Smooth patchy control Lyapunov functions ⋆

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Abstract

A smooth patchy control Lyapunov function for a nonlinear system consists of an ordered family of smooth local control Lyapunov functions, whose open domains form a locally finite cover of the state space of the system, and which satisfy certain further increase or decrease conditions. We prove that such a control Lyapunov function exists for any asymptotically controllable nonlinear system. We also show a construction, based on such a control Lyapunov function, of a stabilizing hybrid feedback that is robust to measurement noise.

Key words: Hybrid feedback, control Lyapunov function, nonlinear system, asymptotic controllability, asymptotic stability.

1 Introduction

When a nonlinear control system has a compact set that is robustly asymptotically stabilizable by locally bounded state feedback, the control system admits a smooth "control Lyapunov function" (CLF). See Clarke et al. (1998) or Ledyaev and Sontag (1999). Given a continuously differentiable CLF for a nonlinear control system that is affine in its controls, formulas exist for continuous (and thus robust) feedback stabilizers. See, e.g., Sontag (1989). Similar results are available for systems with restricted controls Lin and Sontag (1991), Lin and Sontag (1995). The first results on stabilization using a CLF, which involve relaxed controls for nonaffine systems, can be found in Artstein (1983).

While every asymptotically controllable nonlinear control system admits a locally Lipschitz, semiconcave CLF (Sontag (1983), Rifford (2002)), not every such system admits a continuously differentiable CLF, even when the system is affine in the control variable. Systems that do not admit a continuously differentiable CLF include systems that fail Brockett's condition, see Ryan (1994).

In the absence of a continuously differentiable CLF, discontinuous feedback stabilizers have been developed. For example, see Clarke et al. (1997) and Ancona and Bressan (1999). Typically, these feedbacks produce asymptotic stability with some robustness to additive disturbances but no robustness to measurement noise. To guarantee the latter robustness, the feedback of Clarke et al. (1997) can be implemented using sample and hold, and then the robustness margins decrease to zero as the sampling period decreases to zero; see Sontag (1999), Clarke et al. (2000). As shown in Ancona and Bressan (2003), the patchy feedbacks of Ancona and Bressan (1999) have some robustness, for the purposes of semiglobal practical stabilization, to measurement noise with small total variation, but not to just small, locally bounded noise.

Sample and hold implementation of state feedback is a special type of hybrid feedback: at certain time instants, components of the state (a timer and the control value) change discontinuously (jump). In Prieur (2003), Prieur (2005), and Prieur et al. (2007) a different line of "hybridization" was followed, with the goal of robustness to measurement noise and additive disturbances. The patchy feedbacks of Ancona and Bressan (1999) were implemented there using hysteresis, an alternative kind of hybrid feedback control, the power of which has been already recognized for example in Hespanha and Morse (1999). The closed-loop system resulting from the feedback of Prieur et al. (2007) essentially fits the form of the general class of hybrid systems studied in Goebel and Teel (2006). For the latter, general converse Lyapunov results were obtained by Cai et al. (2007), Cai et al.

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(2008). These converse results suggest that the closed-loop system in Prieur et al. (2007) admits a smooth "patchy" Lyapunov function. This is the point of departure for the present paper which has a fourfold purpose:

(1) to introduce the notion of a smooth patchy control Lyapunov function ("smooth patchy CLF", "smooth PCLF", or just "PCLF" for short);
(2) to show that every asymptotically controllable nonlinear control system admits a smooth PCLF;
(3) to show how, given a smooth PCLF, a stabilizing hybrid feedback can be constructed (that is patchwise continuous if the nonlinear system meets some mild assumptions on the admissible controls);
(4) to highlight that the Artstein's circles and the Brockett's integrator examples admit a smooth patchy CLF with a finite number of patches, while they do not admit a smooth CLF.

