REFERENCES

[1] X. Cao, Z. Ren, S. Bhatnagar, M. Fu, and S. Marcus, "A time-aggregation approach to Markov decision processes," *Automatica*, vol. 38, no. 6, pp. 929–943, 2002.

[2] R. Howard, *Dynamic Probabilistic Systems Volume II: Semi-Markov and Decision Processes*. New York: Wiley, 1971.

[3] M. L. Puterman, *Markov Decision Processes: Discrete Stochastic Dynamic Programming*. New York: Wiley, 1994.

[4] Z. Ren and B. H. Krogh, "Adaptive control of Markov chains with average cost," *IEEE Trans. Autom. Control*, vol. 46, no. 4, pp. 613–617, Apr. 2001.

[5] ——, "Switching control in multimode Markov decision processes," presented at the 40th IEEE Conf. Decision and Control, Orlando, FL, 2001.

[6] R. F. Serfozo, "An equivalence between continuous and discrete time Markov decision processes," *Oper. Res.*, vol. 27, no. 3, pp. 616–620, 1979.

[7] J. N. Tsitsiklis, "Asynchronous stochastic approximation and Q-learning," *Mach. Learn.*, vol. 16, pp. 185–202, 1994.

Stabilizing a Linear System With Saturation Through Optimal Control

Rafal Goebel

***Abstract*—We construct a continuous feedback for a saturated system $\dot{x}(t) = Ax(t) + B\sigma(u(t))$. The feedback renders the system asymptotically stable on the whole set of states that can be driven to 0 with an open-loop control. The trajectories of the resulting closed-loop system are optimal for an auxiliary optimal control problem with a convex cost and linear dynamics. The value function for the auxiliary problem, which we show to be differentiable, serves as a Lyapunov function for the saturated system. Relating the saturated system, which is nonlinear, to an optimal control problem with linear dynamics is possible thanks to the monotone structure of saturation.**

***Index Terms*—Convex Lyapunov function, feedbacl stabilization, linear system, optimal control, saturating actuator.**

I. INTRODUCTION

Global asymptotic stabilization of a linear system with saturating actuators

$$\dot{x}(t) = Ax(t) + B\sigma(u(t)) \tag{1}$$

cannot, in general, be achieved with a linear feedback. Moreover, if an eigenvalue of A has a positive real part and σ is bounded, the set X_0 consisting of all states that can be driven to 0 with an open-loop control will not equal the whole state–space. If such eigenvalues are excluded, continuous feedbacks globally stabilizing (1) exist under mild assumptions on σ, as shown by Sontag and Sussmann [19] and Sontag *et al.* [20]. Also then, semiglobal stabilization can be achieved with linear feedback possessing additional properties like robustness and disturbance rejection; see [18]. For the general case, much work has been devoted to estimating X_0 and to semiglobal stabilization on X_0 (that is, to constructing feedbacks stabilizing (1) on any *a priori* given compact subset of X_0); see [13] and the numerous references therein. A positive result on semiglobal stabilization with a continuous feedback of a linear system under both input and state constraints was recently shown by Stoorvogel *et al.* [21].

To summarize, the existence of a continuous static feedback that renders the saturated system (1) asymptotically stable on the whole set X_0 has not been established. We prove it here, by exhibiting a feedback which guarantees that the resulting trajectories of (1) are optimal for the following linear-convex regulator problem:

$$\mathcal{LCR}(x_0): \quad \begin{array}{ll} \text{minimize} & \displaystyle\int_0^\infty \frac{1}{2}x(t)\cdot Qx(t) + r(w(t))\,dt \\ \text{s.t.} & \begin{cases} \dot{x}(t) = Ax(t) + Bw(t) \\ x(0) = x_0. \end{cases} \end{array} \tag{2}$$

This problem has no saturation but information about σ is captured by the convex penalty function r. The stabilizing feedback for the saturated system (1) will turn out to be closely related to the optimal feedback for the $\mathcal{LCR}$. In (2), the control variable is denoted $w(\cdot)$ to distinguish it from $u(\cdot)$ in (1)—these are not the same, and Q is any symmetric and positive–definite matrix.

Before describing the relationship between the saturation σ and the convex function r appearing in (2), we state the assumptions, which are posed throughout this note.

- A1) The pair (A, B), where $A \in I\!R^{n\times n}$, $B \in I\!R^{n\times k}$, is controllable.
- A2) The saturation function $\sigma : I\!R^k \mapsto I\!R^k$ has the form $\sigma(u) = (\sigma_1(u_1), \sigma_2(u_2), \dots, \sigma_k(u_k))$, where $\sigma_i(0) = 0$, σ_i is nondecreasing on $I\!R$ and strictly increasing on a neighborhood of 0, $i = 1, 2, \dots, k$.

Under A2), there exists a convex function $s : I\!R^k \mapsto I\!R$ with $s(0) = 0$ and with the gradient $\nabla s = \sigma$. Then, r is taken to be the convex function conjugate to s in the sense of convex analysis; see [14]. We explain this in detail in Section II.

