mathcal{O}\backslash\mathcal{A}_0$ subordinate to the cover. To each $\overline{\mathcal{U}}_i$ associate the number $\delta_i \in (0,1]$ using Claim 5.1 applied to σ_c. Then define

$$\sigma(x) := \begin{cases} \sigma_c(x)\cdot\min\left\{|x|_{\mathcal{A}_0}, \sum_{i=1}^{\infty}\kappa_i(x)\delta_i\right\} & \text{for } x\in\mathcal{O}\backslash\mathcal{A}_0 \\ 0 & \text{for } x\in\mathcal{A}_0. \end{cases}$$

The function σ is continuous since σ_c is continuous, the κ_i are continuous, and $\sigma(x) \to 0$ as $x \to \mathcal{A}_0$. The function σ is positive on $\mathcal{O}\backslash\mathcal{A}_0$ since $\delta_i \in (0,1]$ for all i and σ_c is positive everywhere. The function σ is bounded by σ_c since $\delta_i \in (0,1]$ for each i. Finally, to see that (14) holds, note that since $\sigma(x) = 0$ for $x \in \mathcal{A}_0$, when $x \in \mathcal{A}_0$ the conditions (14) follow from the fact that V is a smooth Lyapunov function for $(\mathcal{O}, F, G, C, D, \omega_1, \omega_2)$. Let $x \in \mathcal{O}\backslash\mathcal{A}_0$. Then note that $\sigma(x) \le \delta_{i^*}\sigma(x)$ for some index i^* satisfying $x \in \overline{\mathcal{U}}_{i^*}$. By the construction of δ_{i^*} and the result of Claim 5.1 it follows that (14) holds.

Finally, for the last step, assume $\mathcal{H}_\sigma$ is forward complete on $\mathcal{O}$ and let $x \in \mathcal{O}$. By (14), each solution of $\phi \in \mathcal{S}_\sigma(x)$ satisfies

$$V(\phi(t,j)) \le e^{(-t-j)/2}V(x) \qquad \forall (t,j)\in\mathrm{dom}\,\phi\,. \tag{18}$$

For all $x \in \mathcal{O}$, since $\alpha_1(\omega_1(x)) \le V(x) \le \alpha_2(\omega_2(x))$, it follows that each $\phi \in \mathcal{S}_\sigma(x)$ satisfies

$$\omega_1(\phi(t,j)) \le \alpha_1^{-1}\left(\alpha_2(\omega_2(x))e^{(-t-j)/2}\right) \;\; \forall (t,j)\in\mathrm{dom}\,\phi\,.$$

So with $\gamma(s,t,j) := \alpha_1^{-1}\left(\alpha_2(s)e^{-(t+j)/2}\right)$, item 1 of the robust $\mathcal{KLL}$-stability assumption holds.

To see that $\mathcal{A}_\sigma = \mathcal{A}_0$, we first note that $\mathcal{A}_\sigma \subset \mathcal{A}_0$ since the solutions of $\mathcal{H}$ are also solutions of $\mathcal{H}_\sigma$. On the other hand, if $x \in \mathcal{A}_0$ then, by assumption, $V(x) = 0$. Then, by (18), $V(\phi(t,j)) = 0$ for all $\phi \in \mathcal{S}_\sigma(x)$ and all $(t,j) \in \mathrm{dom}\,\phi$. Then, by the bound $\alpha_1(\omega_1(z)) \le V(z)$ for all $z \in \mathcal{O}$, it follows that $\omega_1(\phi(t,j)) = 0$ for all $\phi \in \mathcal{S}_\sigma(x)$ and all $(t,j) \in \mathrm{dom}\,\phi$. Thus $x \in \mathcal{A}_\sigma$. Therefore, item 2 of the robust $\mathcal{KLL}$-stability assumption holds. Finally, item 3 of the robust $\mathcal{KLL}$-stability assumption holds since σ is positive where V is positive.

VI. From (B) to (A) in Theorem 3.2; Outline of the Construction

Throughout this section we assume that $C \cup D = \mathcal{O}$ and that the system $\mathcal{H}$ is robustly $\mathcal{KLL}$-stable with respect to (ω_1, ω_2) on $\mathcal{O}$ and we let γ be any $\mathcal{KLL}$ function coming from (10). First, note that as $C \cup D = \mathcal{O}$, forward completeness of $\mathcal{H}_\sigma$, with σ coming from the robust $\mathcal{KLL}$-stability of $\mathcal{H}$, implies that $\mathcal{H}$ is also forward complete.

The three subsections below describe the three main steps in the construction of the (smooth) Lyapunov function. Some of the important intermediate results are included in the subsections, while remaining technical details are filled in Section VII.

A. A Preliminary Lyapunov Function: (Possibly) Nonsmooth V_1

We first state the following lemma, whose proof is provided in Appendix B.

Lemma 6.1: For any $\gamma' \in \mathcal{KLL}$ and $\lambda > 0$ there exist κ_1, $\kappa_2 \in \mathcal{K}_\infty$ such that for all $(s,t,u) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$

$$\kappa_1(\gamma'(s,t,u)) \leq \kappa_2(s) e^{-\lambda(t+u)} . \tag{19}$$

Let $\kappa_1, \kappa_2 \in \mathcal{K}_\infty$ be any functions satisfying (19) with $\gamma' = \gamma$ and $\lambda = 2$. Define the preliminary Lyapunov function $V_1 : \mathcal{O} \to \mathbb{R}_{\geq 0}$ by

$$V_1(x) := \sup_{\phi \in \mathcal{S}_\sigma(x), (t,j) \in \mathrm{dom}\ \phi} \kappa_1(\omega_1(\phi(t,j))) e^{t+j} . \tag{20}$$

The function V_1 is upper semicontinuous – we will show this in Section VII-A – while its basic properties related to inequalities (2)–(4) are established below.

Lemma 6.2: The function $V_1 : \mathcal{O} \to \mathbb{R}_{\geq 0}$ defined in (20) satisfies

$$\kappa_1(\omega_1(x)) \leq V_1(x) \leq \kappa_2(\omega_2(x)) \qquad \forall x \in \mathcal{O} \tag{21}$$

and, for each $x \in \mathcal{O}$, $\phi \in \mathcal{S}_\sigma(x)$ and $(t,j) \in \mathrm{dom}\ \phi$

$$V_1(\phi(t,j)) \leq V_1(x) e^{-t-j} . \tag{22}$$

Proof: The lower bound in (21) comes from the definition of V_1 in (20)

$$V_1(x) \geq \sup_{\phi \in \mathcal{S}_\sigma(x), (t,j) \in \mathrm{dom}\ \phi} \kappa_1(\omega_1(\phi(t,j))) e^{t+j} \Big|_{t=0, j=0} = \kappa_1(\omega_1(x))$$

while the upper bound in (21) comes from the combination of (20), (10) and (19)

$$V_1(x) \leq \sup_{\phi \in \mathcal{S}_\sigma(x), (t,j) \in \mathrm{dom}\ \phi} \kappa_2(\omega_2(x)) e^{-t-j} \leq \kappa_2(\omega_2(x)) .$$