We achieve (2) with the help of a stabilizing patchy feedback, a kind of discontinuous (and not robust to measurement noise) state feedback, guaranteed to exist for asymptotically controllable nonlinear control systems thanks to results of Ancona and Bressan (1999). We add that our hybrid feedback mentioned in (3) is robust to measurement error, by the results in Prieur et al. (2007).

2 Key idea – an example

The main idea behind a patchy control Lyapunov function is to cover the state space of a nonlinear system by a family of "local" control Lyapunov functions.

Example 2.1 (Artstein's circles) Consider the nonlinear control system on $\mathbb{R}^2$, known as Artstein's cicles:

$$\dot{x}_1 = (x_1^2 - x_2^2)u, \quad \dot{x}_2 = 2x_1x_2u. \tag{1}$$

(See Artstein (1983).) Depending on the initial point and the chosen control, solutions to (1) move along circles centered on the x_2-axis and tangent to the x_1-axis, or along the x_1-axis, or stay at the origin. For initial points on the circles just mentioned that are above the x_1-axis, choosing $u > 0$ results in counterclockwise motion, $u < 0$ results in clockwise motion. (The motion below the x_1-axis is symmetric with respect to that axis.)

For (1), the origin cannot be stabilized by continuous feedback (see Artstein (1983)) nor robustly stabilized by locally bounded feedback (see Ledyaev and Sontag (1999)). In Example 4.7 we will show that it can be robustly stabilized via hybrid feedback (and will explicitly show such a feedback). Note that there is no smooth control Lyapunov function for (1). This is easy to see by noting that any smooth function, positive away from the origin and 0 there, has a maximum relative to each circle centered on the x_2-axis and tangent to the x_1-axis.

However, $\mathbb{R}^2 \setminus \{0\}$ can be covered by two open sets and on each of them, there exists a smooth (local) control Lyapunov function. This captures the key idea of the concept of a smooth PCLF. To see a particular example, consider the open set (written in polar coordinates):

$$O_1 = \{x = (r, \theta) \,|\, r > 0,\ -3\pi/4 < \theta < 3\pi/4\}$$

and let $V_1 : O_1 \to (0, \infty)$ be such that $V_1(x)$ is the distance from x to 0, measured along the part of circle centered on the x_2-axis and tangent to the x_1-axis that is contained in O_1 (for points on the x_1-axis, this reduces to $V_1(x) = |x_1|$). Such a function is a (smooth) control Lyapunov function on O_1, this can be seen by choosing $u = -1$. Let $O_2 = -O_1$, i.e., $x \in O_2$ if and only if $-x \in O_1$, and $V_2 : O_2 \to (0, \infty)$ be given by $V_2(x) = V_1(-x)/3$. This V_2 is a (smooth) control Lyapunov function on O_2, as verified by $u = 1$. Finally, for $x \in O_1 \cap O_2$, we have $V_2(x) < V_1(x)$.

In the example above, it was possible to cover $\mathbb{R}^2 \setminus \{0\}$ with finitely many, in fact two, open sets (patches) and furthermore, to pick the local Lyapunov functions so that the resulting PCLF is strict: $V_2(x) < V_1(x)$ for all $x \in O_1 \cap O_2$. We make the following observations:

- For a general asymptotically controllable nonlinear control system, the existence of a smooth PCLF can be shown only if an infinite number of patches is allowed. (See Section 5.1, in particular Theorem 5.2.)
- For some asymptotically controllable nonlinear control systems, finding a PCLF with a finite number of patches may be far easier that finding such a PCLF that is also strict. This is the case, for example, for the Brockett integrator, see Section 6.
- Strictness is not necessary in order to construct a robustly stabilizing hybrid feedback from a PCLF with a finite number of patches. (See Section 4.2 and, in particular, Theorem 4.4.) In the more technical case of a PCLF with infinitely many patches, strictness or some other condition on the allignment of patches and the corresponding local Lyapunov functions appears necessary to obtain a stabilizing hybrid feedback.

