Smooth Lyapunov Functions for Hybrid Systems Part II: (Pre)Asymptotically Stable Compact Sets

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***Abstract*—It is shown that (pre)asymptotic stability, which generalizes asymptotic stability, of a compact set for a hybrid system satisfying mild regularity assumptions is equivalent to the existence of a smooth Lyapunov function. This result is achieved with the intermediate result that asymptotic stability of a compact set for a hybrid system is generically robust to small, state-dependent perturbations. As a special case, we state a converse Lyapunov theorem for systems with logic variables and use this result to establish input-to-state stabilization using hybrid feedback control. The converse Lyapunov theorems are also used to establish semiglobal practical robustness to slowly varying, weakly jumping parameters, to temporal regularization, to the insertion of jumps according to an "average dwell-time" rule, and to the insertion of flow according to a "reverse average dwell-time" rule.**

***Index Terms*—Asymptotic stability, hybrid systems, robustness, smooth Lyapunov functions.**

I. Introduction

A. Background

CONVERSE Lyapunov theorems identify classes of dynamical systems for which asymptotic stability is equivalent to the existence of a smooth Lyapunov function, i.e., a positive definite, radially unbounded function that decreases along the solutions of the dynamical system. Such theorems have proved to be very useful over the years for establishing robustness of asymptotic stability to various types of perturbations (see, for example, [13, Th. 56.4] or [17]), and for making advances in the area of stabilizability for nonlinear control systems (see, for example, [8, Proposition 2], [27, Sec. 2], [32, Th. 1], or [34, Sec. 4].) This paper establishes converse Lyapunov theorems in the setting of hybrid systems.

Hybrid systems are dynamical systems whose states can evolve continuously and/or evolve discontinuously, and they cover many useful and important systems such as hybrid automata, logic-based control systems, rigid mechanical systems with impact collisions, reset control systems, sampled-data control systems, and networked control systems, etc., (see [11, Sec. 2.2], [12], [16], [22], [28], [36], and Section IV-A of the current paper). While various concepts of solutions were introduced for hybrid systems to capture different properties of asymptotic stability and to establish many Lyapunov sufficient conditions for asymptotic stability (see detailed discussions in [4, Sec. I-A], [9, Sec. 1], and [11, Sec. 1]), *smooth* Lyapunov characterizations of asymptotic stability with emphasis on converse Lyapunov theorems for hybrid systems, which are deeply related to generating nominal robustness (to various types perturbations) and achieving hybrid input-to-state stabilization, are missing in the literature. Results on the existence of nonsmooth (typically discontinuous) Lyapunov functions for hybrid systems have appeared in [37].

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B. Contribution

Inspired by the results on the existence of smooth Lyapunov functions for asymptotically stable differential inclusions [6], [23], [33] and difference inclusions [18], [20], in this paper, we present converse Lyapunov theorems for hybrid systems in the framework proposed and developed in [9] and [11]. Some distinguishing features of this framework include allowing for multiple jumps at a time instant (see also, for example, [24]), allowing for (even instantaneous) Zeno solutions, and not insisting on the uniqueness of solutions, but, on the other hand, requiring some weak regularity and closedness properties from the data (see also [7]) as motivated by the pursuit of robustness of asymptotic stability. These regularity and closedness properties, through the results of [11], make possible general invariance principles (see [31]), and lead to results on the equivalence of robust $\mathcal{KLL}$ stability and the existence of a smooth Lyapunov function for $\mathcal{KLL}$ stability. (See [2] or [4] for precise definitions and results.) The current paper, a continuation of [4], uses this equivalence to show that smooth Lyapunov functions always exist for hybrid systems with compact, preasymptotically stable sets, and, in particular, with compact and asymptotically stable sets. The concept of preasymptotic stability we use is equivalent to asymptotic stability if local existence of solutions can be guaranteed, but is more general and allows, for example, for establishing converse Lyapunov theorems for systems with attractive, but not necessarily stable, sets. Finally, we demonstrate some direct applications of converse Lyapunov theorems in robustness analysis and robust stabilization problems.

The paper is organized as follows. Section II reviews the framework of hybrid systems that we use. Section III introduces preasymptotic stability and states the main results. Section IV specializes the converse Lyapunov theorem to systems with logic variables and uses it to establish an input-to-state stabilization result. Section V illustrates how to use the converse Lyapunov theorem to establish robustness of asymptotic stability in hybrid systems to various types of perturbations, including slowly varying, weakly jumping parameters, temporal regularization, jumps inserted according to an "average dwell-time" rule, and flows inserted according to a "reverse average dwell-time" rule. Section VI demonstrates by examples that the converse Lyapunov theorem can fail if the weak assumptions on the data are violated. Sections VII and Section VIII provide the proofs. Section IX states a result on robustness and a converse Lyapunov theorem for a special case of $\mathcal{KLL}$ stability, which covers the temperature control system example reported in [4]. Section X gives some brief conclusions.

C. Preliminaries

1) $\mathbb{R}$ denotes the reals, $\mathbb{Z}$ the integers, $\mathbb{R}_{\geq 0}$ the nonnegative reals, and $\mathbb{Z}_{\geq 0}$ the nonnegative integers.
2) $\mathbb{B}$ is the open unit ball in Euclidean space (of appropriate dimension).
3) Given a vector $v \in \mathbb{R}^n$, $v^\top$ denotes its transpose.
4) The sets $\overline{\mathcal{U}}$ and $\overline{\text{co}}\,\mathcal{U}$ are, respectively, the closure and closed convex hull of $\mathcal{U} \subset \mathbb{R}^n$.
5) Given two sets $\mathcal{U}_1$ and $\mathcal{U}_2$ in $\mathbb{R}^n$, their sum $\mathcal{U}_1 + \mathcal{U}_2$ is defined by $\{u_1 + u_2 : u_1 \in \mathcal{U}_1, u_2 \in \mathcal{U}_2\}$.
6) Given a set $\mathcal{U} \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, $|x|_{\mathcal{U}} := \inf_{y \in \mathcal{U}} |x - y|$.
7) Given an open set $\mathcal{O} \subset \mathbb{R}^n$, the set $\mathcal{U}$ is said to be *bounded with respect to* $\mathcal{O}$ if it is contained in a compact subset of $\mathcal{O}$.
8) Given sets $\mathcal{U}_1 \subset \mathcal{O} \subset \mathbb{R}^n$, $\mathcal{U}_1$ is said to be *relatively closed* (respectively, *relatively open*) in $\mathcal{O}$ if there exists a closed (respectively, an open) set $\mathcal{U}_2$ such that $\mathcal{U}_1 = \mathcal{U}_2 \cap \mathcal{O}$.
9) The *domain* of a set-valued map $M : \mathcal{O} \rightrightarrows \mathbb{R}^n$ is the set $\text{dom}\, M := \{x \in \mathcal{O} : M(x) \neq \emptyset\}$.
10) Given an open set $\mathcal{O}$ containing a compact set $\mathcal{U}$, a continuous function $\omega : \mathcal{O} \to \mathbb{R}_{\geq 0}$ is *proper on* $\mathcal{O}$ if $\omega(x_i) \to \infty$ when x_i converges to the boundary of $\mathcal{O}$ or $|x_i| \to \infty$, and is a *proper indicator for* $\mathcal{U}$ on $\mathcal{O}\}$ if it is proper on $\mathcal{O}$ and satisfies $\{x \in \mathcal{O} : \omega(x) = 0\} = \mathcal{U}$.
11) A set-valued map $M : \mathcal{O} \rightrightarrows \mathbb{R}^n$ is *outer semicontinuous* at $x \in \mathcal{O}$ if for all sequences $x_i \to x$ and $y_i \in M(x_i)$, if $\lim_{i\to\infty} y_i = y$ for some y, then $y \in M(x)$. The map is said to be outer semicontinuous if it is outer semicontinuous at each $x \in \mathcal{O}$. A set-valued map M is outer semicontinuous on $\mathcal{O}$ if and only if the graph of M is relatively closed in $\mathcal{O} \times \mathbb{R}^n$ (see [30, Th. 5.7]).
12) A set-valued map $M : \mathcal{O} \rightrightarrows \mathbb{R}^n$ is *locally bounded* if for any compact $K \subset \mathcal{O}$, there exists a compact set $K' \subset \mathbb{R}^n$ such that $M(K) := \{y : y \in M(x),\ x \in K\} \subset K'$.
13) A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to *class-*$\mathcal{K}$ ($\alpha \in \mathcal{K}$) if it is continuous, zero at zero, and strictly increasing. It is said to belong to *class-*$\mathcal{K}_\infty$ if, in addition, it is unbounded.
14) A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to *class-*$\mathcal{KL}\}$ ($\beta \in \mathcal{KL}$) if it satisfies: 1) for each $t \geq 0$, $\beta(\cdot, t)$ is nondecreasing and $\lim_{s\to 0^+} \beta(s,t) = 0$ and 2) for each $s \geq 0$, $\beta(s, \cdot)$ is nonincreasing and $\lim_{t\to\infty} \beta(s,t) = 0$. Note that $\mathcal{KL}$-functions here are slightly weaker than usual; see [4, Sec. I-C] for details.
15) A function $\gamma : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to *class-*$\mathcal{KLL}$ ($\gamma \in \mathcal{KLL}$) if, for each $r \geq 0$, $\gamma(\cdot, \cdot, r) \in \mathcal{KL}$ and $\gamma(\cdot, r, \cdot) \in \mathcal{KL}$.

II. Hybrid Systems

The hybrid systems we consider can be informally described by

$$\mathcal{H} := \begin{cases} \dot{x} \in F(x) & \text{for } x \in C \\ x^+ \in G(x) & \text{for } x \in D \end{cases} \tag{1}$$

where the variable x evolves in the state space $\mathcal{O}$, the sets $C \subset \mathcal{O}$ and $D \subset \mathcal{O}$ describe where the flow can occur, respectively, from where the jumps can occur, and the (set-valued) maps F and G describe the flow, respectively, the jumps. The state x may include both "continuous" and "discrete" variables, the latter often consisting of logical modes that can be associated with integer values (see Section IV-A). Hybrid automata, logic-based or sampled-data control systems, and impulsive and switched systems can all be modeled in the format of (1); see details in [9, Sec. 2], [11, Sec. 2.2], and Section IV-A of the current paper.

We impose the following standing assumption on (1).

Standing assumption (hybrid basic conditions):

The state space $\mathcal{O} \subset \mathbb{R}^n$ is open. The data (F, G, C, D) of the system $\mathcal{H}$ satisfy:

SA 1) the sets $C \subset \mathcal{O}$ and $D \subset \mathcal{O}$ are relatively closed in $\mathcal{O}$;

SA 2) the (set-valued) map $F : \mathcal{O} \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded, and $F(x)$ is nonempty and convex for all $x \in C$; and

SA 3) the (set-valued) map $G : \mathcal{O} \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded, and for each $x \in D$, $G(x)$ is a nonempty subset of $\mathcal{O}$.

These conditions were also used for the converse Lyapunov theorems in [4] where, in addition, it was assumed that $C \cup D = \mathcal{O}$, which guarantees existence of solutions from every initial condition. We emphasize that we do not assume $C \cup D = \mathcal{O}$ in this paper. In other words, the union of the flow set and the jump set does not need to cover the open state space $\mathcal{O}$.

The solutions to (1) are defined on hybrid time domains, as used in [7], [9], and [11]. A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is a *compact hybrid time domain* if $E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \cdots \leq t_J$. It is a *hybrid time domain* if for all $(T, J) \in E$, $E \cap ([0,T] \times \{0, 1, \ldots, J\})$ is a compact hybrid time domain. A *hybrid arc* is a function ϕ defined on a hybrid time domain, and such that $\phi(\cdot, j)$ is locally absolutely continuous for each j. A hybrid arc can be viewed as a set-valued map from $\mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ whose domain is a hybrid time domain. A hybrid arc $\phi :\ \text{dom}\,\phi \mapsto \mathcal{O}$ is a *solution* (trajectory) to $\mathcal{H}$ if $\phi(0,0) \in C \cup D$ and:

S 1) for all $j \in \mathbb{Z}_{\geq 0}$ and almost all $t \in \mathbb{R}_{\geq 0}$ s.t. $(t, j) \in \text{dom}\,\phi$: $\phi(t,j) \in C, \dot{\phi}(t,j) \in F(\phi(t,j))$ and

S 2) for all $(t,j) \in \operatorname{dom}\phi$ such that $(t,j+1) \in \operatorname{dom}\phi$: $\phi(t,j) \in D, \phi(t,j+1) \in G(\phi(t,j))$.

A solution to the hybrid system is called *maximal* if it cannot be extended, and *complete* if its domain is unbounded. Complete solutions are maximal. We denote by $\mathcal{S}(x)$ the set of all maximal solutions to $\mathcal{H}$ starting from x. Note that $\mathcal{S}(x)$ is empty for each $x \in \mathcal{O}\backslash(C \cup D)$. The hybrid system $\mathcal{H}$ is said to be *forward complete* on $\mathcal{O}$ if for all $x \in \mathcal{O}$, each $\phi \in \mathcal{S}(x)$ is complete. The hybrid system $\mathcal{H}$ is said to be *forward invariant* on a set $\mathcal{O}_1 \subset \mathcal{O}$ if, for all $x \in \mathcal{O}_1$, each $\phi \in \mathcal{S}(x)$ is such that $\phi(t,j) \in \mathcal{O}_1$ for all $(t,j) \in \operatorname{dom}\phi$. Note that forward invariance need not imply the existence of trajectories from all points of $\mathcal{O}_1$.

III. Converse Lyapunov Theorems for Preasymptotic Stability

A. Smooth Lyapunov Functions

In [4], we formalized the concept of a smooth Lyapunov function with respect to two measures for a hybrid system. Here, we specialize it to the case of a single measure.