To see (22) we first note that if $\psi_\phi \in \mathcal{S}_\sigma(\phi(t,j))$ then there exists $\psi_x \in \mathcal{S}_\sigma(x)$ such that $(\tau,k) \in \mathrm{dom}\ \psi_\phi$ implies $(\tau + t, k+j) \in \mathrm{dom}\ \psi_x$ and $\psi_x(\tau+t, k+j) = \psi_\phi(\tau,k)$. Then note that, for each $x \in \mathcal{O}$, $\phi \in \mathcal{S}_\sigma(x)$, and $(t,j) \in \mathrm{dom}\ \phi$

$$\begin{aligned} V_1(\phi(t,j)) &= \sup_{\psi_\phi \in \mathcal{S}_\sigma(\phi(t,j)),\ (\tau,k) \in \mathrm{dom}\ \psi_\phi} \kappa_1(\omega_1(\psi_\phi(\tau,k))) e^{\tau+k} \\ &\leq \sup_{\substack{\psi_x \in \mathcal{S}_\sigma(x), (\tau+t, k+j) \in \mathrm{dom} \psi_x \\ (\tau,k) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}}} \kappa_1(\omega_1(\psi_x(\tau+t, k+j))) e^{\tau+k} \\ &\leq \sup_{\psi_x \in \mathcal{S}_\sigma(x),\ (\bar{\tau},\bar{k}) \in \mathrm{dom}\ \psi_x} \kappa_1(\omega_1(\psi_x(\bar{\tau},\bar{k}))) e^{\bar{\tau}+\bar{k}} e^{-t-j} \\ &= V_1(x) e^{-t-j} . \end{aligned}$$

■

Finally, we note that

$$V_1(x) = 0 \iff x \in \mathcal{A}_0 \tag{23}$$

which comes directly from the definitions of V_1 in (20) and $\mathcal{A}_\sigma$ in (12) and the fact that $\mathcal{A}_0 = \mathcal{A}_\sigma$ (see item 2 of the robust $\mathcal{KLL}$-stability definition).

B. Initial Smoothing of Lyapunov Function: From V_1 to V_s

Let $\psi : \mathbb{R}^n \to [0,1]$ be a smooth function vanishing outside of $\mathbb{B}$ satisfying $\int \psi(\xi) \mathrm{d}\xi = 1$ where the integration (here and in what follows) is over $\mathbb{R}^n$. We will now find a smooth and sufficiently small function $\widetilde{\sigma} : \mathcal{O} \backslash \mathcal{A}_0 \to \mathbb{R}_{>0}$ and define the new Lyapunov function candidate $V_s : \mathcal{O} \to \mathbb{R}_{\geq 0}$ by

$$V_s(x) := \begin{cases} 0, & \text{for } x \in \mathcal{A}_0 \\ \int V_1(x + \widetilde{\sigma}(x)\xi) \psi(\xi) \mathrm{d}\xi, & \text{for } x \in \mathcal{O} \setminus \mathcal{A}_0. \end{cases} \tag{24}$$

With the right choice of $\widetilde{\sigma}$, we will accomplish that

a) V_s is well-defined, continuous on $\mathcal{O}$, smooth and positive on $\mathcal{O} \backslash \mathcal{A}_0$;
b) as much as possible, the conditions (21), (22), and (23) hold with V_s in place of V_1;
c)

$$x \in C \backslash \mathcal{A}_0 \implies \max_{f \in F(x)} \langle \nabla V_s(x), f \rangle \leq -V_s(x) , \tag{25}$$

$$x \in D \implies \max_{g \in G(x)} V_s(g) \leq e^{-1} V_s(x) . \tag{26}$$

Regarding (a), the desired properties follow from [16, Theorem 3.1], due to the following facts: V_1 is upper semicontinuous (see Proposition 7.1), (23) holds, $\mathcal{O} \backslash \mathcal{A}_0$ is open (we show this in Section VII-C), and $\widetilde{\sigma}$ is smooth and sufficiently small (we establish this in Section VII-D).

Regarding (b), we first note that given $\mu_1, \mu_2 \in \mathcal{K}_\infty$ such that

$$\mu_1(s) < s < \mu_2(s) \qquad \forall s > 0 \tag{27}$$

the function $\widetilde{\sigma}$ can be chosen sufficiently small so that (see Lemma 7.7)

$$\kappa_1 \circ \mu_1(\omega_1(x)) \leq V_s(x) \leq \kappa_2 \circ \mu_2(\omega_2(x)) \qquad \forall x \in \mathcal{O} . \tag{28}$$

Thus, the bound (21) will degrade an arbitrarily small amount, as quantified through the functions μ_1 and μ_2. We will pick $\widetilde{\sigma}$ so that (22) holds, with V_s in place of V_1, but with the following degradation: the bound doesn't hold for all solutions of $\mathcal{H}_\sigma$ but rather for all solutions of a "smaller" perturbation $\mathcal{H}_{\sigma_2}$ characterized by a continuous function σ_2 that is positive on $\mathcal{O}\backslash\mathcal{A}_0$ but that satisfies $\sigma_2(x) \leq \sigma(x)$ for all $x \in \mathcal{O}$. To this end, we will construct functions σ_2 and $\widetilde{\sigma}$ such that for each $x \in \mathcal{O}\backslash\mathcal{A}_0$, for each $\phi \in \mathcal{S}_{\sigma_2}(x)$, for each $\xi \in \overline{\mathbb{B}}$ and $(t,j) \in \operatorname{dom}\phi$ such that $\phi(t,j) \in \mathcal{O}\backslash\mathcal{A}_0$, the function defined on $\operatorname{dom}\phi \cap [0,t] \times \{0,\ldots,j\}$ given by

$$(\tau,k) \mapsto \phi(\tau,k) + \widetilde{\sigma}(\phi(\tau,k))\xi$$

can be extended to a complete solution of $\mathcal{H}_\sigma$. (See Lemma 7.2.) In other words, certain perturbations of the solutions to $\mathcal{H}_{\sigma_2}$ are solutions to the perturbed system $\mathcal{H}_\sigma$. To conclude the comments on (b), we note that (23) for V_s comes from the definition of V_s, the condition (23) for V_1, the openness of $\mathcal{O}\backslash\mathcal{A}_0$, and the upper semicontinuity of V_1.

Regarding (c), first note that from (22), for each $x \in \mathcal{O}\backslash\mathcal{A}_0$, each $\phi \in \mathcal{S}_{\sigma_2}(x)$ and each $(t,j) \in \operatorname{dom}\phi$ such that $\phi(t,j) \in \mathcal{O}\backslash\mathcal{A}_0$, we have

$$\begin{aligned} V_s(\phi(t,j)) &= \int V_1(\phi(t,j) + \widetilde{\sigma}(\phi(t,j))\xi)\psi(\xi)\mathrm{d}\xi \\ &\leq e^{-t-j}\int V_1(x+\widetilde{\sigma}(x)\xi)\psi(\xi)\mathrm{d}\xi \\ &= e^{-t-j}V_s(x)\,. \end{aligned} \tag{29}$$

Claim 6.3: Suppose $\sigma_2 : \mathcal{O} \to \mathbb{R}_{\geq 0}$ is continuous, satisfies $\sigma_2(x) \leq \sigma(x)$ for all $x \in \mathcal{O}$ and $\sigma_2(x) = 0$ if and only if $\sigma(x) = 0$. For any $x \in C \setminus \mathcal{A}_0$ and $f \in F(x)$, there exists a solution$\phi \in \mathcal{S}_{\sigma_2}(x)$ such that, for sufficiently small $t > 0$, we have $(t,0) \in \operatorname{dom}\phi$ and $\phi(t,0) = x + tf$.