While Example 2.1 captures the main idea, several details will need to be included in the formal definition of a PCLF. First, we introduce some background material.

3 Preliminaries

Throughout the paper, $\widetilde{O} \subset \mathbb{R}^n$ is an open set and $\mathcal{A} \subset \widetilde{O}$ is compact. We will be interested in hybrid feedback stabilization for the nonlinear system

$$\dot{x}(t) = f(x(t), u(t)), \quad u(t) \in U, \quad \text{for all } t \geq 0, \tag{2}$$

where $U \subset \mathbb{R}^k$ is a set and $f : \widetilde{O} \times U \to \mathbb{R}^n$ is a (nonlinear in general) continuous mapping. The state space for the continuous variable of the hybrid feedback will not necessarily cover $\widetilde{O}$, but will be an open set $\mathcal{O}$ such that $O \subset \mathcal{O} \subset \widetilde{O}$, where $O = \widetilde{O} \setminus \mathcal{A}$.

Definition 3.1 *A hybrid feedback consists of*

- *a totally ordered countable set Q*
- *for each $q \in Q$,*
  - *sets $C_q \subset \mathcal{O}$ and $D_q \subset \mathcal{O}$,*
  - *a function $k_q : C_q \to U$,*
  - *a set-valued mapping $G_q : D_q \rightrightarrows Q$.* [1]

In closed loop with the nonlinear system (2), a hybrid feedback as in Definition 3.1 leads to a hybrid system

$$\begin{aligned} \dot{x} &= f(x, k_q(x)) & x &\in C_q, \\ q^+ &\in G_q(x) & x &\in D_q. \end{aligned} \tag{3}$$

During flow, x evolves according to the differential equation $\dot{x} = f(x, k_q(x))$, q remains constant, and the constraint $x \in C_q$ is satisfied. During jumps, q evolves according to the difference inclusion $q^+ \in G_q(x)$, x remains constant, and before a jump, the constraint $x \in D_q$ is satisfied. The state space for (3) will then be $\mathcal{O} \times Q$.

We now formally define solutions to (3), following Goebel and Teel (2006). A subset $S \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *hybrid time domain* if S is a union of a finite or infinite sequence of intervals $[t_j, t_{j+1}] \times \{j\}$, with the last interval, if it exists, possibly of the form $[t_j, T)$ with T finite or $T = +\infty$. A *solution to the hybrid system* (3) is a function $x : S \to \mathcal{O}$, where S is a nonempty hybrid time domain, with $x(t,j)$ locally absolutely continuous in t for a fixed j and a function $q : S \to Q$ meeting the following conditions: $x(0,0) \in C_{q(0,0)} \cup D_{q(0,0)}$ and

(S1) For all $j \in \mathbb{N}$ and almost all t such that $(t,j) \in S$,

$$\dot{x}(t,j) = f(x(t,j), k_{q(t,j)}(x(t,j))), \quad x(t,j) \in C_{q(t,j)}.$$

(S2) For all $(t,j) \in S$ such that $(t, j+1) \in S$,

$$q(t,j+1) \in G_{q(t,j)}(x(t,j)), \qquad x(t,j) \in D_{q(t,j)}.$$

Given a solution (x,q) to (3) we refer to its domain by $\operatorname{dom}(x,q)$. A solution (x,q) to (3) is *maximal* if it can not be extended, that is, if there does not exist another solution (x',q') such that $\operatorname{dom}(x,q) \subsetneq \operatorname{dom}(x',q')$ and $(x,q)(t,j) = (x',q')(t,j)$ for all $(t,j) \in \operatorname{dom}(x,q)$. In what follows, we will write $\sup_t(S)$ for the supremum of all t such that $(t,j) \in S$ for some j, and $\operatorname{dist}_{\mathcal{A}}(x)$ for the distance of the point x from the set $\mathcal{A}$.