Introducing a $\mathcal{LCR}$ as an auxiliary optimal control problem is a natural idea. Feedbacks stabilizing a linear system $\dot{x}(t) = Ax(t) + Bu(t)$ can be found with the help of a $\mathcal{LQR}$ problem. When σ in (1) is the standard saturation, that is $\sigma_i(u_i)$ equals u_i if $-1 \leq u_i \leq 1$, -1 if $u_i < -1$, and 1 if $u_i > 1$, one can consider a linear-quadratic regulator with a control constraint $|u_i| \leq 1$. With a well-known optimization technique, one can express the constrained $\mathcal{LQR}$ in the $\mathcal{LCR}$ format (2): let r be quadratic if u satisfies the constraint, and equal to $+\infty$ otherwise. The use of value functions of auxiliary problems as Lyapunov functions is possible for general nonlinear systems, but need not result in a smooth function, and the resulting stabilizing feedbacks need not be continuous; see [6] and [5]. The expected lack of continuity of optimal feedbacks for problems with nonlinear dynamics was a part of the motivation for an alternate approach to stabilization of a saturated system in [20].

The special structure of $\mathcal{LCR}$ has important consequences for the value function $V(x_0)$ defined as the optimal value in (2). Most importantly, V is a convex function. It is positive definite, has finite values on the open and convex set X_0 while $V(x_0) = +\infty$ if $x_0 \notin X_0$, and its sublevel sets $\{x | V(x) \leq \alpha\}$ are compact for each $\alpha \geq 0$. Finally, we prove it is differentiable on X_0, and then continuity of ∇V on X_0 [which will be the key to continuity of the stabilizing feedback for (1)] follows from a general property of convex functions. Details are provided in Section III.

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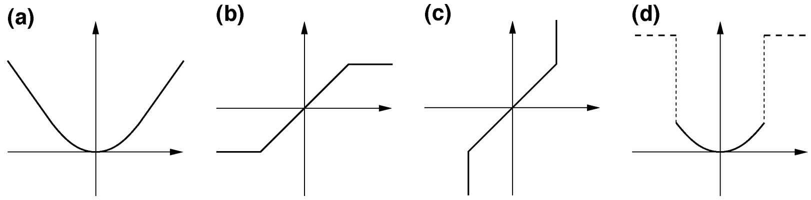

Fig. 1. From a saturation function to a convex cost function (standard saturation).

With the differentiability of V established, standard dynamic programming arguments, outlined following Corollary 3.2, show that the optimal feedback for the $\mathcal{LCR}$ is

$$w = F_{\mathcal{LCR}}(x) = \nabla s\left(-B^*\nabla V(x)\right)$$

which is equivalent to $F_{\mathcal{LCR}}(x) = \arg\max_w\{-\nabla V(x(t))\cdot Bw - r(w)\}$. Optimal trajectories $x(\cdot)$ resulting from applying this optimal feedback to the linear system satisfy

$$\frac{d}{dt}V\left(x(t)\right) \leq -\frac{1}{2}x(t)\cdot Qx(t) \tag{3}$$

and, hence, $x(t) \to 0$ as $t \to \infty$.

Now, the relationship between the saturated system and $\mathcal{LCR}$ should become clear. Since $\nabla s = \sigma$, the *nonsaturated* linear system with the feedback $w = \nabla s(-B^*\nabla V(x))$ is exactly the same as the *saturated* system (1) with the feedback

$$u = F(x) = -B^*\nabla V(x).$$

This means that F is a stabilizing feedback for the saturated system. Moreover, (3) shows that the value function for $\mathcal{LCR}$ serves as a classical Lyapunov function for the saturated system. We state this more precisely in Section IV.

II. Saturation Functions as Gradients

The key to our approach is expressing the saturation function σ of the saturated linear system (1) as a gradient of a convex function. A standard reference for the convex analysis facts we use later is the book by Rockafellar [14].

Example 2.1: Let $\sigma : I\!R \mapsto I\!R$ be continuous and nondecreasing, with $\sigma(0) = 0$. Then

$$s(u) = \int_0^u \sigma(t)dt$$

defines a differentiable convex function $s : I\!R \mapsto I\!R$, with $s(0) = 0$, $s \geq 0$, and, of course, $s' = \sigma$. Other often assumed properties of σ reflect in those of s. For example, if $\sigma(u) = 0$ only for $u = 0$, then s is positive definite. Also, if $\liminf_{u\to 0}(\sigma(u))/u > 0$—equivalently, if for some $\epsilon > 0$, $\delta > 0$, we have $u\sigma(u) \geq \delta u^2$ for $|u| < \epsilon$—then $s(u)$ is bounded below by $(1/2)\delta u^2$ if $|u| < \epsilon$, by $-\delta\epsilon u - (1/2)\delta\epsilon^2$ if $u \leq -\epsilon$, and by $\delta\epsilon u - (1/2)\delta\epsilon^2$ if $\epsilon < u$. Finally, if σ is globally Lipschitz with constant l, then $s(u) \leq (l/2)u^2$. Here, the important relationship is between strict convexity of s on a neighborhood of 0 and σ being strictly increasing on such a neighborhood.

Statements just made can be easily verified for the standard saturation function $\overline{\sigma} : I\!R \mapsto [-1, 1]$, which is the derivative of the following convex function:

$$\overline{s}(u) = \begin{cases} -u - \frac{1}{2}, & \text{for } u < -1 \\ \frac{1}{2}u^2, & \text{for } -1 \leq u \leq 1 \\ u - \frac{1}{2}, & \text{for } 1 < u. \end{cases} \tag{4}$$

Example 2.2: Suppose $\sigma(u) = (\sigma_1(u_1), \sigma_2(u_2), \ldots, \sigma_k(u_k))$, with each σ_i nondecreasing on $I\!R$, and $\sigma(0) = 0$. With each σ_i we can associate a convex function s_i as outlined in Example 2.1. Then, $\sigma = \nabla s$ for $s(u) = \sum_{i=1}^k s_i(u_i)$, which is of course a convex function. Growth properties of s can be analyzed in terms of that of σ_i's. In particular, s is strictly convex on a neighborhood of $0 \in I\!R^k$ if and only if each σ_i is strictly increasing on some neighborhood of $0 \in I\!R$.