Proof: Since $x \in \mathcal{O}\backslash\mathcal{A}_0$, we have, from items 2 and 3 of the robust $\mathcal{KLL}$-stability assumption, that $\sigma(x) > 0$ and thus $\sigma_2(x) > 0$. Then, since $x \in C$ and σ_2 is continuous, there exists $\varepsilon > 0$ such that, for all $z \in x+\varepsilon\overline{\mathbb{B}}$, we have $x \in \left(z + \sigma_2(z)\overline{\mathbb{B}}\right) \cap C$. Then, from the definitions of F_{σ_2} and C_{σ_2} like in (5) and (7) but with σ_2 in place of σ, it follows that $z \in C_{\sigma_2}$ and $f \in F_{\sigma_2}(z)$ for all $z \in \{x\} + \varepsilon\overline{\mathbb{B}}$. These facts imply the claim. ■

As a result of Claim 6.3 and (29), for all $x \in C \setminus \mathcal{A}_0$ and $f \in F(x)$ and small $t \geq 0$, we have $V_s(x+tf) \leq e^{-t}V_s(x)$. This condition gives (25) as $\mathcal{O}\backslash\mathcal{A}_0$ is open and V_s is smooth on $\mathcal{O}\backslash\mathcal{A}_0$. Finally, for each $x \in D$ and $g \in G(x)$ there exists $\phi \in \mathcal{S}_{\sigma_2}(x)$ such that $(0,1) \in \operatorname{dom}\phi$ and $\phi(0,1) = g$. If $g \in \mathcal{O}\backslash\mathcal{A}_0$ then (29) implies $V_s(g) \leq e^{-1}V_s(x)$. If $g \in \mathcal{A}_0$ then $0 = V_s(g) \leq e^{-1}V_s(x)$. Therefore, (26) holds.

C. Final Smoothing of Lyapunov Function: From V_s to V

We now take a smooth function $\rho \in \mathcal{K}_\infty$ such that $\nabla\rho \in \mathcal{K}_\infty$, $\rho(s) \leq s\nabla\rho(s)$ for all $s > 0$, and such that the final Lyapunov function $V : \mathcal{O} \to \mathbb{R}_{\geq 0}$ defined by

$$V(x) := \rho \circ V_s(x) \tag{30}$$

is smooth. Existence of such ρ is guaranteed by [34, Lemma 17] (cf. [20, Lemma 4.3]), which is applicable as V_s is continuous on $\mathcal{O}$ and smooth and positive on $\mathcal{O}\backslash\mathcal{A}_0$.

It remains to establish that V is indeed a Lyapunov function. From (28), we get $\rho \circ \kappa_1 \circ \mu_1(\omega_1(x)) \leq V(x) \leq \rho \circ \kappa_2 \circ \mu_2(\omega_2(x))$ for all $x \in \mathcal{O}$, i.e., we get (2) with

$$\alpha_1 := \rho \circ \kappa_1 \circ \mu_1\,, \qquad \alpha_2 := \rho \circ \kappa_2 \circ \mu_2\,.$$

From (25) it follows that, for all $x \in C\backslash\mathcal{A}_0$

$$\begin{aligned} \max_{f\in F(x)}\langle \nabla V(x), f\rangle &\leq -\nabla\rho(V_s(x))V_s(x) \\ &\leq -\rho(V_s(x)) \\ &= -V(x)\,. \end{aligned}$$

Then, using the fact that, for each $x \in \mathcal{A}_0$, we have $V(x) = 0$ and $\nabla V(x) = 0$, we get

$$\max_{f\in F(x)}\langle \nabla V(x), f\rangle \leq -V(x) \qquad \forall x \in C\,.$$

Also by construction, $\nabla\rho \in \mathcal{K}_\infty$, i.e., ρ is convex. In particular $\rho(e^{-1}s) \leq e^{-1}\rho(s)$ for all $s \geq 0$. It then follows from (26) that, for all $x \in D$:

$$\begin{aligned} \max_{g\in G(x)} V(g) &= \rho\left(\max_{g\in G(x)} V_s(g)\right) \\ &\leq \rho\left(V_s(x)e^{-1}\right) \\ &\leq \rho\left(V_s(x)\right)e^{-1} \\ &= V(x)e^{-1}\,. \end{aligned}$$

VII. From (B) to (A) in Theorem 3.2; The Remaining Technical Details

We now provide the missing technical details. In Section VII-A we show that the function V_1 defined in (20) is upper semicontinuous. In Section VII-B we obtain some preliminary results that will allow us to show, in Section VII-C, that the set $\mathcal{O}\backslash\mathcal{A}_0$ is open and, in Section VII-D, that the function $\widetilde{\sigma}$ can be chosen so that V_s defined in (24) is continuous on $\mathcal{O}$, smooth and positive on $\mathcal{O}\backslash\mathcal{A}_0$, and the bound (28) holds.

A. The Upper Semicontinuity of V_1

Proposition 7.1: The function $V_1 : \mathcal{O} \to \mathbb{R}_{\geq 0}$ defined in (20) is upper semicontinuous.

Proof: First, we claim that for any $x \in \mathcal{O}$ such that $V_1(x) > 0$, the supremum in the definition of V_1 can be taken over $\phi \in \mathcal{S}_\sigma(x)$, $(t,j) \in \operatorname{dom}\phi$, and $t+j \in [0,\mathcal{T}(x)]$ (note the compact hybrid time interval) where

$$\mathcal{T}(x) := -\ln\left(\frac{V_1(x)}{\kappa_2(\omega_2(x))}\right) + 1\,. \tag{31}$$

Indeed, according to (10) and (19), for each $x \in \mathcal{O}$ and $\phi \in \mathcal{S}_\sigma(x)$ we have $\kappa_1(\omega_1(\phi(t,j))) \leq \kappa_2(\omega_2(x))e^{-2(t+j)}$ for all

$(t,j) \in \operatorname{dom} \phi$ and, thus, see the equation at the bottom of the page. Since $1/e < 1$ this verifies the claim.