Definition 3.1: Let $\mathcal{O}_1 \subset \mathcal{O}$ be open and $\omega : \mathcal{O}_1 \to \mathbb{R}_{\geq 0}$ be continuous. A function $V : \mathcal{O}_1 \to \mathbb{R}_{\geq 0}$ is said to be a smooth Lyapunov function for $(\mathcal{O}_1, F, G, C, D, \omega)$ if it is smooth and there exist class-$\mathcal{K}_\infty$ functions α_1, α_2 such that

$$\alpha_1(\omega(x)) \;\leq\; V(x) \;\leq\; \alpha_2(\omega(x)) \qquad \forall x \in \mathcal{O}_1 \tag{2}$$

$$\max_{f \in F(x)} \langle \nabla V(x), f\rangle \;\leq\; -V(x) \qquad \forall x \in \mathcal{O}_1 \cap C \tag{3}$$

$$\sup_{g \in G(x)\cap\mathcal{O}_1} V(g) \;\leq\; e^{-1}V(x) \qquad \forall x \in \mathcal{O}_1 \cap D. \tag{4}$$

We are interested in conditions guaranteeing that such a Lyapunov function exists, for the case where ω is a proper indicator for a compact set $\mathcal{A}$ on an open set $\mathcal{O}_1 \subset \mathcal{O}$. Before presenting such conditions, we give an example of a smooth "strict" Lyapunov function for the hybrid system describing the motion of the bouncing ball. (We note that the often used energy function $V(x) = x_2^2/2 + ax_1$ for the bouncing ball requires the use of LaSalle's invariance principle to establish asymptotic stability of the origin; see [31].)

Example 3.2: The bouncing ball system has the state $x := (x_1, x_2)$, the state space $\mathcal{O} := \mathbb{R}^2$, and the data (f, g, C, D) where

$$\begin{aligned} f(x) &:= (x_2, -a) \\ g(x) &:= (0,\ -\lambda x_2) \\ C &:= \left\{x \in \mathbb{R}^2 \;:\; x_1 \geq 0\right\} \\ D &:= \left\{x \in \mathbb{R}^2 \;:\; x_1 = 0,\ x_2 \leq 0\right\} \end{aligned}$$

with parameters $a > 0$ and $\lambda \in [0,1)$. We take $\omega(x) = |x|$, which is a proper indicator for the origin on $\mathbb{R}^2$ and, instead of asking for (2)–(4), for simplicity, we just ask that

$$\begin{aligned} \alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \qquad & \forall x \in C \cup D \\ \langle \nabla V(x), f(x)\rangle \leq -\rho_1(x) \qquad & \forall x \in C \\ V(g(x)) - V(x) \leq -\rho_2(x) \qquad & \forall x \in D \end{aligned}$$

where ρ_1 and ρ_2 are continuous, positive definite functions and α_1 and α_2 are class-$\mathcal{K}_\infty$ functions. Such a function can be readily converted into a function satisfying (2)–(4).

We choose $V(x) := (1 + \theta \arctan x_2)\left(x_2^2/2 + ax_1\right)$, where $\theta := (1 - \lambda^2)/(\pi + \pi\lambda^2)$. Then

$$\langle \nabla V(x), f(x)\rangle = -\frac{\theta a}{1 + x_2^2}\left(\frac{1}{2}x_2^2 + ax_1\right) \qquad \forall x \in C$$

$$\begin{aligned} & V(g(x)) - V(x) \\ & = \frac{1}{2}x_2^2\left(\lambda^2 + \theta\lambda^2 \arctan(-\lambda x_2) - 1 - \theta \arctan x_2\right) \\ & \leq \frac{1}{2}x_2^2\left(\lambda^2 - 1 + \frac{\theta\pi(\lambda^2+1)}{2}\right) \\ & \leq -\frac{1-\lambda^2}{4}x_2^2 \qquad \forall x \in D\,. \end{aligned}$$

Then, using the fact that $x_1 \geq 0$ for all $x \in C$ and $x_1 = 0$ for all $x \in D$, we can take

$$\begin{aligned} \rho_1(x) &= \frac{\theta a}{1 + x_2^2}\left(\frac{1}{2}x_2^2 + a|x_1|\right) \\ \rho_2(x) &= \frac{1-\lambda^2}{2}\left(\frac{1}{2}x_2^2 + a|x_1|\right). \end{aligned}$$

B. Preasymptotic Stability

In this section, we provide a Lyapunov-based motivation for the notion of preasymptotic stability. While this property is more general than asymptotic stability, we are not using it merely for the sake of generality. Working with preasymptotic stability also allows us to weaken the assumptions used for the existence of smooth Lyapunov functions. For example, in the conference version of this work [2], we employed standard asymptotic stability and needed to assume that the basin of attraction was open in order to develop theorems on the existence of smooth Lyapunov functions. The use of preasymptotic stability permits removing this "openness" assumption, which is fortunate since the assumption is not very natural for some hybrid systems, including systems with logic variables.

When $G(D \cap \mathcal{O}_1) \subset \mathcal{O}_1$ and ω is a proper indicator for a compact set $\mathcal{A}$ on an open set $\mathcal{O}_1 \subset \mathcal{O}$, it is not difficult to see that the existence of a smooth Lyapunov function for $(\mathcal{O}_1, F, G, C, D, \omega)$ implies that: 1) $\mathcal{O}_1$ is forward invariant; 2) each trajectory that starts near $\mathcal{A}$ remains near $\mathcal{A}$ for all time in its domain; and 3) each trajectory that starts in $\mathcal{O}_1$ is bounded with respect to $\mathcal{O}$ over its domain, and if its domain is unbounded, then the trajectory converges to $\mathcal{A}$.

Except for the fact that completeness of trajectories is not assumed,[1] properties (2) and (3) mentioned before resemble the standard notion of asymptotic stability for the set $\mathcal{A}$ with basin of attraction containing $\mathcal{O}_1$. Because completeness is not assumed, we will call the properties, taken together, "preasymptotic stability" with "basin of preattraction" containing $\mathcal{O}_1$. Our

[1]When all solutions are bounded with respect to $\mathcal{O}$, completeness is guaranteed by local existence of solutions from $C \cup D$. This happens, for example, when $C \cup D = \mathcal{O}$, but much weaker conditions can be given; see [11, Proposition 2.4].

main result is that, when ω is a proper indicator for a compact set $\mathcal{A}$ on an open set $\mathcal{O}_1 \subset \mathcal{O}$, the existence of a smooth Lyapunov function for $(\mathcal{O}_1, F, G, C, D, \omega)$ and the assumption $G(D \cap \mathcal{O}_1) \subset \mathcal{O}_1$ is *equivalent* to forward invariance of $\mathcal{O}_1$ and preasymptotic stability of $\mathcal{A}$ with basin of preattraction containing $\mathcal{O}_1 \cap (C \cup D)$. The latter property is formally defined as follows. Let $\mathcal{A} \subset \mathcal{O}$ be compact.

1) $\mathcal{A}$ is *prestable* for $\mathcal{H}$ if for each $\varepsilon > 0$, there exists $\delta > 0$ such that any solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_{\mathcal{A}} \leq \delta$ satisfies $|\phi(t,j)|_{\mathcal{A}} \leq \varepsilon$ for all $(t,j) \in \operatorname{dom}\phi$;
2) $\mathcal{A}$ is *preattractive* for $\mathcal{H}$ if there exists $\delta > 0$ such that any solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_{\mathcal{A}} \leq \delta$ is bounded with respect to $\mathcal{O}$, and if it is complete, then $\phi(t,j) \to \mathcal{A}$ as $t + j \to \infty$[2];
3) $\mathcal{A}$ is *preasymptotically stable* if it is both prestable and preattractive; and
4) ($\mathcal{A}$ is *asymptotically stable* if it is preasymptotically stable and there exists $\delta > 0$ such that any maximal solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_{\mathcal{A}} \leq \delta$ is complete.)

The set of all $x \in \mathcal{O}$ from which all solutions are bounded with respect to $\mathcal{O}$ and the complete ones converge to $\mathcal{A}$ is called the *basin of preattraction* of $\mathcal{A}$, denoted $\mathcal{O}^p_{\mathcal{A}}$.

The subsequent facts about preasymptotic stability can be combined with later results to generate novel converse Lyapunov theorems.

Proposition 3.3: Let $\mathcal{A} \subset \mathcal{O}$ be compact and suppose that $C \cup D$ is disjoint from $\mathcal{A}$. For the system (F, G, C, D), suppose that the set $\mathcal{O}_0 \subset \mathcal{O}$ is such that each solution starting in $\mathcal{O}_0$ is bounded with respect to $\mathcal{O}$ but not complete. Then, for the system (F, G, C, D), the set $\mathcal{A}$ is preasymptotically stable with basin of preattraction containing $\mathcal{O}_0$.

Proof: Prestability and preattractivity of $\mathcal{A}$ follow from the fact that $\mathcal{A}$ is disjoint from $C \cup D$. That the basin of preattraction contains $\mathcal{O}_0$ follows by definition. ■

The next four corollaries follow from Proposition 3.3.

Corollary 3.4: Let $\mathcal{A}$ and C be disjoint, compact subsets of $\mathcal{O}$. Suppose that the set-valued maps F and G are such that G is arbitrary, and there does not exist a complete solution to $\dot{x} \in F(x)$, $x \in C$. Then, for the "continuous-time" system $(F, G, C, \emptyset)$, the set $\mathcal{A}$ is preasymptotically stable with basin of preattraction equal to $\mathcal{O}$.

Corollary 3.5: Let $\mathcal{A}$ and D be disjoint, compact subsets of $\mathcal{O}$. Suppose that the set-valued maps F and G are such that F is arbitrary, and there does not exist a complete solution to $x^+ \in G(x)$, $x \in D$. Then, for the "discrete-time" system $(F, G, \emptyset, D)$, the set $\mathcal{A}$ is preasymptotically stable with basin of preattraction equal to $\mathcal{O}$.

Corollary 3.6: For the system (F, G, C, D), suppose that for each $x \in \mathcal{O}_0 \subset \mathcal{O}$, each corresponding solution $\phi \in \mathcal{S}(x)$ is bounded with respect to $\mathcal{O}$, and if complete, then there exists $(t,j) \in \operatorname{dom}\phi$ such that $\phi(t,j) \in K \subset \mathcal{O}$. Then, for each open set $\mathcal{G} \subset \mathcal{O}$ containing the union of K and a compact set $\mathcal{A}$ and for each system $(F, G, C \backslash \mathcal{G}, D \backslash \mathcal{G})$, the set $\mathcal{A}$ is preasymptotically stable with basin of preattraction containing the set $\mathcal{O}_0$.

[2]Complete solutions that converge to $\mathcal{A}$ are automatically bounded with respect to $\mathcal{O}$.

Corollary 3.7: Suppose, for the system (F, G, C, D), that the compact set $\mathcal{A}$ is preattractive with basin of preattraction $\mathcal{O}^p_{\mathcal{A}}$. Then, for each $\varepsilon > 0$ such that $\mathcal{A} + \varepsilon\mathbb{B} \subset \mathcal{O}$ and each system $(F, G, C \backslash (\mathcal{A} + \varepsilon\mathbb{B}), D \backslash (\mathcal{A} + \varepsilon\mathbb{B}))$, the set $\mathcal{A}$ is preasymptotically stable with basin of preattraction containing $\mathcal{O}^p_{\mathcal{A}}$.

The next proposition asserts that a forward invariant, uniformly preattractive set is preasymptotically stable; the proof relies on [11, Proposition 6.1] and is given in Section VIII.

Proposition 3.8: Let the compact set $\mathcal{A} \subset \mathcal{O}$ be forward invariant and uniformly preattractive, i.e., there exists $\delta > 0$, and for each $\varepsilon > 0$, there exists $\mathcal{T} > 0$ such that any solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_{\mathcal{A}} \leq \delta$ is bounded with respect to $\mathcal{O}$ and $|\phi(t,j)|_{\mathcal{A}} \leq \varepsilon$ for all $(t,j) \in \operatorname{dom}\phi$ satisfying $t + j \geq \mathcal{T}$ (the set of $(t,j) \in \operatorname{dom}\phi$ such that $t + j \geq \mathcal{T}$ may be empty). Under these conditions, the set $\mathcal{A}$ is preasymptotically stable.

The next proposition shows how to construct a forward invariant, uniformly preattractive set from a reachable set; the proof relies on [11, Corollary 4.7] and is provided in Section VIII. First, we set some notation. For a compact set $K \subset \mathcal{O}$ and a real number $\mathcal{T} > 0$, we define the reachable set from K in finite hybrid time $\mathcal{T}$ for the hybrid system $\mathcal{H}$ as follows

$$\mathcal{R}_{\leq \mathcal{T}}(K) := \{\xi \in \mathcal{O} \;:\; \xi = \phi(t,j),\ (t,j) \in \operatorname{dom}\phi \\ t + j \leq \mathcal{T},\ \phi \in \mathcal{S}(x),\ x \in K\}.$$

We also define the "infinite horizon" reachable set as $\mathcal{R}_\infty(K) := \bigcup_{\mathcal{T} \geq 0} \mathcal{R}_{\leq \mathcal{T}}(K)$.

Proposition 3.9: Let K and K_0 be compact subsets of $\mathcal{O}$ and suppose there exists $\delta > 0$ such that $\mathcal{R}_\infty(K) + \delta\mathbb{B} \subset K_0$ and there exists $\mathcal{T} > 0$ such that, for each $x \in K_0$ and each $\phi \in \mathcal{S}(x)$, either $t + j < \mathcal{T}$ for all $(t,j) \in \operatorname{dom}\phi$ and ϕ is bounded with respect to $\mathcal{O}$, or else, there exists $(t,j) \in \operatorname{dom}\phi$ such that $\phi(t,j) \in K$ and $t + j \leq \mathcal{T}$. Then, the set $\mathcal{R}_\infty(K)$ is compact and preasymptotically stable with basin of preattraction containing K_0.

We note that Proposition 3.9 is similar to [33, Proposition 4]. The assumptions for [33, Proposition 4] are stronger. There is no Lipschitz condition on F here, and no assumption that solutions remain in K after time $\mathcal{T}$. On the other hand, the set that is identified as being asymptotically stable is smaller in [33, Proposition 4]. There it is a subset of K, whereas here it is a superset of K. We also point out that an attractor that is similar to but typically smaller than $\mathcal{R}_\infty(K)$ in Proposition 3.9 can be expressed in terms of the omega-limit set: for a discussions of this notion for hybrid systems, see [1].

The next proposition is a consequence of the definition of solutions to hybrid systems.