[1] The double arrow notation is used to distinguish a set-valued mapping from a function.

Definition 3.2 *The set $\mathcal{A}$ is stable for the hybrid system (3) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that any solution (x,q) with $\operatorname{dist}_{\mathcal{A}}(x(0,0)) \leq \delta$ satisfies $\operatorname{dist}_{\mathcal{A}}(x(t,j)) \leq \varepsilon$ for all $(t,j) \in \operatorname{dom}(x,q)$. The set $\mathcal{A}$ is globally attractive for (3) if for any maximal solution (x,q) to (3) we have $\operatorname{dist}_{\mathcal{A}}(x(t,j)) \to 0$ as $t \to \sup_t(\operatorname{dom}(x,q))$. The set $\mathcal{A}$ is globally asymptotically stable for (3) if it is both stable and globally attractive.*

This concept of asymptotic stability refers only to the behavior of the "continuous" part of solutions. We are allowing for solutions approaching $\mathcal{A}$ in finite (hybrid) time. Below, an admissible perturbation radius is a continuous function $\rho : \widetilde{O} \to [0, \infty)$ such that $\rho(x) = 0$ if and only if $x \in \mathcal{A}$ and $x + \rho(x)\mathbb{B} \subset O$ for all $x \in O$. [2]

Definition 3.3 *A hybrid feedback renders $\mathcal{A}$ asymptotically stable, robustly with respect to measurement noise and external disturbances, if there exists an admissible perturbation radius ρ such that the set $\mathcal{A}$ is asymptotically stable, with the basin of attraction equal to O, for the system $\mathcal{H}^\rho$:*

$$\begin{aligned} \dot{x} &\in F_q^\rho(x) & x &\in C_q^\rho, \\ q^+ &\in G_q^\rho(x) & x &\in D_q^\rho, \end{aligned} \tag{4}$$

with the data

$$\begin{aligned} F_q^\rho(x) &:= \operatorname{con} f(x, k_q((x + \rho(x)\mathbb{B}) \cap C_q) + \rho(x)\mathbb{B}, \\ G_q^\rho(x) &:= G_q((x + \rho(x)\mathbb{B}) \cap D_q), \\ C_q^\rho &:= \{x \in \mathcal{O} \mid (x + \rho(x)\mathbb{B}) \cap C_q \neq \emptyset\}, \\ D_q^\rho &:= \{x \in \mathcal{O} \mid (x + \rho(x)\mathbb{B}) \cap D_q \neq \emptyset\}. \end{aligned} \tag{5}$$

In (5), $\operatorname{con} f(x, k_q((x + \rho(x)\mathbb{B}) \cap C_q))$ is the closed convex hull of the set $\bigcup_{\xi \in (x+\rho(x)\mathbb{B}) \cap C_q} f(x, k_q(\xi))$. Solutions to (4) are understood similarly to those to (3).

4 Finite number of patches and a sufficient condition for robust feedback stabilization

4.1 PCLF with finite number of patches

Below, given a set Ω, its boundary is denoted by $\partial\Omega$. By a proper indicator of $\mathcal{A}$ with respect to $\widetilde{O}$ we will understand a function $\omega : \widetilde{O} \to \mathbb{R}_{\geq 0}$ that is continuous, positive definite with respect to $\mathcal{A}$, and that approaches ∞ if its argument approaches the boundary of $\widetilde{O}$ or the norm of its argument approaches ∞.

[2] Here and in what follows, $\mathbb{B}$ is the closed unit ball in $\mathbb{R}^n$.