Now pick any $x \in \mathcal{O}$ and $\{x_i\}_{i=1}^{\infty}$ with $x_i \in \mathcal{O}$ such that $x_i \to x$ as $i \to \infty$. Without loss of generality, assume that $\lim_{i\to\infty} V_1(x_i)$ exists. We want to show that $\lim_{i\to\infty} V_1(x_i) \leq V_1(x)$. Now either $\lim_{i\to\infty} V_1(x_i) = 0 \leq V_1(x)$ or else, by the definition of $\mathcal{T}$ in (31), we have $\lim_{i\to\infty} \mathcal{T}(x_i) < \infty$. In the latter case, for each i there exist $\phi_i \in \mathcal{S}_\sigma(x_i)$ and $(t_i, j_i) \in \operatorname{dom} \phi_i$ with $t_i + j_i \leq \mathcal{T}(x_i)$ such that

$$V_1(x_i) \leq \kappa_1(\omega_1(\phi_i(t_i,j_i)))e^{t_i+j_i} + \frac{1}{i} \,. \tag{32}$$

By [12, Theorems 4.4, 4.6] there exists a subsequence of ϕ_i's graphically convergent to some $\phi \in \mathcal{S}_\sigma(x)$. As $\lim_{i\to\infty} \mathcal{T}(x_i) < \infty$, one can pick a further subsequence of ϕ_i's along with corresponding (t_i, j_i)'s (we do not change the indices) so that $\lim_{i\to\infty}(t_i, j_i) = (t,j) \in \operatorname{dom} \phi$ and $\lim_{i\to\infty} \phi_i(t_i,j_i) = \phi(t,j)$. Now, it follows from (32) and the continuity of κ_1 and ω_1 that

$$\begin{aligned} \lim_{i\to\infty} V_1(x_i) &\leq \lim_{i\to\infty} \kappa_1(\omega_1(\phi_i(t_i,j_i)))e^{t_i+j_i} + \frac{1}{i} \\ &= \kappa_1(\omega_1(\phi(t,j)))e^{t+j} \\ &\leq V_1(x) \,. \end{aligned} \tag{33}$$

■

B. Solution Perturbations are Perturbed Solutions

Lemma 7.2: Let the open set $\mathcal{O}_1$ satisfy $\mathcal{O}\backslash\mathcal{A}_0 \subset \mathcal{O}_1 \subset \mathcal{O}\backslash\{x : \sigma(x) = 0\}$ and let $\sigma_b : \mathcal{O}_1 \to \mathbb{R}_{>0}$ be bounded away from zero on compact subsets of $\mathcal{O}_1$. There exist a continuous function $\sigma_- : \mathcal{O} \to \mathbb{R}_{\geq 0}$ that is positive at points where σ is positive and a smooth function $\sigma_s : \mathcal{O}_1 \to \mathbb{R}_{>0}$, both bounded above by σ_b and such that, for each $x \in \mathcal{O}_1$, each $\phi \in \mathcal{S}_{\sigma_-}(x)$, each $(t,j) \in \operatorname{dom} \phi$ such that $\phi(t,j) \in \mathcal{O}\backslash\mathcal{A}_0$ and each $\xi \in \overline{\mathbb{B}}$, the function defined on $\operatorname{dom} \phi \cap [0,t] \times \{0,\ldots,j\}$ by

$$(\tau,k) \mapsto \phi(\tau,k) + \sigma_s(\phi(\tau,k))\xi$$

can be extended to a complete solution of $\mathcal{H}_\sigma$.

We start with two claims.

Claim 7.3: For each admissible perturbation radius $\eta_1 : \mathcal{O} \to \mathbb{R}_{\geq 0}$, there exists an admissible perturbation radius $\eta_2 : \mathcal{O} \to \mathbb{R}_{\geq 0}$ that is positive where η_1 is positive and such that

$$\begin{aligned} x + \eta_2(x)\overline{\mathbb{B}} &\subset C_{\eta_1} \quad \forall x \in C_{\eta_2} \\ x + \eta_2(x)\overline{\mathbb{B}} &\subset D_{\eta_1} \quad \forall x \in D_{\eta_2} \\ \left.\begin{aligned} F_{\eta_2}(x) + \eta_2(x)\overline{\mathbb{B}} &\subset F_{\eta_1}(y) \\ \Theta(x) &\subset G_{\eta_1}(y) \end{aligned}\right\} &\quad \forall y \in x + \eta_2(x)\overline{\mathbb{B}}. \end{aligned} \tag{34}$$

where $\Theta(x) := \{v \in \mathcal{O} : v \in g + \eta_2(g)\overline{\mathbb{B}},\ g \in G_{\eta_2}(x)\}$.

Proof: Let

$$\eta'(x) := \frac{1}{2}\min\left\{\eta_1(x), |x|_{\{y\in\mathcal{O}\,|\,\eta_1(y)=0\}}\right\} \quad \forall x \in \mathcal{O}$$

and define a (continuous) function η_2 by

$$\eta_2(x) := \frac{1}{2}\min\left\{\eta'(x), \min_{z\in x+\eta'(x)\overline{\mathbb{B}}} \eta_1(z)\right\} \,.$$

Then $0 \leq \eta_2(x) \leq \eta_1(x)$ and $\eta_2(x) > 0$ if and only if $\eta_1(x) > 0$. Furthermore, for all $y \in x + \eta_2(x)\overline{\mathbb{B}}$, we have, from the definition of η_2, that $\eta_2(x) \leq (1/2)\eta_1(y)$ and

$$\begin{aligned} x + \eta_2(x)\overline{\mathbb{B}} &\subset y + \eta_1(y)\overline{\mathbb{B}} \\ y + \eta_2(y)\overline{\mathbb{B}} &\subset x + \eta_1(x)\overline{\mathbb{B}} \,. \end{aligned} \tag{35}$$

To see the first inclusion in (35), note that $x + \eta_2(x)\overline{\mathbb{B}} \subset y + 2\eta_2(x)\overline{\mathbb{B}}$ and then use $2\eta_2(x) \leq \eta_1(y)$. For the second inclusion in (35), note that if $\eta_2(y) \leq \eta_2(x)$, then $y + \eta_2(y)\overline{\mathbb{B}} \subset y + \eta_2(x)\overline{\mathbb{B}} \subset x + 2\eta_2(x)\overline{\mathbb{B}}$ and the inclusion follows from $\eta_2(x) \leq (1/2)\eta_1(x)$. If $\eta_2(y) > \eta_2(x)$ then $x \in y + \eta_2(y)\overline{\mathbb{B}} \subset y + \eta'(y)\overline{\mathbb{B}}$ and, thus, $\eta_2(y) \leq (1/2)\eta_1(x)$. Therefore $y + \eta_2(y)\overline{\mathbb{B}} \subset x + \eta_2(x)\overline{\mathbb{B}} + (1/2)\eta_1(x)\overline{\mathbb{B}} \subset x + \eta_1(x)\overline{\mathbb{B}}$.

Suppose that $x \in C_{\eta_2}$ and recall that this means that $\left(x + \eta_2(x)\overline{\mathbb{B}}\right) \cap C \neq \emptyset$. For any $y \in x + \eta_2(x)\overline{\mathbb{B}}$, by the first inclusion in (35), we have $\left(y + \eta_1(y)\overline{\mathbb{B}}\right) \cap C \neq \emptyset$, which in turn means that $y \in C_{\eta_1}$. This shows the first inclusion in (34). The second inclusion is shown in the same fashion. To show the third inclusion in (34), note that

$$F_{\eta_2}(x) + \eta_2(x)\overline{\mathbb{B}} = \overline{\operatorname{co}}\, F\left(\left(x + \eta_2(x)\overline{\mathbb{B}}\right) \cap C\right) + 2\eta_2(x)\overline{\mathbb{B}} \,.$$