Proposition 3.10: If the compact set $\mathcal{A}$ is preasymptotically stable for $\mathcal{H} := (F, G, C, D)$, then $\mathcal{A}$ is also preasymptotically stable for both the "continuous-time" system $\mathcal{H}_c := (F, G, C, \emptyset)$ and the "discrete-time" system $\mathcal{H}_d := (F, G, \emptyset, D)$.

The converse of Proposition 3.10 is not true, as the next two examples show. The second example also illustrates the difference between preasymptotic stability and asymptotic stability.

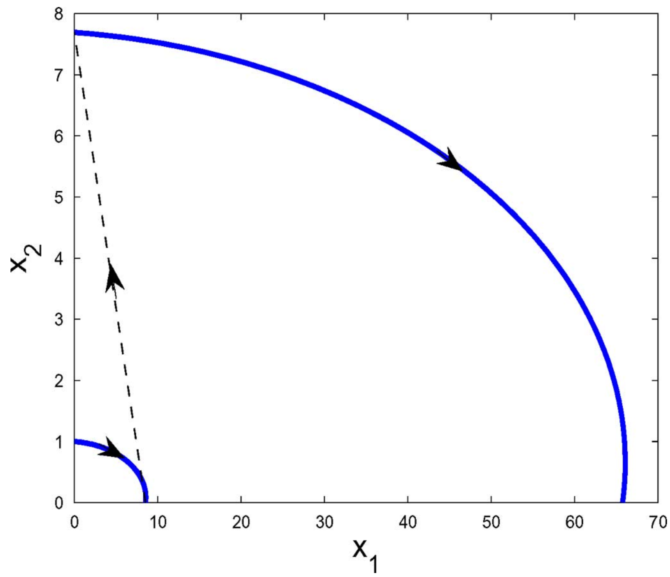

Fig. 1 Possible trajectories for the hybrid system in Example 3.11.

Example 3.11: Let $x = (x_1, x_2)$, $\mathcal{O} = \mathbb{R}^2$, $C = D = \mathcal{O}$, $F(x) = A_c x$, and $G(x) = A_d x$, where

$$A_c = \begin{bmatrix} -\theta & 100 \\ -1 & -\theta \end{bmatrix}, \qquad A_d = \begin{bmatrix} 0 & -\lambda \\ \lambda & 0 \end{bmatrix}$$

$\theta = 1$, and $\lambda = 0.9$. The eigenvalues of A_c and A_d are $-\theta \pm 10\mathrm{i}$ and $\pm\lambda\mathrm{i}$, respectively. Hence the origin is asymptotically stable for both $\mathcal{H}_c = (F, G, C, \emptyset)$ and $\mathcal{H}_d = (F, G, \emptyset, D)$. However, for $\mathcal{H}$, the origin is not asymptotically stable: solutions starting from $\{x_1 = 0, x_2 > 0\}$ may remain in the first quadrant (by flowing to $\{x_1 > 0, x_2 = 0\}$ and then jumping to $\{x_1 = 0, x_2 > 0\}$, and so on) and go unbounded. See Fig. 1. The switching between flow and jump in this example is similar to examples of destabilization by arbitrary switching between stable systems, which have appeared in the switched systems literature.

Example 3.12: In Example 3.11, redefine $C := \{x_2 \geq 0\}$, $D := \{x_2 \leq 0\}$, $\theta := -1$, and $\lambda := -2$. For both $\mathcal{H}_c = (F, G, C, \emptyset)$ and $\mathcal{H}_d = (F, G, \emptyset, D)$, the origin is preasymptotically stable with $\mathcal{O}^p_{\mathcal{A}} = \mathbb{R}^2$, since the solutions of these systems are incomplete and bounded. The origin is not preasymptotically stable for $\mathcal{H}$, since there are complete, unbounded solutions.

C. Main Results: Converse Lyapunov Theorems for Preasymptotic Stability

We now state our main result about the existence of smooth Lyapunov functions for preasymptotically stable compact sets in a hybrid system framework. This result will be used in later sections to establish various forms of robustness for hybrid control systems, and to establish input-to-state stabilization with respect to additive disturbances at the control input when using hybrid feedback (cf. [32]).

Theorem 3.13: Let $\mathcal{A}$ be compact and $\mathcal{O}_1$ be open, with $\mathcal{A} \subset \mathcal{O}_1 \subset \mathcal{O}$, and let ω be a proper indicator for $\mathcal{A}$ on $\mathcal{O}_1$. The following statements are equivalent.

1) There exists a smooth Lyapunov function for $(\mathcal{O}_1, F, G, C, D, \omega)$ and $G(D \cap \mathcal{O}_1) \subset \mathcal{O}_1$.
2) The set $\mathcal{A}$ is preasymptotically stable for $\mathcal{H}$, its basin of preattraction $\mathcal{O}^p_{\mathcal{A}}$ contains $\mathcal{O}_1$, and $\mathcal{O}_1$ is forward invariant for $\mathcal{H}$.

To see the implication 1$\Rightarrow$2 in Theorem 3.13, let ϕ be any solution to $\mathcal{H}$ such that $\phi(0,0) \in \mathcal{O}_1$. Then, $\phi(t,j) \in \mathcal{O}_1$ for all $(t,j) \in \operatorname{dom}\phi$; otherwise, due to the assumption $G(D \cap \mathcal{O}_1) \subset \mathcal{O}_1$, the solution ϕ must flow outside $\mathcal{O}_1$, which immediately contradicts (3) and the property of ω. Thus, we deduce from the existence of a smooth Lyapunov function for $(\mathcal{O}_1, F, G, C, D, \omega)$ that

$$\omega(\phi(t,j)) \leq \alpha_1^{-1}\left(\alpha_2(\omega(\phi(0,0)))e^{-t-j}\right) \qquad \forall (t,j) \in \operatorname{dom}\phi.$$

This implies the remaining conclusions of statement (1) in Theorem 3.13. We add that, in order to conclude standard asymptotic stability in statement (2), one must add to (1) conditions that guarantee local existence of solutions from $C \cup D$. See, for example, [11, Proposition 2.4].

The implication 2$\Rightarrow$1 will be proved in Section VII by the following steps: 1) restrict the hybrid system $\mathcal{H}$ to the set $\mathcal{O}_1$; 2) augment this restricted system to get a hybrid system (even if the original system was not hybrid by virtue of $C = \emptyset$ or $D = \emptyset$) with set $\mathcal{A}$ that is globally asymptotically stable, i.e., its basin of attraction is equal to $\mathcal{O}_1$; 3) establish that global asymptotic stability is robust; and 4) invoke results from [4] that link robustness to the existence of the desired smooth Lyapunov function. In Section VI, we show that the implication may fail if we relax the hybrid basic conditions.

Note that Theorem 3.13 implies that the converse direction in Proposition 3.10 holds if and only if $\mathcal{H}_c$ and $\mathcal{H}_d$ share a common smooth Lyapunov function, that is, there exists a smooth Lyapunov function for $\mathcal{H}$.

The next result, proved in Section VIII, not only asserts useful properties on basins of preattraction (cf. [11, Proposition 6.4]), but also gives a converse Lyapunov theorem for preasymptotically stable hybrid systems (cf. [2, Theorem 1]).

Theorem 3.14: For the hybrid system $\mathcal{H}$, if the compact set $\mathcal{A} \subset \mathcal{O}$ is preasymptotically stable, then its basin of preattraction $\mathcal{O}^p_{\mathcal{A}}$ is open and forward invariant. Furthermore, for each ω as a proper indicator for $\mathcal{A}$ on $\mathcal{O}^p_{\mathcal{A}}$, there exists a smooth Lyapunov function for $(\mathcal{O}^p_{\mathcal{A}}, F, G, C, D, \omega)$.

As we already noted, when $C \cup D = \mathcal{O}$, preasymptotic stability agrees with asymptotic stability and the basin of preattraction becomes the basin of attraction in Theorem 3.14. Thus, by setting $C = \mathcal{O}$ and $D = \emptyset$, Theorem 3.14 captures converse Lyapunov theorems for local asymptotic stability in differential inclusions; see [6, Th. 1.2] and [33, Sec. 3.3]. By setting $D = \mathcal{O}$ and $C = \emptyset$, Theorem 3.14 captures converse Lyapunov theorems for local asymptotic stability in difference inclusions; see [6, Ch. 7]. In fact, Theorem 3.14 is also applicable to differential (respectively, difference) inclusions or equations even when C (respectively, D) is a strict subset of $\mathcal{O}$; for example, the corollary applies to the systems $\mathcal{H}_c$ and $\mathcal{H}_d$ in Example 3.12, even though the origin is unstable for both $\widetilde{\mathcal{H}}_c = (F, G, \mathbb{R}^2, \emptyset)$ and $\widetilde{\mathcal{H}}_d = (F, G, \emptyset, \mathbb{R}^2)$. It also applies, for example, to differential equations defined on the positive orthant with an asymptotically stable origin, which is on the boundary of the positive orthant. Converse Lyapunov theorems for attractivity, respectively "practical" convergence, without stability can be derived easily from combining Theorem 3.14 with Corollary 3.7, respectively Corollary 3.6, or Proposition 3.9. We do not do this explicitly here due to space limitations.

IV. Applications: Systems With Logic Variables and Input-to-State Stabilization

A. Converse Lyapunov Theorem for Systems with Logic Variables

Let $Q \subset \mathbb{Z}^m$ be a set of logic modes. In this section, we consider systems with logic variables

$$\mathcal{H} := \begin{cases} \dot{\xi} \in F_q(\xi) & \text{for } \xi \in C_q \\ \begin{bmatrix} \xi^+ \\ q^+ \end{bmatrix} \in G_q(\xi) & \text{for } \xi \in D_q \end{cases} \tag{5}$$

under the following assumption, which parallels the hybrid basic conditions.

Assumption 4.1: For each $q \in Q$

1) $\mathcal{O}_q \subset \mathbb{R}^n$ is open, and $C_q \subset \mathcal{O}_q$, $D_q \subset \mathcal{O}_q$ are relatively closed in $\mathcal{O}_q$;
2) $F_q : \mathcal{O}_q \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded, and $F_q(\xi)$ is nonempty and convex for all $\xi \in C_q$; and
3) $G_q : \mathcal{O}_q \rightrightarrows \mathbb{R}^{n+m}$ is outer semicontinuous and locally bounded, and for each $\xi \in D_q, G_q(\xi)$ is a nonempty subset of $\bigcup_{\hat{q} \in Q} \mathcal{O}_{\hat{q}} \times \{\hat{q}\}$.

The rocking block model in [25, Sec. II.E] is an example of a system fitting the form considered in this section. This section also addresses the canonical model of a closed-loop hybrid control system. A common special case is when F_q and G_q are single-valued continuous maps for each q, and the first n components of G_q equal ξ so that only the variable q jumps.

We take the composite state space to be the open set $\mathcal{O} := \bigcup_{q \in Q} \mathcal{O}_q \times \{q + \varepsilon\mathbb{B}\}$ for some $\varepsilon \in (0, 1/2)$, and we define $C := \bigcup_{q \in Q} C_q \times \{q\}$ and $D := \bigcup_{q \in Q} D_q \times \{q\}$.

Theorem 4.2: For (5), let Assumption 4.1 hold and assume that the nonempty and compact set $\mathcal{A} \subset \mathcal{O}$ is preasymptotically stable with basin of preattraction $\mathcal{O}^p_{\mathcal{A}}$. Then:

1) for each $q \in Q$, there exists a compact (possibly empty) set $\mathcal{A}_q$ and a (possibly empty) set $\mathcal{X}_q$ open in $\mathcal{O}_q$ such that $\mathcal{A}_q \subset \mathcal{X}_q$, $\mathcal{A} = \bigcup_{q \in Q} \mathcal{A}_q \times \{q\}$,[3] and $\mathcal{O}^p_{\mathcal{A}} = \bigcup_{q \in Q} \mathcal{X}_q \times \{q + \varepsilon\mathbb{B}\}$ and
2) for each $q \in Q$, let $\omega_q : \mathcal{X}_q \to \mathbb{R}_{\geq 0}$ be a proper indicator for $\mathcal{A}_q$ on $\mathcal{X}_q$. Then, there exist class-$\mathcal{K}_\infty$ functions α_1, α_2, and, for each $q \in Q$, a smooth function $V_q : \mathcal{G}_q \to \mathbb{R}_{\geq 0}$, such that

$$\begin{aligned} \alpha_1(\omega_q(\xi)) \;\leq\; V_q(\xi) \leq \alpha_2(\omega_q(\xi)) \quad & \forall \xi \in \mathcal{X}_q \\ \max_{f \in F_q(\xi)} \langle \nabla V_q(\xi), f \rangle \leq -V_q(\xi) \quad & \forall \xi \in \mathcal{X}_q \cap C_q \\ \max_{\left[\begin{smallmatrix} g_1 \\ g_2 \end{smallmatrix}\right] \in G_q(\xi)} V_{g_2}(g_1) \leq e^{-1} V_q(\xi) \quad & \forall \xi \in \mathcal{X}_q \cap D_q. \end{aligned} \tag{6}$$

[3]The structure of $\mathcal{A}$ is similar to that of the global compact attractor in [5, Th. 2]

Proof: Define the state $x := (\xi, q)$, the set $\mathcal{O}^0 := \bigcup_{q \in Q} \mathcal{O}_q \times \{q\}$, and the set-valued maps $\widetilde{F}, \widetilde{G} : \mathcal{O} \rightrightarrows \mathbb{R}^{n+m}$ as $\widetilde{F}(x) := F_q(\xi) \times \{0\}$ and $\widetilde{G}(x) := G_q(\xi)$ for all $x \in \mathcal{O}^0$, and $\widetilde{F}(x) = \widetilde{G}(x) := \emptyset$ for all $x \in \mathcal{O} \setminus \mathcal{O}^0$. Consider a new hybrid system $\widetilde{\mathcal{H}} := (\widetilde{F}, \widetilde{G}, C, D)$ on $\mathcal{O}$. Due to Assumption 4.1, the data of $\widetilde{\mathcal{H}}$ satisfy the hybrid basic conditions on $\mathcal{O}$; furthermore, the sets of (nontrivial) maximal solutions for (5) on $\mathcal{O}^0$ and for $\widetilde{\mathcal{H}}$ on $\mathcal{O}$ are equal, which implies that, for $\widetilde{\mathcal{H}}$, the set $\mathcal{A}$ is preasymptotically stable with basin of preattraction $\mathcal{O}^p_{\mathcal{A}}$. Thus, the first statement follows from the compactness of $\mathcal{A}$ and the fact that $\mathcal{O}^p_{\mathcal{A}}$ is open for $\mathcal{H} = (F, G, C, D)$ (see Theorem 3.14). The application of Theorem 3.14 also gives the second statement. ∎

B. Logic-Based Continuous Stabilization ⇒ Logic-Based Continuous Input-to-State Stabilization

In this section, we show how one of the main results of [32] that smooth stabilization implies smooth input-to-state stabilization with respect to input additive disturbances generalizes to hybrid control systems. Our result can be combined with the hybrid feedback results of [10], [29] to establish logic-based smooth input-to-state stabilization with respect to input additive disturbances for systems like the nonholonomic integrator or Artstein's circles. This application is related to the results of [26] where input-to-state stabilization was established using discontinuous, continuous-time feedback, and nonsmooth Lyapunov functions.