Definition 4.1 *A smooth patchy control Lyapunov function, PCLF, (with a finite number of patches) for (2) with the attractor $\mathcal{A}$ consists of a set Q and a collection of functions V_q and sets Ω_q, Ω'_q for each $q \in Q$, such that*

(i) $Q \subset \mathbb{Z}$ *is a finite set;*

(ii) $\{\Omega_q\}_{q\in Q}$ *and* $\{\Omega'_q\}_{q\in Q}$ *are families of nonempty open subsets of* $\widetilde{O}$ *such that*

$$O \subset \mathcal{O} \subset \widetilde{O}, \quad \text{where } \mathcal{O} := \bigcup_{q\in Q} \Omega_q = \bigcup_{q\in Q} \Omega'_q,$$

and for all $q \in Q$, the unit (outward) normal vector to $\partial\Omega_q$ is continuous on $\left(\partial\Omega_q \setminus \bigcup_{r>q} \Omega'_r\right) \cap \mathcal{O}$, and

$$\overline{\Omega'_q} \cap \mathcal{O} \subset \Omega_q;$$

(iii) for each q, V_q is a smooth function defined on a (relative to $\mathcal{O}$) neighborhood of $\overline{\Omega_q \setminus \bigcup_{r>q} \Omega'_r}$;

and the following conditions are met: there exist a continuous function $\alpha : (0,\infty) \to (0,\infty)$, class-$\mathcal{K}_\infty$ functions $\underline{\gamma}$, $\overline{\gamma}$, and a function ω which is a proper indicator of $\mathcal{A}$ with respect to $\widetilde{O}$ such that:

(iv) for all $q \in Q$, all $x \in \Omega_q \setminus \bigcup_{r>q} \Omega'_r$,

$$\underline{\gamma}(\omega(x)) \le V_q(x) \le \overline{\gamma}(\omega(x)) \;;$$

(v) for all $q \in Q$, all $x \in \Omega_q \setminus \bigcup_{r>q} \Omega'_r$, there exists $u_{q,x} \in U$ such that

$$\nabla V_q(x) \cdot f(x, u_{q,x}) \le -\alpha(\omega(x));$$

(vi) for all $q \in Q$, all $x \in \left(\partial\Omega_q \setminus \bigcup_{r>q} \Omega'_r\right) \cap \mathcal{O}$, the $u_{q,x}$ of (v) can be chosen such that

$$n_q(x) \cdot f(x, u_{q,x}) \le -\alpha(\omega(x)),$$

where $n_q(x)$ is the unit (outward) normal vector to $\overline{\Omega_q}$ at x.

Next we add an extra condition to the definition of a smooth PCLF, arriving at "strict" and "almost strict" PCLFs. It is important to note that neither strict nor almost strict PCLFs will be needed or used to construct stabilizing feedbacks in the case of a finite number of patches. Their definitions are provided here for the purposes of comparison and because these concept will play a role later in the case of necessity of PCLFs and in the case of an infinite number of patches.

Definition 4.2 *A smooth patchy control Lyapunov function is almost strict if the following condition holds:*

(vii) for all $q, r \in Q$, $r > q$, all $x \in \Omega_q \cap \partial\Omega'_r$, $V_r(x) \le V_q(x)$.

If this inequality is strict, the smooth patchy control Lyapunov function is strict.

Example 4.3 (Artstein's circles, revisited) We now return to the system (1) and display a strict smooth PCLF for it. Let O_q, V_q, $q = 1, 2$ be as in Example 2.1. Pick any two angles $\beta' < \beta$ in $(\pi/2, 3\pi/4)$ and let

$$\Omega'_1 = \{x = (r,\theta) \,|\, r > 0,\ -\beta' < \theta < \beta'\},$$

$$\Omega_1 = \{x = (r,\theta) \,|\, r > 0,\ -\beta < \theta < \beta\},$$

while $\Omega'_2 = -\Omega'_1$, $\Omega_2 = -\Omega_1$. The sets O_1, Ω_1 and Ω'_1 are sketched in Figure 1. These sets and functions form