Now the very definition of F_{η_1}, the first inclusion in (35), and the fact that $2\eta_2(x) \leq \eta_1(y)$ when $y \in x + \eta_2(x)\overline{\mathbb{B}}$ finish the argument. To see the last inclusion in (34), note that, by the definition of G_{η_2}, the set $\Theta(x)$ equals

$$\begin{aligned} \{v \in \mathcal{O} : v \in g + \eta_2(g)\overline{\mathbb{B}},\ g \in w + \eta_2(w)\overline{\mathbb{B}}, \\ w \in G\left(\left(x + \eta_2(x)\overline{\mathbb{B}}\right) \cap D\right)\} \end{aligned}$$

which, by the second inclusion in (35) with $x = w$ and $y = g$, is a subset of

$$\left\{v \in \mathcal{O} : v \in w + \eta_1(w)\overline{\mathbb{B}}, w \in G\left(\left(x + \eta_2(x)\overline{\mathbb{B}}\right) \cap D\right)\right\} \,.$$

The last inclusion in (34) then follows from the definition of G_{η_1} and the first inclusion in (35). ■

Claim 7.4: Let $\mathcal{O}_1$ be open. For each function $\eta_3 : \mathcal{O}_1 \to \mathbb{R}_{>0}$ that is bounded away from zero on compact subsets of $\mathcal{O}_1$ there exists a smooth $\eta_4 : \mathcal{O}_1 \to \mathbb{R}_{>0}$ such that for all $x \in \mathcal{O}_1$, $\max\{\eta_4(x), |\nabla\eta_4(x)|\} \leq \eta_3(x)$.

$$\begin{aligned} V_1(x) &\leq \max\left\{\sup_{\phi\in\mathcal{S}_\sigma(x),(t,j)\in\operatorname{dom}\phi,t+j\in[0,\mathcal{T}(x)]} \kappa_1(\omega_1(\phi(t,j)))e^{t+j}, \kappa_2(\omega_2(x))e^{-\mathcal{T}(x)}\right\} \\ &= \max\left\{\sup_{\phi\in\mathcal{S}_\sigma(x),(t,j)\in\operatorname{dom}\phi,t+j\in[0,\mathcal{T}(x)]} \kappa_1(\omega_1(\phi(t,j)))e^{t+j}, V_1(x)\frac{1}{e}\right\} \,. \end{aligned}$$

Proof: Let $\{\mathcal{U}_i\}_{i=1}^{\infty}$ be a locally finite open cover of $\mathcal{O}_1$ with $\overline{\mathcal{U}}_i$ a compact subset of $\mathcal{O}_1$ and let $\{\kappa_i\}_{i=1}^{\infty}$ be a smooth partition of unity on $\mathcal{O}_1$ subordinate to the cover. The function

$$\eta_4(x) := \sum_{i=1}^{\infty} \frac{\inf_{z\in\mathcal{U}_i} \eta_3(z)}{2^i \sup_{z\in\mathcal{U}_i} \max\{\kappa_i(z), |\nabla\kappa_i(z)|\}}\kappa_i(x)$$

has the requested properties. ■

We are now ready to prove Lemma 7.2.

Proof: Let $\mathcal{O}_1$ be open and satisfy $\mathcal{O}\backslash\mathcal{A}_0 \subset \mathcal{O}_1 \subset \mathcal{O}\backslash\{x : \sigma(x) = 0\}$. Apply Claim 7.3 with $\eta_1 := \sigma$ to get the continuous function $\eta_2 : \mathcal{O} \to \mathbb{R}_{\geq 0}$ that is positive where σ is positive and satisfies (34). Apply Claim 7.4 with η_3 given by $\eta_3(x) = \min\{\eta_2(x), \sigma_b(x)\} / \left(1 + \max_{f\in F_{\eta_2}(x)} |f|\right)$. This gives a smooth function $\eta_4 : \mathcal{O}_1 \to \mathbb{R}_{>0}$ satisfying

$$\left.\begin{array}{l}\eta_4(x) \leq \min\{\eta_2(x), \sigma_b(x)\}\,, \\ \max_{f\in F_{\eta_2}(x)} |\langle\nabla\eta_4(x), f\rangle| \leq \eta_2(x)\end{array}\right\} \quad \forall x \in \mathcal{O}_1\,. \quad (36)$$

Now define $\sigma_- := \eta_2$ and $\sigma_s := \eta_4$. Let $x \in \mathcal{O}_1$ and $\phi \in \mathcal{S}_{\sigma_-}(x)$, $(t,j) \in \operatorname{dom}\phi$ and $\phi(t,j) \in \mathcal{O}\backslash\mathcal{A}_0$. Since $\mathcal{A}_\sigma = \mathcal{A}_0$ and the set $\mathcal{A}_\sigma$ is forward invariant, it follows that $\phi(\tau,k) \in \mathcal{O}\backslash\mathcal{A}_0 \subset \mathcal{O}_1$ for all $(\tau,k) \in \operatorname{dom}\phi$ with $\tau + k \leq t + j$. For each $\xi \in \overline{\mathbb{B}}$, consider the function defined on $\operatorname{dom}\phi \cap [0,t] \times \{0,\ldots,j\} =: E$ by

$$(\tau,k) \mapsto \phi(\tau,k) + \sigma_s(\phi(\tau,k))\xi =: \psi(\tau,k)\,.$$

Suppose (τ,k) and $(\tau,k+1) \in E$. By the definition of a solution, we have

$$\phi(\tau,k) \in D_{\sigma_-}\,,\ \phi(\tau,k+1) \in G_{\sigma_-}(\phi(\tau,k))\,.$$

Then, using $\sigma_-(\phi(\tau,k)) = \eta_2(\phi(\tau,k))$ and $\sigma(\phi(\tau,k)) = \eta_1(\phi(\tau,k))$, (34), and the first part of (36) to get $\sigma_s(\phi(\tau,k)) \leq \eta_2(\phi(\tau,k))$, we have $\psi(\tau,k) \in D_\sigma$ and

$$\begin{aligned}\psi(\tau,k+1) &\in \{g + \sigma_s(g)\xi : g \in G_{\sigma_-}(\phi(\tau,k))\} \\ &\subset G_\sigma(\phi(\tau,k) + \sigma_s(\phi(\tau,k))\xi) \\ &= G_\sigma(\psi(\tau,k)).\end{aligned}$$

Next suppose there exist $0 \leq \tau_a < \tau_b$ such that $(\tau_a,k), (\tau_b,k) \in E$. Then, by the definition of a solution, for almost all $t \in [\tau_a,\tau_b]$, we have

$$\phi(\tau,k) \in C_{\sigma_-}\,,\ \dot{\phi}(\tau,k) \in F_{\sigma_-}(\phi(\tau,k))\,.$$

Then, using $\sigma_-(\phi(\tau,k)) = \eta_2(\phi(\tau,k))$ and $\sigma(\phi(\tau,k)) = \eta_1(\phi(\tau,k))$, (34), and the second part of (36), we have for almost all $\tau \in [\tau_a,\tau_b]$, $\psi(\tau,k) \in C_\sigma$ and

$$\begin{aligned}\dot{\psi}(\tau,k) &\in F_{\sigma_-}(\phi(\tau,k)) + \sigma_-(\phi(\tau,k))\overline{\mathbb{B}} \\ &\subset F_\sigma(\phi(\tau,k) + \sigma_s(\phi(\tau,k))\xi) \\ &= F_\sigma(\psi(\tau,k)).\end{aligned}$$

This establishes the lemma. ■

C. The First Implication: The Set $\mathcal{O}\backslash\mathcal{A}_0$ is Open

Lemma 7.5: For each $x \in \mathcal{O}\backslash\mathcal{A}_0$ there exist $\delta > 0$ and $\varepsilon > 0$ such that $V_1(z) \geq \varepsilon$ for all $z \in \{x\} + \delta\overline{\mathbb{B}}$. In particular, because of (23), the set $\mathcal{O}\backslash\mathcal{A}_0$ is open.