Consider the hybrid control system

$$\begin{cases} \dot{\xi} = f_q(\xi) + \eta_q(\xi)(\mathbf{u}_q + v_q \mathbf{d}) & \text{for } \xi \in C_q \\ \left[\begin{smallmatrix} \xi^+ \\ q^+ \end{smallmatrix}\right] \in G_q(\xi) & \text{for } \xi \in D_q \end{cases} \tag{7}$$

where f_q, $\eta_q : \mathcal{O}_q \to \mathbb{R}^n$ are continuous functions, $\mathcal{O}_q$, C_q, D_q, and G_q satisfy Assumption 4.1, $\mathbf{u}_q$ is the control, and $\mathbf{d}$ is the disturbance and the constant vectors v_q are uniformly bounded over Q. Suppose $\mathcal{H}$ is stabilizable by logic-based continuous feedback; that is, for the case where $\mathbf{d} = 0$, there exist continuous functions k_q defined on C_q such that, with $\mathbf{u}_q := k_q(\xi)$, the nonempty and compact set $\mathcal{A} = \bigcup_{q \in Q} \mathcal{A}_q \times \{q\}$ is asymptotically stable with basin of attraction $\mathcal{O}^p_{\mathcal{A}} = \bigcup_{q \in Q} \mathcal{X}_q \times \{q + \varepsilon\mathbb{B}\}$, where $\varepsilon \in (0, 1/2)$ comes from Theorem 4.2. According to Theorem 4.2, the sets $\mathcal{X}_q$ are open in $\mathcal{O}_q$. For each $q \in Q$, let $\omega_q : \mathcal{X}_q \to \mathbb{R}_{\geq 0}$ be a proper indicator for $\mathcal{A}_q$ on $\mathcal{X}_q$.

Proposition 4.3: Under the assumptions of the preceding paragraph, there exists a logic-based continuous feedback that renders the system input-to-state stable with respect to $\mathbf{d}$ as measured through the function $(\xi, q) \mapsto \omega_q(\xi)$. More specifically, there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and for each $\varepsilon > 0$ and $q \in Q$, there exists a continuous function $\kappa_{q,\varepsilon}$ defined on C_q such that, with $\mathbf{u}_q = \kappa_{q,\varepsilon}(\xi)$, the following property holds for the system (7): For each initial condition $(\xi(0,0), q(0,0)) \in \mathcal{O}^p_{\mathcal{A}}$, each corresponding solution, and each (t, j)

in its hybrid time domain

$$\omega_{q(t,j)}(\xi(t,j)) \le \max\left\{\alpha_1^{-1}\left(\frac{\sup_{q\in Q}|v_q|^2}{2\varepsilon}||\mathbf{d}||_\infty^{\mathbf{2}}\right)\right. \\ \left.\alpha_1^{-1}\left(2\exp(-t-j)\alpha_2(\omega_{q(0,0)}(\xi(0,0)))\right)\right\}$$

where $||\mathbf{d}||_\infty = \text{ess. sup.}_{(t,j)\in\text{dom}\,\phi}|\mathbf{d}(t,j)|$. In particular, one can take

$$\kappa_{q,\varepsilon}(\xi) = k_q(\xi) - \varepsilon\eta_q^\top(\xi)\nabla V_q(\xi)$$

where the smooth functions V_q come from Theorem 4.2.

Proof: (Sketch) By construction

$$\alpha_1(\omega_q(\xi)) \le V_q(\xi) \le \alpha_2(\omega_q(\xi)) \qquad \forall\xi\in\mathcal{X}_q \tag{8}$$

and

$$\begin{aligned} \langle\nabla V_q(\xi), f_q(\xi)+\eta_q(\xi)k_q(\xi)\rangle &\le -V_q(\xi) && \forall\xi\in\mathcal{X}_q\cap C_q \\ \max_{\left[\begin{smallmatrix} g_1\\ g_2\end{smallmatrix}\right]\in G_q(\xi)} V_{g_2}(g_1) &\le e^{-1}V_q(\xi) && \forall\xi\in\mathcal{X}_q\cap D_q. \end{aligned} \tag{9}$$

It follows from the first inequality in (9) that, for all $\xi\in\mathcal{X}_q\cap C_q$

$$\left\langle\nabla V_q(\xi),\ f_q(\xi)+\eta_q(\xi)\left(k_q(\xi)-\varepsilon\eta_q^\top(\xi)\nabla V_q(\xi)+v_q\mathbf{d}\right)\right\rangle \\ \le -V_q(\xi)+\frac{\sup_{q\in Q}|v_q|^2}{4\varepsilon}|\mathbf{d}|^2.$$

It then follows that, for each solution starting in $\mathcal{O}_{\mathcal{A}}^p$

$$V_{q(t,j)}(\xi(t,j)) \\ \le \exp(-t-j)V_{q(0,0)}(\xi(0,0)) + \frac{\sup_{q\in Q}|v_q|^2}{4\varepsilon}||\mathbf{d}||_\infty^2.$$

The result then follows using (8) and the fact that $\alpha_1^{-1}(a+b)\le\max\left\{\alpha_1^{-1}(2a),\alpha_1^{-1}(2b)\right\}$. ■

V. Applications: (Semiglobal Practical) Robustness of Stability

A. General Observations

In the companion paper [4], the existence of smooth Lyapunov functions was shown to be equivalent to robustness of stability. In this section, we wish to elaborate on how converse Lyapunov theorems for preasymptotic stability of compact sets can be used to understand robustness. Let V be a smooth Lyapunov function for $(\mathcal{O}_1, F, G, C, D, \omega)$, where ω is a proper indicator for a compact set $\mathcal{A}$ on the open set $\mathcal{O}_1$. Since there exist $\alpha_1,\alpha_2\in\mathcal{K}_\infty$, such that

$$\alpha_1(\omega(x)) \le V(x) \le \alpha_2(\omega(x)) \qquad \forall x\in\mathcal{O}_1 \tag{10}$$

it follows that for each pair (ℓ_1,ℓ_2) with $0<\ell_1<\ell_2<\infty$, the set $\{x\in\mathcal{O}_1 : \ell_1\le V(x)\le\ell_2\}$ is a compact subset of $\mathcal{O}_1\backslash\mathcal{A}$. Then, the result of [4, Claim 5.1] can be used to conclude readily that, given $0<\ell_1<\ell_2<\infty$, there exists $\rho>0$, such that

$$\begin{aligned} \max_{f\in F_\rho(x)}\langle\nabla V(x), f\rangle &\le -\frac{1}{2}V(x) \\ &\qquad \forall x\in C_\rho\cap\{x\in\mathcal{O}_1 : \ell_1\le V(x)\le\ell_2\} \\ \max_{g\in G_\rho(x)} V(g) &\le e^{-1/2}V(x) \\ &\qquad \forall x\in D_\rho\cap\{x\in\mathcal{O}_1 : \ell_1\le V(x)\le\ell_2\} \end{aligned} \tag{11}$$

where [cf. (22)–(25)]

$$F_\rho(x) := \overline{\text{co}}F((x+\rho\overline{\mathbb{B}})\cap C)+\rho\overline{\mathbb{B}},\ \forall x\in\mathcal{O} \tag{12}$$

$$G_\rho(x) := \{v\in\mathcal{O} : v\in g+\rho\overline{\mathbb{B}},\ g\in G((x+\rho\overline{\mathbb{B}})\cap D)\} \\ \forall x\in\mathcal{O} \tag{13}$$

$$C_\rho := \{x\in\mathcal{O} : (x+\rho\overline{\mathbb{B}})\cap C\neq\emptyset\} \tag{14}$$

$$D_\rho := \{x\in\mathcal{O} : (x+\rho\overline{\mathbb{B}})\cap D\neq\emptyset\}. \tag{15}$$

Using the same arguments as in the proof of [14, Claim 5.1], we can assume, without loss of generality, that $\rho>0$ also satisfies

$$\max_{g\in G_\rho(x)} V(g)\le\ell_1 \qquad \forall x\in D_\rho\cap\{x\in\mathcal{O}_1 : V(x)\le\ell_1\}\ . \tag{16}$$

It follows from (11) and (16) that, for the system

$$\mathcal{H}_\rho := \begin{cases} \dot{x}\in F_\rho(x) & \text{for } x\in C_\rho \\ x^+\in G_\rho(x) & \text{for } x\in D_\rho \end{cases} \tag{17}$$

the compact set $\{x\in\mathcal{O}_1 : V(x)\le\ell_1\}$ is preasymptotically stable with basin of preattraction containing the forward invariant open set $\{x\in\mathcal{O}_1 : V(x)<\ell_2\}$. It also follows from (10) that the set $\{x\in\mathcal{O}_1 : V(x)\le\ell_1\}$ converges to $\mathcal{A}$ as $\ell_1\to 0$, and the set $\{x\in\mathcal{O}_1 : V(x)<\ell_2\}$ fills out $\mathcal{O}_1$ as $\ell_2\to\infty$. Using terminology from parameterized differential equations, we say that the set $\mathcal{A}$ is *semiglobally* (with respect to $\mathcal{O}_1$) *practically preasymptotically stable in the parameter* ρ. Now, we apply these ideas to robustness with respect to slowly varying, weakly jumping parameters, to temporal regularization and to "average dwell-time" perturbations.

B. Slowly Varying, Weakly Jumping Parameters

Consider a parameterized hybrid system

$$\mathcal{H} := \begin{cases} \begin{bmatrix}\dot{\xi}\\ \dot{p}\end{bmatrix} \in \begin{bmatrix}F(\xi,p)\\ 0\end{bmatrix} & \text{for } \begin{bmatrix}\xi\\ p\end{bmatrix}\in C \\ \begin{bmatrix}\xi^+\\ p^+\end{bmatrix} \in \begin{bmatrix}G(\xi,p)\\ p\end{bmatrix} & \text{for } \begin{bmatrix}\xi\\ p\end{bmatrix}\in D \end{cases} \tag{18}$$

where the state is taken to be (ξ,p) belonging to the open state space $\mathcal{O}$ under the hybrid basic conditions. Suppose that this system has the compact set $\mathcal{A}$ preasymptotically stable with basin of preattraction $\mathcal{O}_{\mathcal{A}}^p$. Also, assume that for each $(\xi,p)\in D$ and each $\eta\in G(\xi,p)$, we have $(\eta,p)\in C\cup D$. Since $\mathcal{A}$ is asymptotically stable and p does not change along solutions, the parameter vector p is restricted to a compact set. One could easily write down a converse theorem for preasymptotic stability for the system (18) and use the resulting Lyapunov function to establish robustness to slow variations in the parameter during flows and small jumps in the parameter during jumps of the hybrid system. In fact, robustness can also be established for small jumps in the parameter that are not synchronized with jumps of the hybrid system. This observation is relevant, for example, for parameter jumps in differential equations. We establish this extra robustness as follows.

Let $\delta \geq 0$ and $\tau^* > 0$ and consider the related system $\mathcal{H}_{\delta,\tau^*} := (\widetilde{F}, \widetilde{G}, \widetilde{C}, \widetilde{D})$ with the state $x = (\xi, p, \tau)$ and the state space $\mathcal{O} \times \mathbb{R}$, where $\widetilde{C} := C \times \mathbb{R}_{\geq 0}$, $\widetilde{D} := (D \times \mathbb{R}_{\geq 0}) \cup ((C \cup D) \times \mathbb{R}_{\geq \tau^*})$

$$\widetilde{F}(x) := \begin{bmatrix} cF(\xi,p) \\ \delta\mathbb{B} \\ 1-\tau+\tau^* \end{bmatrix}$$

$$\widetilde{G}(x) := \begin{cases} \widetilde{G}_1(x) & \text{for} \quad x \in D \times [0,\tau^*) \\ \widetilde{G}_1(x) \bigcup \widetilde{G}_2(x) & \text{for} \quad x \in D \times \mathbb{R}_{\geq \tau^*} \\ \widetilde{G}_2(x) & \text{for} \quad x \in (C \backslash D) \times \mathbb{R}_{\geq \tau^*} \end{cases}$$

where

$$\widetilde{G}_1(x) = \begin{bmatrix} G(\xi,p) \\ p+\delta\overline{\mathbb{B}} \\ \tau \end{bmatrix}, \qquad \widetilde{G}_2(x) = \begin{bmatrix} \xi \\ p+\delta\overline{\mathbb{B}} \\ 0 \end{bmatrix}.$$

When $\delta = 0$, the parameter p is constant along solutions, and all of the solutions of (18) are enabled as the (ξ, p) component of the solution. The new enabled solutions are those containing "jumps" via $\widetilde{G}_2(\cdot)$, but these jumps are separated by a flow with at least $\ln((1+\tau^*)/\tau^*)$ seconds, since that is the amount of time required for $\dot{\tau} = 1-\tau+\tau^*$ to increase from 0 to τ^*. So, $\mathcal{H}_{0,\tau^*}$ has the set $\widetilde{\mathcal{A}} := \mathcal{A} \times [0, 1+\tau^*]$ preasymptotically stable with basin of preattraction $\widetilde{\mathcal{O}}^p_{\mathcal{A}} := \mathcal{O}^p_{\mathcal{A}} \times \mathbb{R}$.