Now, we explain how the convex function r, representing the control cost in the linear-convex regulator (2), is related to σ. Given a convex function s with $\nabla s = \sigma$ and $s(0) = 0$, we set r to be the convex function conjugate to s in the sense of convex analysis [14, Ch. 12]

$$r(w) = \sup_{u\in I\!R^k}\left\{w\cdot u - s(u)\right\}. \tag{5}$$

This function is convex and lower semicontinuous. It need not be finite everywhere—for some w, we may have $r(w) = +\infty$. Also, r need not be differentiable. Its subdifferential ∂r defined by $\partial r(w) = \{z \in I\!R^k \mid r(w') \geq r(w) + z\cdot(w' - w) \ \ \forall w' \in I\!R^k\}$ is the set-valued inverse of ∇s [14, Ch. 23]. The latter equals σ, and need not be invertible in the classical sense.

In many cases of practical interest, r can be found directly. First, observe that the very definition (5) implies that when $s(u) = \sum_{i=1}^k s_i(u_i)$, as in Example 2.2, $r(w)$ is given by

$$\sup_{u\in I\!R^k}\{w\cdot u - s(z)\} = \sum_{i=1}^k \sup_{u_i\in I\!R}\{w_i\cdot u_i - s_i(u_i)\} = \sum_{i=1}^k r_i(w_i)$$

where r_i is the convex conjugate of s_i. That is, r can be found componentwise. We now give some one-dimensional examples.

Example 2.3: Consider the standard saturation $\overline{\sigma}$, shown in Fig. 1(b). The function $\overline{s}$ given by (4), and shown in Fig. 1(a), can be used to calculate $\overline{r}$ directly from the definition (5). An alternate approach is to look at the set-valued inverse of $\overline{\sigma}$, equal to $\partial\overline{r}$, which is shown in Fig. 1(c). Then, it remains to "integrate" $\partial\overline{r}$ to obtain $\overline{r}$, shown in Fig. 1(d).

Explicit formulas for $\partial\overline{r}$ and $\overline{r}$ are as follows:

$$\partial\overline{r}(w) = \begin{cases} \emptyset, & \text{for } w < -1 \\ (-\infty, 1], & \text{for } w = -1 \\ w, & \text{for } -1 < w < 1 \\ [1, +\infty), & \text{for } w = 1 \\ \emptyset, & \text{for } w > 1. \end{cases}$$

$$\overline{r}(w) = \begin{cases} \frac{1}{2}w^2, & \text{for } w \in [-1, 1] \\ +\infty, & \text{for } w \notin [-1, 1]. \end{cases}$$

Example 2.4: Consider $\sigma(u) = u/\sqrt{u^2+1}$, which is a derivative of $s(u) = \sqrt{u^2+1} - 1$. The conjugate r can be found through (5). Alternatively, $\sigma^{-1}(w) = r'(w) = w/\sqrt{1-w^2}$ for $w \in (-1, 1)$, while for $w \notin (-1, 1)$, $\sigma^{-1}(w) = r'(w) = \emptyset$. Then, $r(w)$ can be found, for any $w \in [-1, 1]$, by integrating r'. This leads to $r(w) = 1 - \sqrt{1-w^2}$ on $[-1, 1]$, while $r(w) = +\infty$ for $w \notin [-1, 1]$. Fig. 2(a) shows s, Fig. 2(a) shows σ, Fig. 2(a) displays $\sigma^{-1} = r'$, and r is in

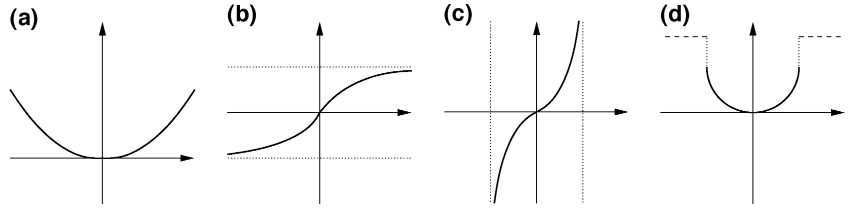

Fig. 2. From a saturation function to a convex cost function (saturation of Example 2.4).

Fig. 2(a). Note a slight discrepancy between the set of points where r is finite and the set of points where ∂r, which reduces to r', is nonempty.

The set $\operatorname{dom} r = \{w \in I\!R^n | r(w) < +\infty\}$ need not equal $I\!R^n$. In fact $r(w) = +\infty$ whenever $w \notin \overline{\operatorname{rge} \sigma}$ (the closure of the range of σ). Infinite values of r introduce a control constraint to the linear-convex regulator: feasible controls must satisfy $w(t) \in \operatorname{dom} r$. For the standard saturation this yields $w(t) \in \operatorname{dom} \overline{r} = \operatorname{rge} \overline{\sigma} = [-1, 1]$. In general $\operatorname{rge} \sigma \subset \operatorname{dom} r$. (The equality $\operatorname{rge} \sigma = \operatorname{dom} r$ fails in Example 2.4.) For details, see the beginning of [14, Sec. 24].

To summarize this section, we state the following.