Proof: Use Lemma 7.2 with $\mathcal{O}_1 = \mathcal{O}\backslash\{x : \sigma(x) = 0\}$ and $\sigma_b : \mathcal{O}_1 \to \mathbb{R}_{>0}$ defined by $\sigma_b(x) = 1$ for all $x \in \mathcal{O}_1$ to get σ_- and σ_s. By the definition of $\mathcal{A}_0$, there exists a solution $\phi \in \mathcal{S}(x)$ and $(t,j) \in \operatorname{dom}\phi$ such that $\omega_1(\phi(t,j)) =: c > 0$. Since ω_1 is continuous at $\phi(t,j)$, there exists $\varrho \in (0,1]$ such that, for all $\xi \in \overline{\mathbb{B}}$,

$$\omega_1(\phi(t,j) + \sigma_s(\phi(t,j))\varrho\xi) \geq \frac{c}{2}\,. \quad (37)$$

By the definition of solutions, we immediately have that $\phi \in \mathcal{S}(x)$ implies $\phi \in \mathcal{S}_{\sigma_-}(x)$. Let $\xi \in \overline{\mathbb{B}}$ be arbitrary and, for each $(\tau,k) \in \operatorname{dom}\phi$ with $\tau \leq t$, $k \leq j$, define

$$\psi(\tau,k) := \phi(\tau,k) + \sigma_s(\phi(\tau,k))\varrho\xi. \quad (38)$$

Since $\varrho \in (0,1]$, it follows from Lemma 7.2 that ψ can be extended to an element of $\mathcal{S}_\sigma(x + \sigma_s(x)\varrho\xi)$. It follows from the definition of V_1 and (37)–(38) that

$$V_1(x + \sigma_s(x)\varrho\xi) \geq \kappa_1(\omega_1(\psi(t,j)))\,e^{t+j} \geq \kappa_1\left(\frac{c}{2}\right)\,.$$

Taking $\delta = \sigma_s(x)\varrho$ and $\varepsilon = \kappa_1(c/2)$ establishes the lemma. ■

D. The Second Implication: $\widetilde{\sigma}$ Can be Chosen as Desired

We will apply Lemma 7.2 with $\mathcal{O}_1 = \mathcal{O}\backslash\mathcal{A}_0$, which is a subset of $\mathcal{O}\backslash\{x : \sigma(x) = 0\}$ because of items 2 and 3 in the definition of robust $\mathcal{KLL}$-stability, and σ_b constructed below in order to get the properties for V_s indicated in (28) and (29), as well as continuity of V_s.

Let $\mu_1, \mu_2 \in \mathcal{K}_\infty$ satisfy (27).

Claim 7.6: The functions $\sigma_u : \mathcal{O}\setminus\mathcal{A}_0 \to \mathbb{R}_{>0}$ and $\sigma_\ell : \mathcal{O}\setminus\mathcal{A}_0 \to \mathbb{R}_{>0}$, defined by

$$\begin{aligned}\sigma_u(x) &:= \sup\{r \in \mathbb{R}_{>0} : |z - x| \leq 2r,\ z \in \mathcal{O} \\ &\qquad\qquad \Rightarrow \omega_2(z) \leq \mu_2(\omega_2(x))\} \\ \sigma_\ell(x) &:= \sup\{r \in \mathbb{R}_{>0} : |z - x| \leq 2r,\ z \in \mathcal{O} \\ &\qquad\qquad \Rightarrow V_1(z) \geq \kappa_1(\mu_1(\omega_1(x)))\}\end{aligned}$$

are bounded away from zero on compact subsets of $\mathcal{O}\backslash\mathcal{A}_0$.

Proof: Pick any compact $K \subset \mathcal{O}\backslash\mathcal{A}_0$ and let $K' \subset \mathcal{O}\backslash\mathcal{A}_0$ be any compact neighborhood of K. For all $x \in K'$, $\omega_2(x) > 0$ and, thus, $\omega_2(x) < \mu_2(\omega_2(x))$. By the compactness of K' and the continuity of ω_2 and μ_2, there exists $\delta > 0$ such that $|x - z| \leq \delta$ implies $\omega_2(z) \leq \mu_2(\omega_2(x))$ for all $x \in K$. Thus, $\sigma_u(x) \geq \delta$ for all $x \in K$.

To deal with σ_ℓ, pick K, K' as above. By Lemma 7.5, there exists $\epsilon > 0$ such that $V_1(x) \geq \epsilon$ for all $x \in K'$. We have two cases to consider. On the set $\{x \in K : \kappa_1(\omega_1(x)) < \epsilon\}$, pick $\delta > 0$ such that $K + \delta\overline{\mathbb{B}} \subset K'$, and hence $|z - x| \leq \delta$ implies $V_1(z) \geq \epsilon \geq \kappa_1(\mu_1(\omega_1(x)))$. On the compact set $\{x \in K : \kappa_1(\omega_1(x)) \geq \epsilon\}$ we have $\kappa_1(\omega_1(x)) > \kappa_1(\mu_1(\omega_1(x)))$; by the compactness of K' and the continuity of ω_1 and μ_1, there exists $\delta > 0$ such that, for all $x \in K$ such that $\kappa_1(\omega_1(x)) \geq \epsilon$ and all z with $|z - x| \leq \delta$, we have $\kappa_1(\omega_1(z)) > \kappa_1(\mu_1(\omega_1(x)))$ and, hence, $V_1(z) > \kappa_1(\mu_1(\omega_1(x)))$. These imply that $\sigma_\ell(x) \geq \delta$ for all $x \in K$. ■

We then define, for each $x \in \mathcal{O} \setminus \mathcal{A}_0$

$$\sigma_b(x) := \min\{\sigma_u(x), \sigma_\ell(x), |x|_{\mathcal{A}_0}, \sigma(x)\}\,. \tag{39}$$

It follows from the claim above, the continuity of $|\cdot|_{\mathcal{A}_0}$ and σ, and items 2 and 3 of the robust $\mathcal{KLL}$-stability assumption, that $\sigma_b(\cdot)$ is bounded away from zero on compact subsets of $\mathcal{O} \setminus \mathcal{A}_0$.