When $\delta > 0$, the parameter p is allowed to change slowly during flows, it is allowed to make small jumps when the hybrid system would be jumping anyway, and it is also allowed to make additional jumps when the timer τ reaches or exceeds the value τ^*. Nevertheless, the set $\widetilde{\mathcal{A}}$ is semiglobally (with respect to $\widetilde{\mathcal{O}}^p_{\mathcal{A}}$) practically preasymptotically stable with respect to δ.

C. *Temporal Regularization*

Zeno behavior is a frequently encountered phenomenon in hybrid or switched control systems (for example, see a simplified version of the hybrid controller for nonholonomic integrator in [9, Sec. 2]). To eliminate Zeno behavior in applications, temporal regularization (i.e., to force the interval between jumps to be at least some amount of time) is an effective recipe. In this section, we show how to recover the result on semiglobal practical robustness under temporal regularization, reported in [11, Example 6.8], via converse Lyapunov theorems.

Suppose one is given a hybrid system $\mathcal{H} := (F, G, C, D)$ on the open state space $\mathcal{O}$, where $F(\xi)$ is nonempty and convex for all $\xi \in \mathcal{O}$, and suppose that the compact set $\mathcal{A}$ is preasymptotically stable with basin of preattraction $\mathcal{O}^p_{\mathcal{A}}$. Now let $\delta \geq 0$ and consider a related system $\mathcal{H}_\delta := (\widetilde{F}, \widetilde{G}, C_\delta, D_\delta)$ with the state $x := (\xi, \tau)$ and the state space $\widetilde{\mathcal{O}} := \mathcal{O} \times \mathbb{R}$, where

$$\begin{aligned} \widetilde{F}(x) &:= F(\xi) \times \{1-\tau\} \\ \widetilde{G}(x) &:= G(\xi) \times \{0\} \\ C_\delta &:= (C \times \mathbb{R}_{\geq 0}) \cup (\mathcal{O} \times [0,\delta]) \\ D_\delta &:= D \times [\delta, \infty). \end{aligned}$$

When $\delta = 0$, flowing is possible only if $\xi \in C$, since $\dot{\tau} = 1 - \tau$ and the flow set for τ when $\xi \notin C$ is the point $\tau = 0$. Thus, the ξ component of the solution with $\delta = 0$ matches the solution of $\mathcal{H}$, and the τ component converges to the interval $[0, 1]$. So, the system $\mathcal{H}_0$ has the compact set $\widetilde{\mathcal{A}} := \mathcal{A} \times [0, 1]$ preasymptotically stable with basin of preattraction $\widetilde{\mathcal{O}}^p_{\mathcal{A}} := \mathcal{O}^p_{\mathcal{A}} \times \mathbb{R}$.

When $\delta > 0$, in each hybrid time domain of each solution, each time interval is at least δ seconds long, since $\dot{\tau} \leq 1$ for all $\tau \in [0, \delta]$. In particular, Zeno solutions, if there were some, have been eliminated. Nevertheless, the set $\widetilde{\mathcal{A}}$ is semiglobally (with respect to $\widetilde{\mathcal{O}}^p_{\mathcal{A}}$) practically preasymptotically stable with respect to δ. This fact follows from the discussion of the previous section and the fact that the sets C_δ and D_δ are contained in the sets C_ρ and D_ρ, respectively, that were defined in (14) and (15), respectively, when we set $\rho = \delta$. It also uses the fact that $\widetilde{G}(D_0) \subset \mathcal{O} \times \{0\} \subset C_0$.

D. *Average Dwell Time*

Consider the differential inclusion $\dot{\xi} \in F(\xi)$ for $\xi \in C$, with open state space $\mathcal{O}$ where the state ξ may contain logical modes that remain constant. Suppose that the compact set $\mathcal{A}$ is preasymptotically stable with basin of preattraction $\mathcal{O}^p_{\mathcal{A}}$ and that we are interested in injecting jumps, on occasion, through a jump inclusion $\xi^+ \in G(\xi)$ for $\xi \in D$ while maintaining (semiglobal practical) preasymptotic stability. In order to achieve this goal, we suppose

$$G(\mathcal{A} \cap D) \subset \mathcal{A}, \qquad G(\mathcal{O}^p_{\mathcal{A}} \cap D) \subset \mathcal{O}^p_{\mathcal{A}}. \tag{19}$$

Let $\delta \geq 0$ and let N be a positive integer and consider a related system $\mathcal{H}_{\delta,N} := (\widetilde{F}, \widetilde{G}, \widetilde{C}, \widetilde{D})$ with the state $x = (\xi, \tau)$ and the state space $\widetilde{\mathcal{O}} := \widetilde{O} \times \mathbb{R}$, where

$$\widetilde{F}(x) := F(\xi) \times \eta_\delta(\tau), \qquad \widetilde{C} := C \times [0, N]$$
$$\widetilde{G}(x) := G(\xi) \times \{\tau - 1\}, \qquad \widetilde{D} := D \times [1, N]$$

where

$$\eta_\delta(\tau) := \begin{cases} \delta & \text{for} \quad \tau \in [0, N) \\ [0,\delta] & \text{for} \quad \tau = N. \end{cases}$$

In the special case where $\delta = 0$, at most N jumps are allowed in the time domain of a solution. Using (19), these jumps do not destroy preasymptotic stability of $\mathcal{A}$ or that its basin of preattraction is $\mathcal{O}^p_{\mathcal{A}}$ for the ξ component of the solution. For the composite system, we have that the set $\widetilde{\mathcal{A}} := \mathcal{A} \times [0, N]$ is preasymptotically stable with basin of preattraction $\widetilde{\mathcal{O}}^p_{\mathcal{A}} := \mathcal{O}^p_{\mathcal{A}} \times \mathbb{R}$.

When $\delta > 0$, the number of jumps may be infinite, but each solution's time domain must satisfy

$$j - i \leq \delta(t-s) + N \quad \forall (t,j),(s,i) \in \operatorname{dom}\phi \quad \text{with} \quad t+j > s+i\,. \tag{20}$$

(In fact, it can be shown that every hybrid time domain that satisfies this constraint can be generated with the system mentioned earlier using the appropriate initial condition for τ. See Proposition 1.1 in Appendix.) The condition (20) corresponds to the notion of "average dwell time," as introduced in [15]. We conclude that the system $\mathcal{H}_{\delta,N}$ has the set $\widetilde{\mathcal{A}}$ semiglobally (with respect to $\widetilde{\mathcal{O}}^p_{\mathcal{A}}$) practically preasymptotically stable with respect to δ. In other words, preasymptotic stability in differential inclusions is robust, in a semiglobal practical sense, to "average dwell-time" jump perturbations.

E. Reverse Average Dwell Time

Consider a jump inclusion $\xi^+ \in G(\xi)$ for $\xi \in D$ with open state space $\mathcal{O}$. Suppose that the compact set $\mathcal{A}$ is preasymptotically stable with basin of preattraction $\mathcal{O}^p_{\mathcal{A}}$, and that we are interested in injecting flows, on occasion, through a flow inclusion $\dot{\xi} \in F(\xi)$ for $\xi \in C$ while maintaining (semiglobal practical) preasymptotic stability. In order to achieve this goal, we suppose, for the flow inclusion, that the sets $\mathcal{A} \cap C$ and $\mathcal{O}^p_{\mathcal{A}} \cap C$ are forward invariant and that trajectories are bounded with respect to $\mathcal{O}^p_{\mathcal{A}}$ on compact time intervals. Let $\delta \geq 0$, $\lambda > 0$, and consider a related system $\mathcal{H}_{\delta,\lambda} := (\widetilde{F}, \widetilde{G}, \widetilde{C}, \widetilde{D})$ with the state $x = (\xi, \tau)$ and the state space $\widetilde{\mathcal{O}} := \widetilde{O} \times \mathbb{R}$, where

$$\widetilde{F}(x) := F(\xi) \times \{1\}\,, \qquad \widetilde{C} := C \times [0,\lambda]$$
$$\widetilde{G}(x) := G(\xi) \times \{\max\{0, \tau - \delta\}\}, \qquad \widetilde{D} := D \times [0,\lambda]\,.$$

In the special case where $\delta = 0$, at most, λ seconds of flow are allowed in the time domain of a solution. Due to the assumptions on the flow, this does not destroy preasymptotic stability of $\mathcal{A}$ or that its basin of preattraction is $\mathcal{O}^p_{\mathcal{A}}$ for the ξ component of the solution. For the composite system, we have that the set $\widetilde{\mathcal{A}} := \mathcal{A} \times [0,\lambda]$ is preasymptotically stable with basin of preattraction given by $\widetilde{\mathcal{O}}^p_{\mathcal{A}} := \mathcal{O}^p_{\mathcal{A}} \times \mathbb{R}$.

When $\delta > 0$, the flow time may be infinite, but each solution's time domain must satisfy

$$t - s \leq \delta(j-i) + \lambda \quad \forall (t,j),(s,i) \in \operatorname{dom}\phi \quad \text{with} \quad t+j > s+i. \tag{21}$$

(In fact, it can be shown that every hybrid time domain that satisfies this constraint can be generated with the system mentioned before using the appropriate initial condition for τ. See Proposition 1.2 in Appendix.) The condition (21) corresponds to the notion of "reverse average dwell time," introduced in [14]. We conclude that the system $\mathcal{H}_{\delta,\lambda}$ has the set $\widetilde{\mathcal{A}}$ semiglobally (with respect to $\widetilde{\mathcal{O}}^p_{\mathcal{A}}$) practically preasymptotically stable with respect to δ. In other words, preasymptotic stability in jump inclusions is robust, in a semiglobal practical sense, to "reverse average dwell-time" flow perturbations.

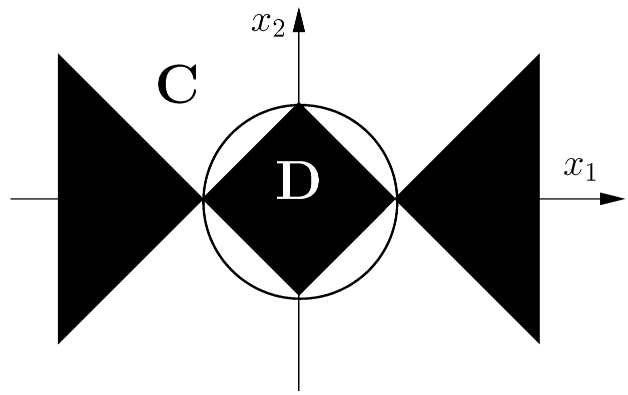

Fig. 2. Flow and jump sets for Example 6.1.

VI. Can the Hybrid Basic Conditions Be Relaxed?

Results for purely continuous-time systems (corresponding to $C = \mathcal{O}$ and $D = \emptyset$) and purely discrete-time systems ($C = \emptyset$ and $D = \mathcal{O}$) show that existence of smooth Lyapunov functions requires F and G being outer semicontinuous; see, for example, [21] for continuous-time systems, and [20] and also [35] for discrete-time systems. Here, in Example 6.1, respectively, 6.2, we illustrate that, in general, (relative) closedness of C, respectively, D, cannot be omitted. For continuous-time systems, under a stronger continuity assumption—that F is a locally Lipschitz set-valued map—convexity of the values of F need not be present in order to guarantee the existence of smooth Lyapunov functions (see [33, Th. 2]). The key behind this is a classical relaxation theorem for differential inclusions. For hybrid systems, this is no longer the case—Example 6.3 shows that F needs to have convex values, unless other quite strong structural assumptions on the data are placed to enable relaxation theorems for hybrid systems (see [3]).

Example 6.1: (Flow set not closed) Define $D_1 := \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$ and the jump set

$$D := D_1 \cup \{x \in \mathbb{R}^2 : x_1 \geq 0,\ |x_1 - 1| \geq |x_2|\} \cup \{x \in \mathbb{R}^2 : x_1 \leq 0,\ |x_1 + 1| \geq |x_2|\}.$$

Consider the following hybrid system with the state $x = (x_1, x_2)^\top \in \mathbb{R}^2$

$$\begin{cases} \dot{x} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} =: F(x) & \text{for} \quad x \in \left(\overline{\mathbb{R}^2 \backslash D}\right) \backslash D_1 =: C \\ x^+ = 0 =: G(x) & \text{for} \quad x \in D \end{cases}.$$

The flow set and jump set are depicted in Fig. 2. One can verify that, for each initial condition, each solution is complete and satisfies[4] the bound $|x(t,j)| \leq e^{2\pi}|x(0,0)|e^{-t-j}$ for all $(t,j) \in \operatorname{dom} x$. This is because flowing in D_1 is not possible, while flowing outside of D_1 is possible until hitting the set D,

[4] We have chosen an example that does not admit a smooth Lyapunov function, and yet, is asymptotically stable even when using the phrase "almost all t" in the condition (S1) of a solution to a hybrid system. It would be simpler to construct an example if "almost all t" were replaced by "all t." Indeed, the distinction is significant when the flow set is not closed, as in this example, whereas the distinction does not affect solutions when the flow set is closed.

which happens in no more than 2π seconds. Trajectories must jump to the origin when hitting D since it is not possible to flow through D. On the other hand, the hybrid system does not admit a smooth Lyapunov function. Indeed, suppose there is one for $(\mathbb{R}^2, F, G, C, D, |\cdot|)$. Then, using (3) and continuity of ∇V and F, we infer that $\langle \nabla V(x), F(x)\rangle \le V(x)$ for all $x \in \overline{C}$. Thus, V is a smooth Lyapunov function for $(\mathbb{R}^2, F, G, \overline{C}, D, |\cdot|)$, which, according to the implication 1⇒2 in Theorem 3.13, implies that the origin of the hybrid system $\widetilde{\mathcal{H}} := (F, G, \overline{C}, D)$ is (pre)asymptotically stable. However, since $\overline{C}$ contains D_1, there exists a complete solution that starts and remains in D_1 on its hybrid time domain $\mathbb{R}_{\ge 0} \times \{0\}$, which gives a contradiction.