Fact 2.5: (Saturation and Convex Functions): Given a saturation function σ as in Assumption A2), there exist convex functions $s : I\!R^k \mapsto [0, +\infty)$ and $r : I\!R^k \mapsto [0, +\infty]$ related to each other by (5) and such that

i) s is differentiable, $\nabla s = \sigma$, $s(0) = 0$, and s is strictly convex on some neighborhood of 0;
ii) r is positive definite and on some neighborhood of 0, it has finite values.

III. Value Function for $\mathcal{LCR}$

The value function of the linear-convex regulator

$$V(x_0) = \inf \left\{ \int_0^\infty \frac{1}{2} x(t) \cdot Qx(t) + r\left(u(t)\right) dt \;\middle|\; \dot{x}(t) = Ax(t) + Bu(t), x(0) = x_0 \right\} \quad (6)$$

with the minimization carried out over all locally integrable controls $u : [0, +\infty)$, is obviously positive definite. It may occur that for some $x_0 \in I\!R^n$, $V(x_0) = +\infty$; this is the case when no control makes the integral in (6) finite.

A key property of V is that it is a convex function on $I\!R^n$. This is a consequence of a general principle that value functions for convex optimization problems are convex; see [16]. Here, since a composition of an affine map with a convex function is convex, and the trajectory $x(\cdot)$ depends affinely on x_0 and $u(\cdot)$, the integral in (6) is a convex function of x_0 and $u(\cdot)$. Minimizing it with respect to $u(\cdot)$ yields a convex function of x_0. Convexity can also be verified directly through the definition of convexity, the infinite values just require some extra care.

A consequence of the value function being convex is that the level sets of V, being $\{x \in I\!R^n \mid V(x) \leq \alpha\}$, are convex and bounded for each $\alpha \in I\!R$. Boundedness follows from the existence of a single nonempty and bounded level set [14, Cor. 8.7.1]: we have $\{x \in I\!R^n | V(x) \leq 0\} = \{0\}$. In turn, boundedness implies that any process $(\bar{x}(\cdot), \bar{u}(\cdot))$ for which the integral in (6) is finite satisfies $\bar{x}(t) \to 0$ as $t \to \infty$. Indeed, from the definition of the value function it follows that

$$V\left(\bar{x}(t)\right) \leq \int_t^\infty \frac{1}{2} \bar{x}(t) \cdot Q\bar{x}(t) + r\left(\bar{u}(t)\right) dt$$

for every $t \geq 0$, and the aforementioned integral tends to 0 as $t \to +\infty$.

We now argue that the set $\operatorname{dom} V = \{x \in I\!R^n \mid V(x) < +\infty\}$ is open. (This also follows from Lemma 4.2 of this note and [21, Lemma 9].) By continuity of r at 0 and controllability of (A, B), $\operatorname{dom} V$ contains some neighborhood N of 0. Pick any $x_0 \in \operatorname{dom} V$, and let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be a process for which the integral in (6) is finite. For some $T > 0$, $\bar{x}(T) \in N$. Thus, any x_0' from some neighborhood of x_0 can be driven into N by the control $\bar{u}(\cdot)$ truncated to $[0, T]$. This shows that $V(x_0')$ is finite and, thus, $\operatorname{dom} V$ is open.

The main result of this section claims the smoothness of V, which will turn out to be the key to the continuity of the stabilizing feedback for the saturated system.

Theorem 3.1: (Differentiability of V): The value function V is differentiable at every point of $\operatorname{dom} V$ and $\|\nabla V(x_i)\| \to +\infty$ for any sequence of points $x_i \in \operatorname{dom} V$ converging to a point not in $\operatorname{dom} V$. The gradient ∇V is continuous on $\operatorname{dom} V$. The function V is strictly convex.

Some results guaranteeing differentiability of value functions in similar settings exist, but do not directly apply here. Benveniste and Scheinkman [2] and Gota and Montrucchio [12] require r to be differentiable and the optimal controls to be interior in some sense; neither assumption is met here.[1] Rockafellar [15] showed that if the (maximized) Hamiltonian is strictly concave in x, strictly convex in p, the value function is differentiable. The Hamiltonian for $\mathcal{LCR}$ is

$$H(x, p) = p \cdot Ax - \frac{1}{2} x \cdot Qx + s(B^* p). \quad (7)$$

It is not strictly convex in p unless B is invertible and s is strictly convex everywhere. Barbu [1] incorporated a controllable linear system to the framework of [15], under additional growth properties of H. Ideas from [15] and [1] can be combined to show that V is differentiable around 0. Then, writing $\mathcal{LCR}$ as a finite time problem with a terminal penalty V and applying [9, Th. 3.1] could be used to obtain a global statement. Instead, we rely on strict convexity of the dual optimal value function and on a duality result of Goebel [8]; see also [10].

Proof: (of Theorem 3.1.): Differentiability properties of V as stated in the first sentence of Theorem 3.1 can be equivalently expressed in terms of the convex function conjugate to V. Theorem 3.4 of [8] describes this conjugate as the value function of a dual optimal control problem

$$W(p_0) = \inf \left\{ \int_0^{+\infty} s\left(B^* p(t)\right) + \frac{1}{2} z(t) \cdot Q^{-1} z(t) dt \;\middle|\; \dot{p}(t) = -A^* p(t) - z(t), p(0) = p_0 \right\}. \quad (8)$$

That is, the convex conjugate of V is $W(-\cdot)$ and *vice versa*: $W(-p_0) = \sup_{x_0}\{p_0 \cdot x_0 - V(x_0)\}$, $V(x_0) = \sup_{p_0}\{x_0 \cdot p_0 - W(-p_0)\}$. In (8), $z(\cdot)$ should be thought of as a (dual) control variable. Arguments similar to those at the beginning of the current section show it is a positive–definite convex function and that processes $(p(\cdot), z(\cdot))$ for which the integral in (8) is finite are such that $p(t) \to 0$ as

[1]These two works represent a wide body of theoretical economics research devoted to optimal control on infinite time intervals, a good source of references is [4].