We now apply Lemma 7.2 with $\mathcal{O}_1 = \mathcal{O}\setminus\mathcal{A}_0$ and σ_b to get σ_- and σ_s. We define $\sigma_2 = \sigma_-$ and $\widetilde{\sigma} = \sigma_s$. According to Lemma 7.2, $\widetilde{\sigma}(x) \leq \sigma_b(x)$ for all $x \in \mathcal{O}\setminus\mathcal{A}_0$. In particular, for each $x_0 \in \mathcal{A}_0$ and $\varepsilon > 0$, if $x \in \mathcal{O}\setminus\mathcal{A}_0$ and $|x - x_0| \leq \varepsilon$ then $\widetilde{\sigma}(x) \leq \sigma_b(x) \leq |x|_{\mathcal{A}_0} \leq |x - x_0| \leq \varepsilon$. Also, since σ is an admissible perturbation radius, so is $\widetilde{\sigma}$. Hence, we verify [16, Assumption 1]. Note that the upper semicontinuity of V_1 and the openness of $\mathcal{O} \setminus \mathcal{A}_0$ verify [16, Assumption 2], and that [16, Assumptions 1 and 2] are the required conditions for [16, Theorem 3.1].

We now continue toward establishing (28) and (29) for the function V_s defined in (24).

Lemma 7.7: For all $x \in \mathcal{O}$, $\kappa_1(\mu_1(\omega_1(x))) \leq V_s(x) \leq \kappa_2(\mu_2(\omega_2(x)))$, i.e., (28) holds.

Proof: If $x \in \mathcal{A}_0$ then $\omega_1(x) = 0$ and $V_s(x) = 0$. Thus

$$0 = \kappa_1(\mu_1(\omega_1(x)) = V_s(x) \leq \kappa_2(\mu_2(\omega_2(x))\,.$$

Thus the bounds hold for $x \in \mathcal{A}_0$. For $x \in \mathcal{O}\setminus\mathcal{A}_0$, since $\widetilde{\sigma}(x) \leq \sigma_u(x)$, we have

$$V_s(x) \leq \sup_{z\in\{x\}+\sigma_u(x)\overline{\mathbb{B}}} \kappa_2(\omega_2(z)) \leq \kappa_2(\mu_2(\omega_2(x)))\,.$$

Thus the upper bound holds for all $x \in \mathcal{O}$. Also, since $\widetilde{\sigma}(x) \leq \sigma_\ell(x)$, we have

$$V_s(x) \geq \inf_{z\in\{x\}+\sigma_\ell(x)\overline{\mathbb{B}}} V_1(z) \geq \kappa_1(\mu_1(\omega_1(x)))\,.$$

Thus the lower bound holds for all $x \in \mathcal{O}$. ■

Finally, given any $\phi \in \mathcal{S}_{\sigma_2}(x)$ and $(t,j) \in \operatorname{dom}\phi$, if $\phi(t,j) \in \mathcal{O} \setminus \mathcal{A}_0$, then by Lemma 7.2 and (22), (29) holds.

VIII. Lyapunov Function for Forward Completeness

Below, we show that forward completeness of a hybrid system can be equivalently characterized via a **smooth Lyapunov function for forward completeness** of $(\mathcal{O}, F, G, C, D)$. This generalizes [1, Theorem 2] and can be used to derive a Lyapunov characterization of "unboundedness observability" for hybrid systems as for continuous-time systems (see [1, Theorem 1]). Unboundedness observability is a weaker property than forward completeness and has proved to be a useful concept in nonlinear control systems; see [1] and references therein. For example, Lyapunov characterization of unboundedness observability has been used to establish a dynamic norm estimator and hence prove that unboundedness observability is necessary for constructing an asymptotic observer for some class of continuous-time systems (see [2, Section 6]).

A function $\omega : \mathcal{O} \to \mathbb{R}_{>0}$ is called proper with respect to $\mathcal{O}$ if $\omega(x_i) \to \infty$ for any sequence of points $x_i \in \mathcal{O}$ with $\lim_{i\to\infty}|x_i| = \infty$ or $\lim_{i\to\infty} x_i \in \partial\mathcal{O}$. We note that the existence of a smooth function $V : \mathcal{O} \to \mathbb{R}_{\geq 0}$ as in (2)–(4) with ω_1 being proper with respect to $\mathcal{O}$, does imply forward completeness of $\mathcal{H}$ on $\mathcal{O}$, if $C \cup D = \mathcal{O}$. Indeed, (2) implies that V is proper with respect to $\mathcal{O}$, the inequalities (3), (4) give that $V(\phi(t,j)) \leq e^{-t-j}V(x)$, and this excludes finite time "blow-up" of solutions. It turns out that far weaker inequalities than (3), (4) also imply forward completeness, and furthermore, they allow for a converse result.

Theorem 8.1: Let $\omega : \mathcal{O} \to \mathbb{R}_{>0}$ be a continuous function which is proper with respect to $\mathcal{O}$. The following statements are equivalent

(A_c) The hybrid system $\mathcal{H}$ is forward complete.

(B_c) $C \cup D = \mathcal{O}$ and there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a smooth function $V : \mathcal{O} \to \mathbb{R}_{>0}$ such that

$$\alpha_1(\omega(x)) \leq V(x) \leq \alpha_2(\omega(x)) \qquad \forall x \in \mathcal{O} \tag{40}$$

$$\max_{f\in F(x)} \langle \nabla V(x), f\rangle \leq V(x) \qquad \forall x \in C \tag{41}$$

$$\max_{g\in G(x)} V(g) \leq eV(x) \qquad \forall x \in D\,. \tag{42}$$

Proof: First, assume (B_c). By (40) and properness of ω, V is proper with respect to $\mathcal{O}$, and in particular, $\{z \in \mathcal{O} \mid V(z) \leq r\}$ is compact for all $r \geq 0$. By (41), (42), we have $V(\phi(t,j)) \leq e^{t+j}V(x)$ for all $x \in \mathcal{O}$, $\phi \in \mathcal{S}(x)$, $(t,j) \in \operatorname{dom}\phi$. In light of Proposition 2.2, all maximal solutions to $\mathcal{H}$ are complete, and hence (A_c) holds.

Now assume (A_c). As we already have shown (see the first step of the proof in Section V), there exists an admissible perturbation radius σ that is positive on $\mathcal{O}$ and such that $\mathcal{H}_\sigma$ is forward complete. Let $0 < \varepsilon := \min_{x\in\mathcal{O}} \omega(x)$, and define $\zeta : \mathbb{R}_{\geq\varepsilon} \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ by

$$\zeta(r,s) := \sup\{\omega(\phi(t,j)) : \phi \in \mathcal{S}_\sigma(x),\ \omega(x) \leq r,\ (t,j) \in \operatorname{dom}\phi,\ t+j \leq s\}.$$

This function is well defined as $\mathcal{H}_\sigma$ satisfies the hybrid basic conditions (see Proposition 3.1) and for such a hybrid system, reachable sets from compact sets are compact (see [12, Corollary 4.7]). By the very definition, ζ is nondecreasing in each argument.