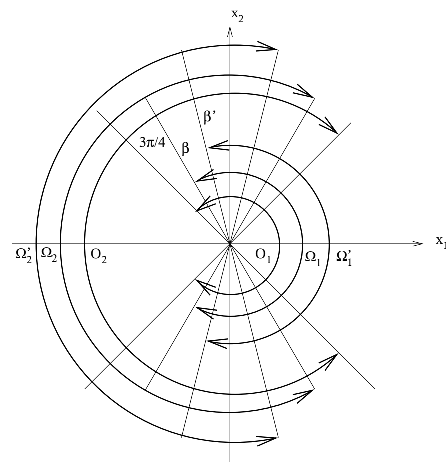

Fig. 1. Sketch of the sets O_1, Ω_1, and Ω'_1

a smooth PCLF. The set $Q = \{1, 2\}$ is ordered by $2 > 1$. The families $\{\Omega_1, \Omega_2\}$, $\{\Omega'_1, \Omega'_2\}$ consist of nonempty and open sets and form a locally finite cover of $\mathcal{O} = \Omega_1 \cup \Omega_2 = \Omega'_1 \cup \Omega'_2 = \mathbb{R}^2 \setminus \{0\}$. We have $\left(\partial\Omega_1 \setminus \bigcup_{r>1} \Omega'_r\right) \cap \mathcal{O} = (\partial\Omega_1 \setminus \Omega'_2) \cap \mathcal{O} = \emptyset$. Furthermore, $\left(\partial\Omega_2 \setminus \bigcup_{r>2} \Omega'_r\right) \cap \mathcal{O} = \partial\Omega_2 \cap \mathcal{O}$ consists of two half lines that do not contain their endpoints, and the (outward) unit normal vector to $\partial\Omega_2$ is constant (so continuous) relative to each of these lines. Obviously, $\overline{\Omega'_q} \cap \mathcal{O} \subset \Omega_q$, $q = 1, 2$. Each V_q is smooth on O_q, a neighborhood of $\overline{\Omega_q}$ relative to $\mathcal{O}$. Verifying (iv) of Definition 4.1 is possible via $\omega(x) = \|x\|$ (Euclidean norm) and noting that, for $q = 1$ and $x \in O_1$ and for $q = 2$ and $x \in O_2$

$$\|x\| \le V_q(x) \le \frac{3\pi}{2\sqrt{2}} \|x\|.$$

Setting $u_{1,x} = -1$, $u_{2,x} = 1$, one gets

$$\nabla V_1(x) \cdot f(x, u_{1,x}) = -\|f(x, u_{1,x})\| = -\|x\|^2$$

and similarly, $\nabla V_2(x) \cdot f(x, u_{2,x}) = -\|x\|^2/3$. This verifies (v). Regarding (vi), there is nothing to check for

$q = 1$. For $q = 2$, we have for each $x \in \partial\Omega_2 \cap \mathcal{O}$ that $n_q(x)$ and $f(x,1)$ have opposite directions, and

$$n_2(x) \cdot f(x,1) = -\|f(x,1)\| = -\|x\|^2.$$

Finally, for $x \in O_1 \cap O_2$, and so for $x \in \Omega_1 \cap \partial\Omega_2'$, $V_2(x) < V_1(x)$. This verifies (vii). Note that if V_2 was defined in Example 2.1 by $V_2(x) = V_1(-x)$, the resulting object would be a smooth PCLF, but not a strict one.

4.2 Robust stabilizing feedback

With a patchy control Lyapunov function, and under a convexity assumption on the nonlinear system, we can design a hybrid feedback on $\mathcal{O}$ for (2) that renders $\mathcal{A}$ robustly globally asymptotically stable.

Theorem 4.4 *Suppose that*

- *there exists a smooth patchy control Lyapunov function (with finitely many patches) for (2) with attractor $\mathcal{A}$;*
- *for any $v \in \mathbb{R}^n$, $c \in \mathbb{R}$, the set $\{u \in U \mid v \cdot f(x,u) \leq c\}$ is convex.*

Then, there exists a hybrid feedback on $\mathcal{O}$ for (2) such that, for each $q \in Q$, the mapping k_q is continuous and $C_q \cup D_q = \mathcal{O}$, which renders $\mathcal{A}$ globally asymptotically stable, robustly with respect to measurement noise and external disturbances.