$t \to 0$; thus [8, Th. 3.4] applies. Now, [14, Th. 26.3] states that if W is strictly convex on $\mathrm{dom}\, W$, then V has the desired differentiability properties.

Convexity of W means that $W((1-\lambda)p_0' + \lambda p_0'') \leq (1-\lambda_0')W(p_0') + \lambda W(p_0'')$ for all p_0', p_0'', all $\lambda \in [0,1]$. If W is not strictly convex on $\mathrm{dom}\, W$, then for some $p_0' \neq p_0''$ with $W(p_0')$, $W(p_0'')$ finite, and some $\lambda \in (0,1)$ we have, for $p_0^\lambda = (1-\lambda)p_0' + \lambda p_0''$, that $W(p_0^\lambda) = (1-\lambda_0')W(p_0') + \lambda W(p_0'')$. Let $(p'(\cdot), z'(\cdot))$, $(p''(\cdot), z''(\cdot))$ be optimal processes for $W(p_0')$, $W(p_0'')$. The last equality and strict convexity of the quadratic function given by Q^{-1} imply that for all $t \geq 0$, $z'(t) = z''(t)$. Indeed, otherwise the process given by $p^\lambda(t) = (1-\lambda)p'(t) + \lambda p''(t)$, $z^\lambda(t) = (1-\lambda)z'(t) + \lambda z''(t)$, with $p^\lambda(0) = p_0^\lambda$, yields a cost lower than $W(p_0^\lambda)$. Similarly, $B^* p'(t) = B^* p''(t)$ for all sufficiently large t (since s is strictly convex on a neighborhood of 0). As $(-A^*, B^*)$ is detectable, for such t's, $p'(t) = p''(t)$. However, then, since $z'(t) = z''(t)$ for all $t \geq 0$, we also have $p'(t) = p''(t)$ for all $t \geq 0$. In particular, $p'(0) = p''(0)$ what contradicts $p_0' \neq p_0''$. Thus, W is strictly convex on $\mathrm{dom}\, W$.

With differentiability of V on $\mathrm{dom}\, V$ established, continuity of ∇V follows, as V is a convex function; see [14, Cor. 25.5.1]. Strict convexity of V can be verified directly, using arguments as those in the previous paragraph. ■

Corollary 3.2: (Optimal Feedback for $\mathcal{LCR}$): The mapping $F_{\mathcal{LCR}} : \mathrm{dom}\, V \to I\!R^k$ defined by $F_{\mathcal{LCR}}(x) = \nabla s(-B^* \nabla V(x))$ is the optimal feedback for $\mathcal{LCR}$. That is, for any $x_0 \in \mathrm{dom}\, V$, the process $(x(\cdot), w(\cdot))$ with $x(\cdot)$ being the solution to $x(0) = x_0, \dot{x}(t) = Ax(t) + Bw(t)$ and $w(t) = F_{\mathcal{LCR}}(x(t))$, is optimal for $\mathcal{LCR}(x_0)$.

We outline the standard argument. The value function V satisfies the Hamilton–Jacobi equation

$$H(x, -\nabla V(x)) = 0, \qquad \text{for all } x \in \mathrm{dom}\, V \tag{9}$$

where H is given by (7). (In fact the value function is the unique positive semidefinite, lower semicontinuous, and convex function satisfying $H(x, -\partial V(x)) = 0$; see [8].) From the definition of r in terms of s in (5), one can see that $r(\nabla s(u)) = \nabla s(u) \cdot u - s(u)$. This and the Hamilton–Jacobi equation show that

$$\frac{d}{dt} V(x(t)) = -\frac{1}{2} x(t) \cdot Qx(t) - r(w(t)) \tag{10}$$

which implies both that $x(t) \to 0$ as $t \to 0$ and that $x(\cdot)$ is optimal for $\mathcal{LCR}(x_0)$. The latter follows from integrating (10) on $[0, +\infty)$ and comparing the result with the definition of $V(x_0)$. Additionally, this shows that the optimal control $w(\cdot)$ is continuous and $w(t) \to 0$ as $t \to \infty$.

We note that the solutions to the closed-loop equation $\dot{x}(t) = Ax(t) + BF_{\mathcal{LCR}}(x(t))$ are unique, even though in general the right-hand side need not be Lipschitz continuous. This is a consequence of the uniqueness of optimal processes for $\mathcal{LCR}$. (Assuming that for some x_0 there exist two different optimal processes for (2) leads to a contradiction, through arguments very similar to those used in the proof of strict convexity of W in the proof of Theorem 3.1.) Finally, we add that the conditions in Corollary 3.2 are not only sufficient, but also necessary for optimality; this follows from Proposition 3.7 and Corollary 3.8 in [8].

IV. Stabilizing Feedback for Saturated Systems

We are now ready to state our main result.

Theorem 4.1: (Stabilizing Feedback for Saturated Systems): Consider the system

$$\dot{x}(t) = Ax(t) + B\sigma(u(t)) \tag{11}$$

under assumptions A1) and A2). Let X_0 be the set of all $x_0 \in I\!R^n$ for which there exists a piecewise continuous control $u : [0, +\infty) \mapsto I\!R^k$ such that the solution of (11) with $x(0) = x_0$ converges to 0. Let $Q \in I\!R^{n \times n}$ be any symmetric and positive–definite matrix.