Claim 8.2: There exist $\kappa_1, \kappa_2 \in \mathcal{K}_\infty$ such that

$$\kappa_1(\zeta(r,s)) \leq \kappa_2(r)e^{s/2} \quad \forall (r,s) \in \mathbb{R}_{\geq\varepsilon} \times \mathbb{R}_{\geq 0}\,.$$

Proof: Let $\alpha : \mathbb{R}_{\geq\varepsilon} \to \mathbb{R}_{\geq 0}$ be given by $\alpha(\tau) := \sup_{r+s\leq\tau} \zeta(r,s)$ for each $\tau \geq \varepsilon$. Clearly, α is nondecreasing. Pick $\kappa_1 \in \mathcal{K}_\infty$ to satisfy $\kappa_1(\alpha(2r)) \leq e^{r/2}$ for each $r \geq \varepsilon$ and $\kappa_2 \in \mathcal{K}_\infty$ to satisfy $\kappa_1(\alpha(2r)) \leq \kappa_2(r)$ for each $r \geq \varepsilon$ and $\kappa_2(\varepsilon) \geq 1$. Then κ_1 and κ_2 have the requested properties: if $\varepsilon \leq r \leq s$, then

$\kappa_1(\zeta(r,s))e^{-s/2} \le \kappa_1(\alpha(2s))e^{-s/2} \le 1 \le \kappa_2(r)$, and if $s \le r$, then $\kappa_1(\zeta(r,s))e^{-s/2} \le \kappa_1(\alpha(2r)) \le \kappa_2(r)$. ■

Define the (nonsmooth) Lyapunov function candidate $V_1 : \mathcal{O} \to \mathbb{R}_{\geq 0}$ by

$$V_1(x) := \sup_{\phi \in \mathcal{S}_\sigma(x), (t,j) \in \mathrm{dom}\ \phi} \kappa_1(\omega(\phi(t,j)))e^{-(t+j)}.$$

Then V_1 is well-defined, upper semicontinuous, satisfies $\kappa_1(\omega(x)) \le V_1(x) \le \kappa_2(\omega(x))$ for all $x \in \mathcal{O}$, and finally, for each $x \in \mathcal{O}$, $\phi \in \mathcal{S}_\sigma(x)$, and $(t,j) \in \mathrm{dom}\ \phi$, $V_1(\phi(t,j)) \le e^{t+j}V_1(x)$. These facts can be verified similarly to those in Proposition 7.1 and Lemma 6.2.

Now, construct $\widetilde{\sigma}$ and σ_2 as in Section VII-D, by considering $\mathcal{A}_0 = \emptyset$ and $\omega = \omega_1 = \omega_2$. (Note that (39) makes sense with empty $\mathcal{A}_0$, as then $|x|_{\mathcal{A}_0} = \infty$ for all x.) Then, define $V : \mathcal{O} \to \mathbb{R}_{\geq 0}$ by $V(x) := \int V_1(x + \widetilde{\sigma}(x)\xi)\psi(\xi)\mathrm{d}\xi$ for all $x \in \mathcal{O}$. By [16, Theorem 3.1], V is smooth on $\mathcal{O}$. As in the proof of Lemma 7.7, one can verify that

$$\kappa_1 \circ \mu_1(\omega(x)) \le V(x) \le \kappa_2 \circ \mu_2(\omega(x)) \quad \forall\, x \in \mathcal{O}$$

Note that $\kappa_1, \kappa_2 \in \mathcal{K}_\infty$ satisfy (27). Defining the functions $\alpha_1 := \kappa_1 \circ \mu_1$ and $\alpha_2 := \kappa_2 \circ \mu_2$ establishes (40). As in Section VI-B, one can verify that, for each $x \in \mathcal{O}$, $\phi \in \mathcal{S}_{\sigma_2}(x)$, and $(t,j) \in \mathrm{dom}\ \phi$, $V(\phi(t,j)) \le e^{t+j}V(x)$, and hence establish (41) and (42). ■

Appendix A
Proof of Proposition 3.1

Let $M_1 : \mathcal{O} \rightrightarrows \mathcal{O}$ and $M_2 : \mathcal{O} \rightrightarrows \mathbb{R}^n$ be outer semicontinuous (osc) mappings. Then $M_2 \circ M_1 : \mathcal{O} \rightrightarrows \mathbb{R}^n$ is osc if M_1 is locally bounded, and locally bounded if both M_1 and M_2 are locally bounded. (See [28, Proposition 5.52] for a global version. The extensions to mappings from and to $\mathcal{O}$ are immediate.)

Let $S : \mathcal{O} \rightrightarrows \mathcal{O}$ be given by $S(x) = x + \sigma(x)\overline{\mathbb{B}}$. Then S is osc (in fact continuous) and locally bounded on $\mathcal{O}$, and consequently, so are $S_C, S_D : \mathcal{O} \rightrightarrows \mathcal{O}$ given by $S_C(x) = S(x) \cap C$, $S_D(x) = S(x) \cap D$.

Now note that $F \circ S_C$ is osc and locally bounded, and by [10, Lemma 16, p. 66], so is $\overline{\mathrm{co}} F \circ S_C$. Now note that $F_\sigma(x) = \overline{\mathrm{co}} F \circ S_C(x) + \sigma(x)\mathbb{B}$, the mapping $x \rightrightarrows \sigma(x)\mathbb{B}$ is continuous (and locally bounded) at each point of $\mathcal{O}$, and thus by [28, Exercise 5.24, Proposition 5.51(a)], F_σ is osc and locally bounded. By [28, Proposition 2.23], it is also convex-valued, and finally, by its definition and nonemptiness of $F(x)$ for $x \in C$, is nonempty on C_σ.

Arguments similar to those above show that $G \circ S_D : \mathcal{O} \rightrightarrows \mathcal{O}$ is osc and locally bounded on $\mathcal{O}$. Now note that $G_\sigma = S \circ (G \circ S_D)$, and thus G_σ osc and locally bounded. Nonemptiness of G_σ on D_σ comes out directly from the definitions.

Finally, extend S to $\mathbb{R}^n$ by setting $S(x) = x$ outside $\mathcal{O}$; such a mapping is osc. Note that $C_\sigma = \mathcal{O} \cap S^{-1}(\overline{C})$. The set $S^{-1}(\overline{C})$ is closed by [28, Theorem 5.25 (b)], and thus C_σ is relatively closed in $\mathcal{O}$. Similar arguments show that so is D_σ.

Appendix B
Proof of Lemma 6.1

The result will follow from [34, Lemma 3] (which is based on [32, Proposition 7] but uses the definition of $\mathcal{KL}$-functions used here) after we establish the existence of $\beta \in \mathcal{KL}$ such that $\gamma'(s,t,u) \le \beta(s,t+u)$ for all $s,t,u \ge 0$. Define $\beta(s,r) := \sup_{t \ge 0, u \ge 0, t+u \ge r} \gamma'(s,t,u)$. The required monotonicity properties of β follow from the analogous properties for γ'. Moreover, $\lim_{s \searrow 0} \beta(s,r) \le \lim_{s \searrow 0} \gamma'(s,0,0) = 0$ and

$$\lim_{r \to \infty} \beta(s,r) \le \lim_{r \to \infty} \sup_{\substack{t \ge 0, u \ge 0 \\ t+u=r}} \min\{\gamma'(s,t,0), \gamma'(s,0,u)\} = 0.$$

This establishes that $\beta \in \mathcal{KL}$ and thus gives the result.

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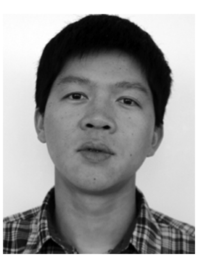

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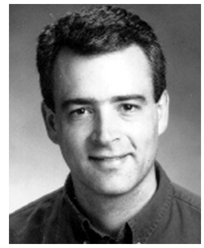

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