Example 6.2: (Jump set not closed) Define $C := (-\infty, 1]$ and consider the hybrid system

$$\begin{cases} \dot{x} = -x \;=:\; F(x) & \text{for } x \in C \\ x^+ = \min\{|x| 1\} \;=:\; G(x) & \text{for } x \in \mathbb{R}\backslash C \;=:\; D\,. \end{cases}$$

One can verify that, for each initial condition, each solution is complete and satisfies $|x(t,j)| \le e|x(0,0)|e^{-t-j}$ for all $(t,j) \in \operatorname{dom} x$. On the other hand, the hybrid system does not admit a smooth Lyapunov function. To see this, suppose there is one for $(\mathbb{R}, F, G, C, D, |\cdot|)$. Then, using (4) and the continuity of V and G, we infer that $V(G(x)) \le e^{-1}V(x)$ for all $x \in \overline{D}$. Thus, V is also a smooth Lyapunov function for $(\mathbb{R}, F, G, C, \overline{D}, |\cdot|)$, which implies that the origin is asymptotically stable for $\widetilde{\mathcal{H}} := (F, G, C, \overline{D})$ according to the implication 1⇒2 in Theorem 3.13. However, since $x^+ = \min\{|x|, 1\}$, there exists a complete solution that starts and remains at the value one on its hybrid time domain $\{0\} \times \mathbb{Z}_{\ge 0}$, which gives a contradiction.

Example 6.3: (Nonconvex set-valued flow map) Let $C := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2\}$ and consider the hybrid system

$$\begin{cases} \dot{x} \in \left\{ \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] \right\} \;=:\; F(x) & \text{for} \quad x \in C \\ x^+ = 0 \;=:\; G(x) & \text{for} \quad x \in \mathbb{R}^2 \;=:\; D\,. \end{cases}$$

Solutions can only jump, and so, satisfy $|x(t,j)| \le |x(0,0)| \cdot \max\{0, 1-j\}$ for all $(t,j) \in \operatorname{dom} x$. On the other hand, the hybrid system does not admit a smooth Lyapunov function. To see this, suppose there is one for $(\mathbb{R}^2, F, G, C, D, |\cdot|)$. Then, using Carathéodory's Theorem [30, Th. 2.29] and (3), we infer that $\max_{f \in \overline{\mathrm{co}} F(x)} \langle \nabla V(x), f\rangle \le -V(x)$ for all $x \in C$. Therefore, V is also a smooth Lyapunov function for $(\mathbb{R}^2, \overline{\mathrm{co}} F, G, C, D, |\cdot|)$, which implies that the origin is asymptotically stable for the new hybrid system $\widetilde{\mathcal{H}} := (\overline{\mathrm{co}} F, G, C, D)$ according to the implication 1⇒2 in Theorem 3.13. However, we have $\left[\begin{smallmatrix}0.5\\0.5\end{smallmatrix}\right] \in \overline{\mathrm{co}} F(x)$ for all $x \in C$, which implies the existence of a complete solution that starts and remains in C on its hybrid time domain $[0, \infty) \times \{0\}$, but blows up, and hence, gives a contradiction.

We note that the asymptotic stability in the previous three counterexamples is not robust (see the definition of robust stability in Section VII; also see [2] for detailed explanations).

VII. Proof of 2⇒1 in Theorem 3.13

We now prove the implication 2⇒1 in Theorem 3.13. First, using generic robustness properties of global asymptotic stability of compact sets and the results of [4], we establish a converse Lyapunov theorem for globally asymptotically stable sets. Second, we give some useful results on truncating hybrid systems. Third, we show how by augmenting a hybrid system, one can pass from preasymptotic to asymptotic stability. Fourth, we put all this together to complete the proof of Theorem 3.13.

A. Converse Lyapunov Result for Global Asymptotic Stability

Using some results of [11] and [4], we will now relate global asymptotic stability with an open basin of attraction to the existence of smooth Lyapunov functions. Statements 1′) and 2′) given after are counterparts of statements 1) and 2) of our main result, Theorem 3.13.

Theorem 7.1: Suppose $C \cup D = \mathcal{O}$ and $\mathcal{A} \subset \mathcal{O}$ is compact. Let ω be any proper indicator of $\mathcal{A}$ with respect to $\mathcal{O}$. Then, the following statements are equivalent:

1′) there exists a smooth Lyapunov function for $(\mathcal{O}, F, G, C, D, \omega)$ and
2′) the set $\mathcal{A}$ is globally asymptotically stable for $\mathcal{H}$.

In particular, if $C \cup D = \mathcal{O} = \mathbb{R}^n$, then $\mathcal{A}$ is globally asymptotically stable for $\mathcal{H}$ if and only if there exists a smooth Lyapunov function for $(\mathbb{R}^n, F, G, C, D, |\cdot|_{\mathcal{A}})$.

The goal of this section is to prove that statement 2′) implies 1′) in the previous result.

Definition 7.2: Let $\omega : \mathcal{O} \to \mathbb{R}_{\ge 0}$ be continuous. The hybrid system $\mathcal{H}$ is said to be $\mathcal{KLL}$*-stable* with respect to ω on $\mathcal{O}$ if $\mathcal{H}$ is forward complete on $\mathcal{O}$, and there exists $\gamma \in \mathcal{KLL}$ such that, for each $x \in \mathcal{O}$, all solutions $\phi \in \mathcal{S}(x)$ satisfy

$$\omega(\phi(t,j)) \le \gamma(\omega(x), t, j) \qquad \forall (t,j) \in \operatorname{dom} \phi.$$

The following result is a consequence of [11, Th. 6.5]. It parallels with known results for differential and difference inclusions that satisfy the analogous basic conditions (for example, see [33, Proposition 3] and [19, Proposition 7.2]), where basins of attraction are automatically open.

Proposition 7.3: Let the compact set $\mathcal{A} \subset \mathcal{O}$ be asymptotically stable for the system $\mathcal{H}$ with the basin of attraction equal $\mathcal{O}$. Then, for each proper indicator ω of $\mathcal{A}$ with respect to $\mathcal{O}$, the system $\mathcal{H}$ is $\mathcal{KLL}$-stable with respect to ω on $\mathcal{O}$.

We now describe different robustness properties of $\mathcal{KLL}$-stability. In what follows, by an *admissible perturbation radius on* $\mathcal{O}\}$, we mean any continuous $\sigma : \mathcal{O} \to \mathbb{R}_{\ge 0}$ such that $x + \sigma(x)\overline{\mathbb{B}} \subset \mathcal{O}$ for all $x \in \mathcal{O}$. For each such function σ, we define the σ-perturbation of $\mathcal{H}$, denoted by $\mathcal{H}_\sigma$, as the hybrid system on $\mathcal{O}$ with the data

$$F_\sigma(x) := \overline{\mathrm{co}} F((x + \sigma(x)\overline{\mathbb{B}}) \cap C) + \sigma(x)\overline{\mathbb{B}} \quad \forall x \in \mathcal{O} \tag{22}$$

$$\begin{aligned} G_\sigma(x) := \{ &v \in \mathcal{O} : v \in g + \sigma(g)\overline{\mathbb{B}} \\ &g \in G((x + \sigma(x)\overline{\mathbb{B}}) \cap D)\} \qquad \forall x \in \mathcal{O} \end{aligned} \tag{23}$$

$$C_\sigma := \{x \in \mathcal{O} : (x + \sigma(x)\overline{\mathbb{B}}) \cap C \neq \emptyset\} \tag{24}$$

$$D_\sigma := \{x \in \mathcal{O} : (x + \sigma(x)\overline{\mathbb{B}}) \cap D \neq \emptyset\} \tag{25}$$

$$\mathcal{H}_\sigma := \begin{cases} \dot{x} \in F_\sigma(x) & \text{for } x \in C_\sigma \\ x^+ \in G_\sigma(x) & \text{for } x \in D_\sigma\,. \end{cases} \tag{26}$$

We write $\mathcal{S}_\sigma(x)$ for the set of maximal solutions to $\mathcal{H}_\sigma$ starting at x.

Definition 7.4: Let $\mathcal{A} \subset \mathcal{O}$ be compact and ω be a proper indicator for $\mathcal{A}$ on $\mathcal{O}$. The system $\mathcal{H}$ is said to be *robustly $\mathcal{KLL}$-stable* with respect to ω on $\mathcal{O}$ if there exists an admissible perturbation radius $\sigma : \mathcal{O} \to \mathbb{R}_{\geq 0}$ that is positive on $\mathcal{O} \setminus \mathcal{A}$ and such that $\mathcal{H}_\sigma$ is $\mathcal{KLL}$-stable with respect to ω on $\mathcal{O}$.

Such robustness was shown to be equivalent to the existence of a desired smooth Lyapunov function in [4]. More precisely, the main result of [4], specialized to the case of a single measure which is a proper indicator, says the following.

Theorem 7.5: Assume $C \cup D = \mathcal{O}$. Let $\mathcal{A} \subset \mathcal{O}$ be compact and let ω be a proper indicator of $\mathcal{A}$ with respect to $\mathcal{O}$. Then, the following are equivalent:

1) there exists a smooth Lyapunov function for $(\mathcal{O}, F, G, C, D, \omega)$ and
2) $\mathcal{H}$ is robustly $\mathcal{KLL}$-stable with respect to ω on $\mathcal{O}$.

Hence, to prove that there exists the desired smooth Lyapunov function, we are going to show that the $\mathcal{KLL}$-stability is automatically robust for the hybrid systems under consideration. A partial result in this direction was given in [11, Th. 6.6], we state it here as Proposition 7.7. First, we need another definition.

Definition 7.6: Let $\mathcal{A} \subset \mathcal{O}$ be compact and ω be a proper indicator for $\mathcal{A}$ on $\mathcal{O}$. The system $\mathcal{H}$ is said to be *semiglobally practically robustly $\mathcal{KLL}$- stable* with respect to ω on $\mathcal{O}$, if for any admissible perturbation radius σ, there exists $\gamma \in \mathcal{KLL}$ such that the following holds: for each $\varepsilon > 0$ and each compact set $K \subset \mathcal{O}$, there exists $\delta \in (0,1)$ such that, for each $x \in K$, each $\phi \in \mathcal{S}_{\delta\sigma}(x)$ [we write $\mathcal{S}_{\delta\sigma}(x)$ for the set of maximal solutions to the $\delta\sigma$-perturbed hybrid system $\mathcal{H}_{\delta\sigma}$ starting at x] is complete and satisfies

$$\omega(\phi(t,j)) \leq \gamma(\omega(x), t, j) + \varepsilon \qquad \forall (t,j) \in \operatorname{dom}\phi\,. \quad (27)$$

Proposition 7.7: Suppose the compact set $\mathcal{A} \subset \mathcal{O}$ is asymptotically stable with the basin of attraction equal to $\mathcal{O}$ for $\mathcal{H}$, and the measure ω is a proper indicator for $\mathcal{A}$ on $\mathcal{O}$. Then, $\mathcal{H}$ is semiglobally practically robustly $\mathcal{KLL}$-stable with respect to ω on $\mathcal{O}$.

A key step to establishing the main result of this paper is the following lemma, which shows that semiglobal practical robust $\mathcal{KLL}$-stability implies robust $\mathcal{KLL}$-stability.

Lemma 7.8: Let $\mathcal{A} \subset \mathcal{O}$ be compact and ω be a proper indicator for $\mathcal{A}$ on $\mathcal{O}$. If $\mathcal{H}$ is semiglobally practically robustly $\mathcal{KLL}$-stable with respect to ω on $\mathcal{O}$, then $\mathcal{H}$ is robustly $\mathcal{KLL}$-stable with respect to ω on $\mathcal{O}$.

Proof: Let σ be any admissible perturbation radius and γ be the $\mathcal{KLL}$ function coming from Definition 7.6. Pick any family of positive numbers $\{r_n\}_{n\in\mathbb{Z}}$ such that $r_{n+1} \geq 4\gamma(r_n, 0, 0)$, so, in particular, $r_{n+1} \geq 4r_n$, $\lim_{n\to-\infty} r_n = 0$, $\lim_{n\to\infty} r_n = \infty$. For each $n \in \mathbb{Z}$, semiglobal practical robustness of $\mathcal{KLL}$ stability implies that there exists $\delta_n \in (0,1)$ such that each $\phi \in \mathcal{S}_{\delta_n\sigma}$ with $\omega(\phi(0,0)) \leq r_n$ satisfies

$$\omega(\phi(t,j)) \leq \gamma(\omega(\phi(0,0)), t, j) + \frac{r_{n-1}}{2} \qquad \forall (t,j) \in \operatorname{dom}\phi. \quad (28)$$

From (28), we infer that, for each $n \in \mathbb{Z}$, each $\phi \in \mathcal{S}_{\delta_n\sigma}$ with $\omega(\phi(0,0)) \leq r_n$ satisfies

$$\omega(\phi(t,j)) \leq \frac{r_{n+1}}{2} \qquad \forall (t,j) \in \operatorname{dom}\phi \quad (29)$$

and that there exists $\tau_n > 0$ such that, each $\phi \in \mathcal{S}_{\delta_n\sigma}$ with $\omega(\phi(0,0)) \leq r_n$ satisfies

$$\omega(\phi(t,j)) \leq r_{n-1} \qquad \forall (t,j) \in \operatorname{dom}\phi \quad \text{with} \quad t+j \geq \tau_n\,. \quad (30)$$

Now, find any continuous function $\delta : \mathcal{O} \to [0,\infty)$ that is positive on $\mathcal{O} \setminus \mathcal{A}$ and such that

$$\delta(x) \leq \min\{\delta_{n-1}, \delta_n\}\sigma(x) \text{ when } r_{n-1} \leq \omega(x) \leq r_n. \quad (31)$$

Note that δ has to be an admissible perturbation radius. Therefore, for each $n \in \mathbb{Z}$, each $\phi \in \mathcal{S}_\delta$ with $\omega(\phi(0,0)) \leq r_n$ satisfies

$$\omega(\phi(t,j)) \leq \frac{r_{n+1}}{2} \qquad \forall (t,j) \in \operatorname{dom}\phi \quad (32)$$

which implies that $\mathcal{A}$ is stable for $\mathcal{H}_\delta$. To see (32), let ϕ be a solution to $\mathcal{H}_\delta$ with $\omega(\phi(0,0)) \leq r_n$. If $r_{n-1} \leq \omega(\phi(t,j))$ for all $(t,j) \in \operatorname{dom}\phi \cap [0,T] \times \{0,1,\ldots,J\}$, then by the choice of δ, ϕ is a solution to $\mathcal{H}_{\delta_n\sigma}$ and $\omega(\phi(T,J)) \leq r_{n+1}/2$ by (29). If $\omega(\phi(T,J)) < r_{n-1}$ for some $(T,J) \in \operatorname{dom}\phi$, then one can consider $\phi'(t,j) := \phi(T+t, J+j)$, which is also a solution to $\mathcal{H}_\delta$, and for which $\omega(\phi'(0,0)) \leq r_{n-1}$.