Then, there exists a continuous mapping $F : X_0 \mapsto I\!R^k$ and a convex, positive–definite, and differentiable function $V : X_0 \mapsto I\!R$ such that, for any $x_0 \in X_0$, the solution $x(\cdot)$ to

$$\dot{x}(t) = Ax(t) + B\sigma(F(x(t))) \tag{12}$$

with $x(0) = x_0$ satisfies

$$\frac{d}{dt} V(x(t)) \leq -\frac{1}{2} x(t) \cdot Qx(t) \tag{13}$$

so that $x(t) \to 0$ as $t \to +\infty$.

As may be now expected, justification of Theorem 4.1 hinges upon translating the optimal feedback for $\mathcal{LCR}$ to a stabilizing feedback for the saturated system. First, we need to relate the set where the value function V is finite to X_0.

Lemma 4.2: $X_0 = \mathrm{dom}\, V$.

Proof: Fix $x_0 \in X_0$. There exists a piecewise continuous control such that the resulting solution of the saturated system, originating at x_0, converges to 0. As σ is continuously invertible around 0 and (A and B) is controllable, x_0 can be steered to 0 by a piecewise continuous control $u(\cdot)$ in finite time, say $T > 0$. Then, the control $w(t) = \sigma(u(t))$ on $[0, T]$ and $w(t) = 0$ for $r > T$ and the resulting trajectory of $\dot{x}(t) = Ax(t) + Bw(t)$ yields a finite cost in (6). Indeed, as $\mathrm{rge}\, \sigma \subset \mathrm{dom}\, r$ (by the discussion at the end of Section II), $r(w(\cdot))$ is piecewise continuous, $x(\cdot)$ is continuous, and both are 0 outside a compact interval. Thus, $V(x_0) < +\infty$ which means that $X_0 \subset \mathrm{dom}\, V$.

On the other hand, if $x_0 \in \mathrm{dom}\, V$, then the solution of

$$\dot{x}(t) = Ax(t) + B\nabla s(-B^* \nabla V(x(t))) \tag{14}$$

converges to 0; see Corollary 3.2. By construction, $\nabla s = \sigma$, so $u(t) = -B^* \nabla V(x(t))$ is the control required by the definition of X_0. Thus $\mathrm{dom}\, V \subset X_0$. ■

Proof: (of Theorem 4.1): Given (11) and a matrix Q as assumed, let V be the value function (6) with the convex function r given by (5) and s such that $s(0) = 0, \nabla s = \sigma$. Corollary 3.2 and the discussion following it show that for any point $x_0 \in \mathrm{dom}\, V$, so by Lemma 4.2, for any point $x_0 \in X_0$, the solution $x(\cdot)$ to (14) with $x(0) = x_0$ satisfies (13). As by construction $\nabla s = \sigma$, the mapping $F : X_0 \mapsto I\!R^k$ defined by

$$F(x) = -B^* \nabla V(x) \tag{15}$$

satisfies the conclusions of Theorem 4.1. Continuity was established in Theorem 3.1. ■

V. Comments and Extensions

We now make several comments regarding our main result, Theorem 4.1, and the constructions leading up to it.

i) The stabilizing feedback F for the saturated system is not the same as the optimal feedback for $\mathcal{LCR}$. However, by construction, trajectories of the saturated system with $u(t) = F(x(t))$ agree with optimal trajectories for the linear-convex regulator.

ii) The optimal feedback $F_{\mathcal{LCR}}$ for the linear-convex regulator is related to the stabilizing feedback F by $F_{\mathcal{LCR}}(x) = \sigma(F(x))$, and when σ is invertible, $F(x) = \sigma^{-1}(F_{\mathcal{LCR}}(x))$. When σ is not invertible, the

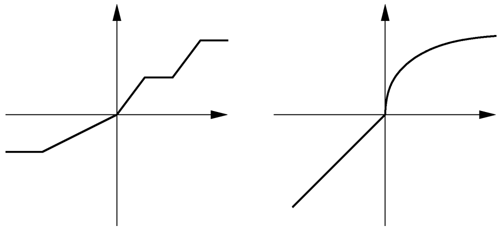

Fig. 3. Examples of admissible saturations

relationship $F(x) = \sigma^{-1}(F_{\mathcal{LCR}}(x))$ is not valid even in the set-valued sense, as then σ^{-1} is not single-valued.

iii) Our construction of F does not rely on considering σ^{-1}, not even on a subset of $\operatorname{rge}\sigma$ on which σ is invertible (as was, for example, the approach of [19]). Partly due to this, F (and not just $\sigma(F(\cdot))$) is continuous even when the saturation σ is not invertible on "large" subsets of $\operatorname{rge}\sigma$. Furthermore, we do not request that σ be Lipschitz, differentiable at 0, or bounded. Examples of saturations we allow are sketched in Fig. 3.

iv) When σ is an identity on some neighborhood of 0, which is the case for the standard saturation $\overline{\sigma}$, then ∇V, and consequently $F_{\mathcal{LCR}}$, is locally Lipschitz. This can be shown through strong (not just strict) convexity of the dual value function W used in the proof of Theorem 3.1 and a conjugacy relationship in Proposition 12.60 of [17], or by writing V in terms of a finite-horizon problem and using results of [7].

v) When σ is invertible, and generally, when there exists a continuous mapping η on $\operatorname{rge}\sigma$ such that $\sigma(\eta(w)) = w$ for $w \in \operatorname{rge}\sigma$ (as is the case, for example, for the standard saturation, but not for the saturation sketched above on the left), then other choices of the function r in (2) can be considered. Precisely, if s and r are functions as described by Fact 2.5, with the condition $\nabla s = \sigma$ replaced by $\operatorname{rge}\nabla s = \operatorname{rge}\sigma$, then the resulting $\mathcal{LCR}$ fits our framework and $\eta(\nabla s(-B^*\nabla V(x)))$ is a continuous stabilizing feedback for the saturated system.

vi) $\mathcal{LCR}$ is a convex optimization problem. From the numerical computation viewpoint, such problems have many advantages over their nonconvex counterparts; see [3]. A seemingly more obvious choice of an auxiliary problem, with a convex or even quadratic cost and the dynamics provided by the saturated system, does not lead to a convex problem and is unlikely to yield a regular feedback or even a regular value function (which need not be convex); time-optimal control has similar drawbacks. Convexity also yields a global description of ∇V in terms of a Hamiltonian dynamical system associated with $\mathcal{LCR}$ [8, Prop. 3.7]. This suggests a numerical method for computing the feedback which does not require the calculation of optimal values of $\mathcal{LCR}$. For details and numerical examples for the standard saturation case, see [11].

vii) An approach different from ours, but with some convex structure, would be to find a Lyapunov function $\tilde{V}$ for the saturated system as a solution to the Hamilton–Jacobi inequality

$$\inf_u \nabla\tilde{V}(x)\cdot(Ax + B\sigma(u)) \leq -\frac{1}{2}x\cdot Qx$$

which translates to $\widetilde{H}(x, -\nabla\tilde{V}(x)) \geq 0$ for $\widetilde{H}(x,p) = p\cdot Ax - (1/2)x\cdot Qx + \sup_{w\in\operatorname{rge}\sigma} p\cdot Bw \geq 0$. This Hamiltonian is concave in x, convex in p, similarly to (7) corresponding to $\mathcal{LCR}$. However, it is not finite everywhere unless σ is bounded. Also, it is not clear if solutions are smooth (since $\widetilde{H}$ is not strictly convex in p anywhere). Furthermore, recovering the stabilizing feedback for the saturated system would need to involve σ^{-1} in some way.

viii) The componentwise structure of σ as in assumption A2) is not necessary for our main result, as long as the conclusions of Fact 2.5 remain valid. Corollary 5.1 makes this precise, and Example 5.2 shows a saturation function without the componentwise structure.

Corollary 5.1: The conclusions of Theorem 4.1 hold for any σ such that functions s, r as described in Fact 2.5 exist.

This is true since the statements in Section III and the proof of Theorem 3.1 only invoke Fact 2.5. Lemma 4.2 requires that σ^{-1} be continuous around 0. However, $\sigma^{-1} = \nabla r$ there, as differentiability of r around 0 is implied by strict convexity of s around 0, and ∇r is continuous, as the gradient of any differentiable convex function is. We now give an example of σ which satisfies the assumption of Corollary 5.1, but does not have the componentwise structure. For such saturation functions, calculating s and r is less simple, and makes use of calculus rules for conjugate convex functions; see [14] or [17, Ch. 11].

Example 5.2: (Projection Onto a Convex Set): The standard saturation $\overline{\sigma}$ on $\mathbb{R}$ can be thought of as a projection of u onto $[-1, 1]$—for any u, $\overline{\sigma}(u)$ is the point in $[-1, 1]$ closest to u. In general, if C is a nonempty, closed, and convex set in $\mathbb{R}^k$, the projection onto it, denoted P_C, is a well-defined continuous mapping, with Lipschitz constant 1; see, for example, [17, Cor. 12.20]. Then also $P_C = \nabla s$ for a convex function r given by

$$s(u) = \inf_{z\in\mathbb{R}^k}\left\{\sup_{c\in C} z\cdot c + \frac{1}{2}\|u - z\|^2\right\}. \tag{16}$$

This formula becomes much clearer for particular choices of C. For example, consider C to be the unit ball in $\mathbb{R}^k$. The map P_C is an identity for points in C, and a radial projection onto the unit sphere for points outside it (that is, $P_C(u) = u/\|u\|$). Then, $\sup_{c\in C} u\cdot c = \|u\|$, and $s(u) = (1/2)\|u\|^2$ for $\|u\| \leq 1$, $\|u\| - 1/2$ for $\|u\| > 1$. When $k = 1$, this reduces to the function $\overline{r}$ corresponding to standard saturation. Note also that this s is strictly convex around 0. In fact, this property is present whenever 0 is in the interior of C.

The conjugate r of (16) can be found through Example 11.4 and Theorem 11.23 in [17]

$$r(w) = \begin{cases} \frac{1}{2}\|w\|^2, & \text{for } w \in C \\ +\infty, & \text{for } w \notin C. \end{cases}$$

The standard saturation is to a special instance of the aforementioned formula.

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References

[1] V. Barbu, "Convex control problems and Hamiltonian systems on infinite intervals," *SIAM J. Control Optim.*, vol. 16, no. 6, pp. 895–911, 1978.

[2] L. M. Benveniste and J. A. Scheinkman, "On the differentiability of the value function in dynamic models of economics," *Econometrica*, vol. 47, pp. 727–732, 1979.

[3] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge, U.K.: Cambridge Univ. Press, 2004.

[4] D. A. Carlson, A. Haurie, and A. Leizarowitz, *Infinite Horizon Optimal Control: Deterministic and Stochastic Systems*. New York: Springer-Verlag, 1991.