Also then, for each $n \in \mathbb{Z}$, for any $\phi \in \mathcal{S}_{\delta\sigma}$ with $\omega(\phi(0,0)) \leq r_n$, there exists $(t,j) \in \operatorname{dom}\phi$, $t+j \leq \tau_n$ so that $\omega(\phi(t,j)) \leq r_{n-1}$. Indeed, either $r_{n-1} \leq \omega(\phi(t,j))$ for all $(t,j) \in \operatorname{dom}\phi$, $t+j \leq \tau_n$, in which case, by (30), $\omega(\phi(T,J)) = r_{n-1}$ for $(T,J) \in \operatorname{dom}\phi$ with $T+J = \tau_n$, or $\omega(\phi(t,j)) < r_{n-1}$ for some $(t,j) \in \operatorname{dom}\phi$ with $t+j < \tau_n$. The convergence property just shown, in light of stability, implies that each solution to $\mathcal{H}_\delta$ converges to $\mathcal{A}$ (in fact uniformly).

Since $C \cup D = \mathcal{O}$, we also have $C_\delta \cup D_\delta = \mathcal{O}$, and hence, each maximal solution to $\mathcal{H}_\delta$ is either complete or eventually leaves any compact subset of $\mathcal{O}$; see [11, Proposition 2.4]. The latter option is impossible in light of (32), and thus, $\mathcal{H}_\delta$ is forward complete. We have shown earlier that $\mathcal{A}$ is asymptotically stable for $\mathcal{H}_\delta$, with the basin of attraction equal to $\mathcal{O}$. Now, Proposition 7.3 implies that $\mathcal{H}_\delta$ is $\mathcal{KLL}$-stable with respect to ω. ■

An immediate consequence of the result just proved is that stability of compact attractors with open basins of attraction is automatically robust.

Theorem 7.9: Suppose that the compact set $\mathcal{A} \subset \mathcal{O}$ is asymptotically stable with the basin of attraction $\mathcal{O}_\mathcal{A} = \mathcal{O}$ for $\mathcal{H}$, and the measure ω is a proper indicator for $\mathcal{A}$ on $\mathcal{O}$. Then, $\mathcal{H}$ is robustly $\mathcal{KLL}$-stable with respect to ω on $\mathcal{O}$.

Now, combining Theorems 7.5 and 7.9 gives the implication $(2')\Rightarrow(1')$ in Theorem 7.1.

B. Truncations of Hybrid Systems

To pass from global results and Lyapunov functions on the open $\mathcal{O}_\mathcal{A}$ to local results on invariant subsets of $\mathcal{O}_\mathcal{A}$, we will need to consider truncations of hybrid systems. Suppose that $\mathcal{O}_1 \subset C \cup D$ is open and $\mathcal{H}$ is forward invariant on $\mathcal{O}_1$. Consider the system $\mathcal{H}|_{\mathcal{O}_1}$, which is a "truncation" of $\mathcal{H}$ to $\mathcal{O}_1$, with the data (F_1, G_1, C_1, D_1) defined as

$$F_1 := F|_{\mathcal{O}_1}, \qquad C_1 := C \cap \mathcal{O}_1,$$
$$G_1 := G|_{\mathcal{O}_1}, \qquad D_1 := D \cap \mathcal{O}_1. \quad (33)$$

Since $O_1 \subset C \cup D$, we have $(C \cap \mathcal{O}_1) \cup (D \cap \mathcal{O}_1) = \mathcal{O}_1$. Forward invariance implies that $G(x) \subset \mathcal{O}_1$ for all $x \in D \cap \mathcal{O}_1$. This guarantees that the basic hybrid conditions hold for $\mathcal{H}|_{\mathcal{O}_1}$.

Proposition 7.10: Let $\mathcal{O}_1 \subset \mathcal{O}$ be open and assume that $G(x) \subset \mathcal{O}_1$ for all $x \in D \cap \mathcal{O}_1$. Then, the system $\mathcal{H}|_{\mathcal{O}_1}$ with state space $\mathcal{O}_1$ and data in (33) satisfies the hybrid basic conditions.

Proof: For (SA1), note that $C = \overline{C} \cap \mathcal{O}$, and hence, $C_1 = \overline{C} \cap \mathcal{O}_1$, which shows that C_1 is relatively closed in $\mathcal{O}_1$. Similarly, for D_1, (SA2) is obvious, and so is (SA3) in light of the assumption that $G(x) \subset \mathcal{O}_1$ for $x \in \mathcal{O}_1$. ■

We add that if $\mathcal{A}$ is locally asymptotically stable for the system $\mathcal{H}$, then it is locally asymptotically stable for $\mathcal{H}|_{\mathcal{O}_1}$, and forward invariance of $\mathcal{O}_1$ for $\mathcal{H}$ implies that the basin of attraction of $\mathcal{A}$ for $\mathcal{H}|_{\mathcal{O}_1}$ equals $\mathcal{O}_1$. In other words, $\mathcal{A}$ is globally asymptotically stable for $\mathcal{H}|_{\mathcal{O}_1}$. Preasymptotic stability carries over in the same manner, as we show later.

Lemma 7.11: Suppose that a compact set $\mathcal{A} \subset \mathcal{O}$ is preasymptotically stable for $\mathcal{H}$. Let $\mathcal{O}_1 \subset \mathcal{O}$ be an open set that contains $\mathcal{A}$ and is forward invariant for $\mathcal{H}$. Then, $\mathcal{A}$ is preasymptotically stable for the truncation of $\mathcal{H}$ to $\mathcal{O}_1$, that is, for the system $\mathcal{H}|_{\mathcal{O}_1}$, given on the state space $\mathcal{O}_1$ by the data (F_1, G_1, C_1, D_1) defined in (33).

Proof: The only thing to verify is whether a maximal solution ϕ to $\mathcal{H}|_{\mathcal{O}_1}$ that is not complete is bounded with respect to $\mathcal{O}_1$. As in Proposition 7.10, one can check that $\mathcal{H}|_{\mathcal{O}_1}$ satisfies the basic conditions. Thus, the maximal and not complete solution ϕ has a closed, hence compact, domain $\operatorname{dom}\phi$. This solution is also a solution to $\mathcal{H}$, and so, $\phi(t,j) \in \mathcal{O}_1$ for all $(t,j) \in \operatorname{dom}\phi$, by the invariance of $\mathcal{O}_1$ for $\mathcal{H}$. But, since $\operatorname{dom}\phi$ is compact, its range is compact too, and a subset of $\mathcal{O}_1$. Thus, ϕ is bounded with respect to $\mathcal{O}_1$.

C. From Preasymptotic to Asymptotic Stability

The key idea behind passing from preasymptotic stability, and also, from a relatively open to an open basin of attraction is captured in the following lemma.

Lemma 7.12: Suppose that $\mathcal{A}$ is preasymptotically stable for $\mathcal{H}$ with basin of preattraction $\mathcal{O}^p_{\mathcal{A}}$. Consider an augmented system $\mathcal{H}^* = (F, G^*, C, D^*)$ where

$$D^* = O, \quad G^*(x) = \begin{cases} G(x) \cup \mathcal{A} & \text{for } x \in D \\ \mathcal{A} & \text{for } x \in O \setminus D. \end{cases}$$

Then, $\mathcal{A}$ is asymptotically stable for $\mathcal{H}^*$ with basin of attraction $\mathcal{O}_{\mathcal{A}} = \mathcal{O}^p_{\mathcal{A}}$. Furthermore, the set $\mathcal{O}^p_{\mathcal{A}}$ is open in $\mathcal{O}$ for $\mathcal{H}$.

Proof: Since $\mathcal{H} = (F, G, C, D)$ satisfies hybrid basic conditions, so does its augmented system $\mathcal{H}^* = (F, G^*, C, D^*)$. Next, we show the stability of $\mathcal{A}$ for $\mathcal{H}^*$. Pick any $\varepsilon > 0$ and take $\delta > 0$ from the prestability of $\mathcal{A}$ for $\mathcal{H}$. Take any solution ψ to $\mathcal{H}^*$ with $\psi(0,0) \in \mathcal{A} + \delta\mathbb{B}$. If $(T,J) \in \operatorname{dom}\psi$ is such that for all $(t,j) \in \operatorname{dom}\psi$ with $t + j \le T + J$, we have $\phi(t,j) \notin \mathcal{A}$, then ψ is a solution to $\mathcal{H}$ on $\operatorname{dom}\psi \cap ([0,T] \times \{0,1,\ldots,J\})$, and as such, it must satisfy $\psi(t,j) \in \mathcal{A} + \varepsilon\mathbb{B}$ for all $(t,j) \in \operatorname{dom}\psi \cap ([0,T] \times \{0,1,\ldots,J\})$ by stability of $\mathcal{A}$ for $\mathcal{H}$. If $(T,J) \in \operatorname{dom}\psi$ is such that $\psi(T,J) \in \mathcal{A}$, then $\psi(t,j) \in \mathcal{A}$ for all $(t,j) \in \operatorname{dom}\psi$ with $T + J \le t + j$.

Now, we show the attractivity of $\mathcal{A}$ for $\mathcal{H}^*$. Pick $\varepsilon > 0$ so that $\mathcal{A} + \varepsilon\mathbb{B} \subset \mathcal{O}$ and take $\delta_1 > 0$ from prestability of $\mathcal{A}$ for $\mathcal{H}^*$. Pick $\delta > 0$ from preattractivity of $\mathcal{A}$ for $\mathcal{H}$ and such that $\delta < \delta_1$. Any maximal solution ψ to $\mathcal{H}^*$ with $\psi(0,0) \in \mathcal{A} + \delta\mathbb{B}$ is then bounded with respect to $\mathcal{O}$ (by stability and the choice of ε), and thus, is complete. This comes from $C \cup D^* = \mathcal{O}$ and [11, Proposition 2.4]. If ψ is a solution to $\mathcal{H}$, then by preattractivity of $\mathcal{A}$ for $\mathcal{H}$, $\psi(t,j) \to \mathcal{A}$ as $t + j \to \infty$. If ψ is not a solution to $\mathcal{H}$, then for some $(T,J) \in \operatorname{dom}\psi$, we have $\psi(T,J) \in \mathcal{A}$, and then, $\psi(t,j) \in \mathcal{A}$ for all $(t,j) \in \operatorname{dom}\psi$ with $T + J \le t + j$, and hence, $\psi(t,j) \to \mathcal{A}$ as $t + j \to \infty$.

Therefore, $\mathcal{A}$ is asymptotically stable for $\mathcal{H}^*$. Let $\mathcal{O}_{\mathcal{A}}$ be the basin of attraction for $\mathcal{H}^*$. Next, we show that $\mathcal{O}^p_{\mathcal{A}} = \mathcal{O}_{\mathcal{A}}$. If $x \in \mathcal{O}_{\mathcal{A}}$, then any solution to $\mathcal{H}^*$ from x is bounded (since any of its completions converges to $\mathcal{A}$), and, if it is complete, then it converges to $\mathcal{A}$. Since any solution to $\mathcal{H}$ is also a solution to $\mathcal{H}^*$, any solution to $\mathcal{H}$ from x is bounded, and converges to $\mathcal{A}$ if it is also complete. Therefore, $\mathcal{O}_{\mathcal{A}} \subset \mathcal{O}^p_{\mathcal{A}}$. To see that $\mathcal{O}^p_{\mathcal{A}} \subset \mathcal{O}_{\mathcal{A}}$, take any $x \in \mathcal{O}^p_{\mathcal{A}}$, and any maximal solution ψ to $\mathcal{H}^*$ with $\psi(0,0) = x$. As $C \cup D^* = O$, ψ is either complete or "blows-up," and so, is unbounded and not complete. If ψ is also a solution to $\mathcal{H}$, then it must be a maximal solution to $\mathcal{H}$. Now, it cannot be unbounded, and thus, must be complete. As such, it converges to $\mathcal{A}$. If ψ is not a solution to $\mathcal{H}$, then for some $(T,J) \in \operatorname{dom}\psi$, $\psi(T,J) \in \mathcal{A}$, and then, $\psi(t,j) \in \mathcal{A}$ for all $(t,j) \in \operatorname{dom}\psi$ with $T + J \le t + j$. Either way, ψ is complete and converges to $\mathcal{A}$. Therefore, $\mathcal{O}^p_{\mathcal{A}} \subset \mathcal{O}_{\mathcal{A}}$.

Finally, since $C \cup D^* = \mathcal{O}$, the set $\mathcal{O}_{\mathcal{A}}$ is open in $\mathcal{O}$ by [11, Proposition 6.4]. Thus, since $\mathcal{O}^p_{\mathcal{A}} = \mathcal{O}_{\mathcal{A}}$, the set $\mathcal{O}^p_{\mathcal{A}}$ is open in $\mathcal{O}$.

Corollary 7.13: Under the assumptions of Lemma 7.12, if additionally $\mathcal{O}^p_{\mathcal{A}} = \mathcal{O}$, then $\mathcal{O}_{\mathcal{A}}$, the basin of attraction of $\mathcal{A}$ for $\mathcal{H}^*$, equals $\mathcal{O}$.