[5] F. H. Clarke, Y. S. Ledyaev, L. Rifford, and R. J. Stern, "Feedback stabilization and Lyapunov functions," *SIAM J. Control Optim.*, vol. 39, no. 1, pp. 25–48, 2000.

[6] F. H. Clarke, Y. S. Ledyaev, E. D. Sontag, and A. I. Subbotin, "Asymptotic controllability implies feedback stabilization," *IEEE Trans Autom. Control*, vol. 42, no. 10, pp. 1394–1407, Oct. 1997.
[7] R. Goebel, "Convex optimal control problems with smooth Hamiltonians," *SIAM J. Control Optim.*, to be published.
[8] ——, "Duality and uniqueness of convex solutions to stationary Hamilton-Jacobi equations," *Trans. Amer. Math. Soc.*, vol. 357, pp. 2187–2203, 2005.
[9] ——, "Hamiltonian dynamical systems for convex problems of optimal control: implications for the value function," in *Proc. 41st IEEE Conf. Decision and Control*, Las Vegas, NV, 2002, pp. 728–732.
[10] ——, "Stationary Hamilton-Jacobi equations for convex control problems—uniqueness and duality of solutions," in *Optimal Control, Stabilization, and Nonsmooth Analysis*, ser. Lecture Notes in Control and Information Sciences. Heidelberg, Germany: Springer-Verlag, 2004.
[11] R. Goebel and M. Subbotin, "Continuous time constrained linear quadratic regulator—convex duality approach," in *Proc. 24th Amer. Control Conf.*, 2005, to be published.
[12] M. L. Gota and L. Montrucchio, "On Lipschitz continuity of policy functions in continuous-time optimal growth models," *Econom. Theory*, vol. 14, pp. 479–488, 1999.
[13] T. Hu and Z. Lin, *Control Systems With Actuator Saturation*. Boston, MA: Birkhäuser, 2001.
[14] R. T. Rockafellar, *Convex Analysis*. Princeton, NJ: Princeton Univ. Press, 1970.
[15] ——, "Saddle points of Hamiltonian systems in convex problems of Lagrange," *J. Optim. Theory Appl.*, vol. 12, no. 4, 1973.
[16] ——, *Conjugate Duality and Optimization*. Philadelphia, PA: SIAM, 1974.
[17] R. T. Rockafellar and R. J.-B. Wets, *Variational Analysis*. New York: Springer-Verlag, 1998.
[18] A. Saberi, Z. Lin, and A. R. Teel, "Control of linear systems with saturating actuators," *IEEE Trans. Autom. Control*, vol. 41, no. 3, pp. 368–378, Mar. 1996.
[19] E. D. Sontag and H. J. Sussman, "Nonlinear output feedback design for linear systems with saturating controls," in *Proc. 29 IEEE Conf. Decision Control*, Honolulu, HI, 1990, pp. 3414–3416.
[20] E. D. Sontag, H. J. Sussmann, and Y. D. Yang, "A general result on the stabilization of linear systems using bounded controls," *IEEE Trans. Autom. Control*, vol. 39, no. 12, pp. 2411–2424, Dec. 1994.
[21] A. A. Stoorvogel, A. Saberi, and G. Shi, "Properties of recoverable region and semi-global stabilization in recoverable region for linear systems subject to constraints," *Autmomatica*, vol. 40, no. 9, pp. 1481–1494, 2004.

Consensus Seeking in Multiagent Systems Under Dynamically Changing Interaction Topologies

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***Abstract*—This note considers the problem of information consensus among multiple agents in the presence of limited and unreliable information exchange with dynamically changing interaction topologies. Both discrete and continuous update schemes are proposed for information consensus. This note shows that information consensus under dynamically changing interaction topologies can be achieved asymptotically if the union of the directed interaction graphs have a spanning tree frequently enough as the system evolves.**

***Index Terms*—Cooperative control, graph theory, information consensus, multiagent systems, switched systems.**

I. Introduction

The study of information flow and interaction among multiple agents in a group plays an important role in understanding the coordinated movements of these agents. As a result, a critical problem for coordinated control is to design appropriate protocols and algorithms such that the group of agents can reach consensus on the shared information in the presence of limited and unreliable information exchange and dynamically changing interaction topologies. Consensus problems have recently been addressed in [1]–[7], to name a few. In this note, we extend the results of [2] to the case of directed graphs and present conditions for consensus of information under dynamically changing interaction topologies.

In contrast to [2], directed graphs will be used to represent the interaction (information exchange) topology between agents, where information can be exchanged via communication or direct sensing. A preliminary result for information consensus is presented in [8], where a linear update scheme is proposed for directed graphs. However, the analysis in [8] was not able to utilize all available communication links. A solution to this issue was presented in [4] for time-invariant communication topologies. Information consensus for dynamically evolving information was addressed in [9] in the context of spacecraft formation flying where the exchanged information is the configuration of the virtual structure associated with the (dynamically evolving) formation.

In many applications, the interaction topology between agents may change dynamically. For example, communication links between agents may be unreliable due to disturbances and/or subject to communication range limitations. If information is being exchanged by direct sensing, the locally visible neighbors of a vehicle will likely change over time. In [2], a theoretical explanation is provided for the observed behavior of the Vicsek model [10]. Possible changes over time in each agent's nearest neighbors is explicitly taken into account, and is an example of information consensus under dynamically changing interaction topologies. Furthermore, it is shown in [2] that consensus can be achieved if the union of the interaction graphs for the team are connected frequently enough as the system evolves.

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