D. Proof of (B)⇒(A) in Theorem 3.13

Proof: By Lemma 7.11, $\mathcal{A}$ is preasymptotically stable for $\mathcal{H}|_{\mathcal{O}_1}$, given on $\mathcal{O}_1$ by (F_1,G_1,C_1,D_1) as in (33). By invariance of $\mathcal{O}_1$ for $\mathcal{H}$ and Proposition 7.10, $\mathcal{H}|_{\mathcal{O}_1}$ satisfies the basic conditions. Since the basin of preattraction of $\mathcal{A}$ for $\mathcal{H}$ contains $\mathcal{O}_1$, the basin of preattraction of $\mathcal{A}$ for $\mathcal{H}|_{\mathcal{O}_1}$ is $\mathcal{O}_1$. Now, consider the extension $\mathcal{H}|^*_{\mathcal{O}_1}$ of $\mathcal{H}|_{\mathcal{O}_1}$, as described in Lemma 7.12. By Lemma 7.12, $\mathcal{A}$ is asymptotically stable for $\mathcal{H}|^*_{\mathcal{O}_1}$, and its basin of attraction, by Corollary 7.13, is $\mathcal{O}_1$. In other words, $\mathcal{A}$ is globally asymptotically stable for $\mathcal{H}|^*_{\mathcal{O}_1}$. Now, Theorem 7.1 implies the existence of a Lyapunov function for $(\mathcal{O}_1, F_1, G^*_1, C_1, D^*_1, \omega)$. Such a function is also the desired Lyapunov function for $(\mathcal{O}_1, F, G, C, D, \omega)$. ■

VIII. Remaining Technical Details

We now prove Propositions 3.8, 3.9, and Theorem 3.14. For the first two proofs, the augmented system as used in Lemma 7.12 again proves very useful.

Proof: (Proposition 3.8) Consider $\mathcal{H}^*$ as in Lemma 7.12. By assumptions, the set $\mathcal{A}$ is forward invariant for $\mathcal{H}^*$. It is also uniformly attractive, in fact, given $\delta > 0$, any $\epsilon > 0$, and $\mathcal{T} > 0$ as in the assumptions, each maximal solution ϕ to $\mathcal{H}^*$ with $\phi(0,0) \in \mathcal{A} + \delta\overline{\mathbb{B}}$ satisfies $\phi(t,j) \in \mathcal{A} + \epsilon\overline{\mathbb{B}}$ for all $(t,j) \in$

$\operatorname{dom}\phi$ with $t+j \geq \mathcal{T}+1$. Then, [11, Proposition 6.1] gives asymptotic stability of $\mathcal{A}$ for $\mathcal{H}^*$. This guarantees that $\mathcal{A}$ is preasymptotically stable for $\mathcal{H}$. ■

Proof: (Proposition 3.9) From the properties of K and $\mathcal{T}$, we obtain that $\mathcal{R}_\infty(K) = \mathcal{R}_{\leq\mathcal{T}}(K)$. Now, consider $\mathcal{H}^*$ as in Lemma 7.12 with K in place of $\mathcal{A}$, and denote by $\mathcal{R}^*_{\leq\mathcal{T}}(K)$ the reachable set from K in finite time $\mathcal{T}$ for $\mathcal{H}^*$. Directly from the definitions, $\mathcal{R}^*_{\leq\mathcal{T}}(K) = \mathcal{R}_{\leq\mathcal{T}}(K)$. Since K_0 is compact and contains $\mathcal{R}_\infty(K)+\delta\mathbb{B}$, all solutions to $\mathcal{H}^*$ from K are bounded. Thus, since for $\mathcal{H}^*$, the union of the flow set and the jump set equals $\mathcal{O}$, the maximal solutions from K are complete; see [11, Proposition 2.4]. So, $\mathcal{H}^*$ is forward complete at each $x \in K$, and then, [11, Corollary 4.7] says that $\mathcal{R}^*_{\leq\mathcal{T}}(K)$ is compact. Thus, $\mathcal{R}_\infty(K)$ is compact. By definition, it is forward invariant. Also, by construction, it is uniformly preattractive. The result now follows from Proposition 3.8. ■

Proof: (Theorem 3.14) That $\mathcal{O}^p_\mathcal{A}$ is open in $\mathcal{O}$ is shown in Lemma 7.12. Let ϕ be any solution to $\mathcal{H}$ with $\phi(0,0) \in \mathcal{O}^p_\mathcal{A}$. For each $(t,j) \in \operatorname{dom}\phi$, there does not exist an unbounded or a complete but not convergent to $\mathcal{A}$ solution to $\mathcal{H}$ starting at $\phi(t,j)$, i.e., $\phi(t,j) \in \mathcal{O}^p_\mathcal{A}$; otherwise, concatenation of such a solution and ϕ would contradict $\phi(0,0) \in \mathcal{O}^p_\mathcal{A}$. Therefore, the set $\mathcal{O}^p_\mathcal{A}$ is forward invariant for $\mathcal{H}$. Finally, the implication 2⇒1 in Theorem 3.13 gives the existence of a smooth Lyapunov function for $(\mathcal{O}^p_\mathcal{A}, F, G, C, D, \omega)$. ■

IX. Robust $\mathcal{KLL}$ Stability With Respect to Two Measures: A Special Case

In the paper [4], the notion of $\mathcal{KLL}$ stability with respect to two measures was used, and it was shown that robustness of $\mathcal{KLL}$ stability was equivalent to the existence of a smooth Lyapunov function that could be used to establish $\mathcal{KLL}$ stability with respect to the two measures. In general, it is not known when $\mathcal{KLL}$ stability with respect to two measures is robust. This paper has shown that it is robust when the two measures are the same and correspond to a proper indicator for a compact set. We can use this result to give some other related cases where $\mathcal{KLL}$ stability with respect to two measures is robust, including a case that covers the temperature control example reported in [4]. The following definition is a slight generalization of the corresponding definition in [4]. In particular, it incorporates the notion of "pre" stability.

Definition 9.1: Let the hybrid system $\mathcal{H} = (F,G,C,D)$, with open state-space $\mathcal{O}$, and two continuous functions $\omega_1, \omega_2 : \mathcal{O} \to \mathbb{R}_{\geq 0}$ be given. The system $\mathcal{H}$ is said to be $\mathcal{KLL}$ prestable with respect to (ω_1, ω_2) if there exists $\gamma \in \mathcal{KLL}$ such that, for each $x \in \mathcal{O}$ and each $\phi \in \mathcal{S}(x)$

$$\omega_1(\phi(t,j)) \leq \gamma(\omega_2(x), t, j) \qquad \forall (t,j) \in \operatorname{dom}\phi \tag{34}$$

and if ϕ is not complete, then ϕ is bounded with respect to $\mathcal{O}$.

We now make our assumptions explicit.

Assumption 9.2: The system $\mathcal{H}$ is $\mathcal{KLL}$ prestable with respect to (ω_1, ω_2), and the set

$$\mathcal{A} := \left\{ x \in \mathcal{O} : \sup_{\phi \in \mathcal{S}(x), (t,j) \in \operatorname{dom}\phi} \omega_1(\phi(t,j)) = 0 \right\}$$

is compact and preasymptotically stable with basin of preattraction $\mathcal{O}$.

In Assumption 9.2, we observe from the forward invariance of $\mathcal{A}$ and Proposition 3.8 that uniform preattractivity is sufficient for preasymptotic stability. In particular, one way to guarantee preasymptotic stability would be for all trajectories starting near $\mathcal{A}$ to enter $\mathcal{A}$ in finite time. This happens for the temperature control system described in [4]. Also, note that $\mathcal{KLL}$-prestability does not necessarily imply prestability or preattractivity unless extra conditions (for example, on growth of ω_1) are assumed. However, due to (34), preasymptotic stability of $\mathcal{A}$ is reasonable.

It is not difficult to see that the augmented system described in Lemma 7.12 would be $\mathcal{KLL}$ stable with respect to (ω_1, ω_2) on $\mathcal{O}$, in the sense defined in [4]. Moreover, for this augmented system, the set $\mathcal{A}$ would be asymptotically stable with basin of attraction equal to $\mathcal{O}$. Thus, according to the main result of this paper, for function ω that was a proper indicator for $\mathcal{A}$ on $\mathcal{O}$, there would exist a smooth Lyapunov function for $(\mathcal{O}, F, G, C, D, \omega)$. Now, by the definition of the set $\mathcal{A}$, it follows that there exists $\widetilde{\alpha}_1 \in \mathcal{K}_\infty$ such that $\widetilde{\alpha}_1(\omega_1(x)) \leq \omega(x)$ for all $x \in \mathcal{O}$.

The other assumption we make is the following.

Assumption 9.3: The function ω_2 is proper on $\mathcal{O}$.

This assumption is satisfied for the temperature control example in [4]. With this assumption, it follows that there exists $\widetilde{\alpha}_2 \in \mathcal{K}_\infty$ such that $\omega(x) \leq \widetilde{\alpha}_2(\omega_2(x))$ for all $x \in \mathcal{O}$.

The preceding discussion can be summarized as follows.

Theorem 9.4: Under Assumptions 9.2 and 9.3, there exists a smooth function $V : \mathcal{O} \to \mathbb{R}_{\geq 0}$ and class-$\mathcal{K}_\infty$ function κ_1 and κ_2 such that

$$\begin{aligned} \kappa_1(\omega_1(x)) \leq V(x) \leq \kappa_2(\omega_2(x)) \qquad & \forall x \in \mathcal{O} \\ \max_{f \in F(x)} \langle \nabla V(x), f \rangle \leq -V(x) \qquad & \forall x \in C \\ \max_{g \in G(x)} V(g) \leq e^{-1} V(x) \qquad & \forall x \in D\,. \end{aligned}$$

The main result of [4] now gives that the assumed $\mathcal{KLL}$ stability with respect to two measures (technically, for the augmented system in Lemma 7.12) is robust in the sense defined in [4].

X. Conclusion

Hybrid systems with preasymptotically stable compact sets admit smooth Lyapunov functions. The key to establishing this property is the fact that preasymptotic stability is robust. Smooth Lyapunov functions are equivalent to robustness and are a convenient tool for encoding robustness. We illustrated this for several classes of systems, including those subject to slowly varying, weakly jumping parameters, to temporal regularization, to jumps inserted under an average dwell-time condition, and to flows inserted under a reverse average dwell-time condition. Hopefully, converse Lyapunov theorems for hybrid systems will lead to a better understanding of robustness in hybrid control systems and to additional insights into robust stabilization by hybrid feedback.

APPENDIX

DWELL-TIME CONDITIONS PRODUCED BY HYBRID SYSTEMS

Proposition 1.1: Let $\delta \geq 0$ and let N be a positive integer. A hybrid time domain E satisfies

$$\begin{aligned} & j-i \leq \delta(t-s)+N \\ & \forall (t,j),(s,i) \in E \quad \text{with} \quad t+j > s+i \end{aligned} \tag{35}$$

if and only if $E = \operatorname{dom}\tau$ for some solution τ to the hybrid system

$$\begin{cases} \dot{\tau} \in \eta_\delta(\tau) & \text{for } \tau \in C := [0, N] \\ \tau^+ = \tau - 1 & \text{for } \tau \in D := [1, N] \end{cases} \tag{36}$$

where

$$\eta_\delta(\tau) := \begin{cases} \delta & \text{for } \tau \in [0, N) \\ [0,\delta] & \text{for } \tau = N. \end{cases}$$

Proof: We observe that the interval $[0, N]$ is forward invariant for (36). Suppose τ is a solution to (36). Then, for each $(s,i),(t,j) \in \operatorname{dom}\ \tau$ with $t+j > s+i$

$$\begin{aligned} 0 &\leq \tau(t,j) \\ &\leq \tau(s,i) + \delta(t-s) - (j-i) \\ &\leq N + \delta(t-s) - (j-i)\ . \end{aligned}$$

Rearranging the previous inequality gives (35). On the other hand, suppose that a hybrid time domain E satisfies (35). Then, take a solution τ to (36) with $\tau(0,0) = N$ to track the hybrid time domain E by flowing or jumping as appropriate. This is possible unless there is a jump in E, at time (t_{i+1}, i), such that $\tau(t_{i+1}, i) < 1$. Suppose that this happens. Since $\tau = N$ is an equilibrium point of the flow of the hybrid system, without loss of generality, we can assume that $(0,1) \in E$ and that $\tau(t,j) < N$ for all $(t,j) \in \operatorname{dom}\tau$ satisfying $t_{i+1} + i \geq t + j > 0$. Because of this, we have

$$1 > \tau(t_{i+1}, i) = \tau(0,0) + \delta t_{i+1} - i = N + \delta t_{i+1} - i$$

i.e., $i + 1 > \delta t_{i+1} + N$. This implies that E does not satisfy (37), which is a contradiction. ∎

Proposition 1.2: Let $\delta \geq 0$ and $\lambda > 0$. A hybrid time domain E satisfies

$$\begin{aligned} & t-s \leq \delta(j-i)+\lambda \\ & \forall (t,j),(s,i) \in E \quad \text{with} \quad t+j > s+i \end{aligned} \tag{37}$$

if and only if $E = \operatorname{dom}\tau$ for some solution τ to the hybrid system

$$\begin{cases} \dot{\tau} = 1 & \text{for} \quad \tau \in C := [0, \lambda] \\ \tau^+ = \max\{0, \tau - \delta\} & \text{for} \quad \tau \in D := [0, \lambda]. \end{cases} \tag{38}$$

Proof: We observe that the interval $[0, \lambda]$ is forward invariant for (38). Suppose τ is a solution to (38). Then, for each $(s,i),(t,j) \in \operatorname{dom}\ \tau$ with $t+j > s+i$

$$\begin{aligned} \lambda \geq \tau(t,j) &\geq \tau(s,i) + (t-s) - \delta(j-i) \\ &\geq (t-s) - \delta(j-i). \end{aligned}$$

Rearranging the previous inequality gives (37). On the other hand, suppose that a hybrid time domain E satisfies (37). Then, take a solution τ to (38) with $\tau(0,0) = 0$ to track the hybrid time domain E by flowing or jumping as appropriate. This is possible unless there exist $(t,j),(s,i) \in E$ with $t > s$ such that $\tau(s,i) = \lambda$. Suppose that this happens. Since $\tau = 0$ is an equilibrium of the jump map of the hybrid system, without loss of generality, we can assume that $(\varepsilon, 0) \in E$ for some $\varepsilon > 0$ and that $\tau(t,j) > 0$ for all $(t,j) \in \operatorname{dom}\tau$ with $s + i \geq t + j > 0$. This implies

$$\lambda = \tau(s,i) = \tau(0,0) + s - \delta i = s - \delta i < t - \delta i$$

which contradicts that E satisfies (37). ∎

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