The convexity assumption is satisfied if the system (2) is affine with respect to the control variable and the set U is convex. It is included to ensure that, for each $q \in Q$, the local feedback k_q, constructed from the local Lyapunov function V_q on a particular subset of Ω_q, is continuous. This continuity, and the property that $C_q \cup D_q = \mathcal{O}$ implies in particular that for each initial point in $\mathcal{O} \times Q$ there exists a nontrivial solution to (3) and furthermore, that each maximal solution is either complete or leaves any compact subset of $\mathcal{O} \times Q$ in finite (hybrid) time; see Proposition 2.4 in Goebel and Teel (2006).

For systems failing the convexity assumption, the work in Ledyaev and Sontag (1999) has established the existence of a robust, stabilizing, but possibly discontinuous feedback when a smooth, classical CLF exists. See also Artstein (1983). For PCLF-based feedback synthesis in the absence of the convexity assumption, the ideas in Ledyaev and Sontag (1999) can be applied to each patch of a PCLF to construct a hybrid feedback with possibly discontinuous k_q's. We do not pursue this here.

The proof of Theorem 4.4 has three steps. First, for each $q \in Q$, we construct a "local" continuous feedback on each patch. (The proof of Lemma 4.5 is in the appendix.)

Lemma 4.5 *Under the assumptions of Theorem 4.4, for each $q \in Q$ there exists a continuous mapping*

$$k_q : \overline{\Omega_q \setminus \bigcup_{r>q} \Omega_r'} \cap \mathcal{O} \to U$$

such that

(a) for all $x \in \overline{\Omega_q \setminus \bigcup_{r>q} \Omega_r'} \cap \mathcal{O}$,

$$\nabla V_q(x) \cdot f(x, k_q(x)) \leq -\alpha(\omega(x))/2;$$

(b) for all $x \in \left(\partial\Omega_q \setminus \bigcup_{r>q} \Omega_r'\right) \cap \mathcal{O}$,

$$n_q(x) \cdot f(x, k_q(x)) \leq -\alpha(\omega(x))/2.$$

Second, we explicitly define the remaining data that is needed to turn the collection of (continuous time) feedbacks k_q into a hybrid feedback:

$$\begin{aligned} C_q &= \overline{\Omega_q \setminus \textstyle\bigcup_{r>q} \Omega_r'} \cap \mathcal{O} \\ D_q &= \textstyle\bigcup_{r>q} \left(\overline{\Omega_r'} \cap \mathcal{O}\right) \cup (\mathcal{O} \setminus \Omega_q) \\ G_q(x) &= \begin{cases} \{r \in Q \mid x \in \overline{\Omega_r'} \cap \mathcal{O}, r > q\} \\ \qquad \text{if } x \in \left(\bigcup_{r>q} \overline{\Omega_r'}\right) \cap \Omega_q \\ \{r \in Q \mid x \in \overline{\Omega_r'} \cap \mathcal{O}\} \\ \qquad \text{if } x \in \mathcal{O} \setminus \Omega_q \end{cases} \end{aligned} \tag{6}$$

The "switching logic" for the hybrid system described by such data and by the differential equations $\dot{x} = f(x, k_q(x))$ is essentially as follows. When a solution (x,q) is flowing with $x \in C_q$, a jump from q to r can only occur if $r > q$ and $x \in \overline{\Omega_r'} \cap \mathcal{O}$. Such jumps leads to $V_q(x)$ nonincreasing in the case of an almost strict PCLF, and decreasing in the strict case. A different kind of a jump, to any r with $x \in \overline{\Omega_r'} \cap \mathcal{O}$, may also be needed before any flow occurs, in the case of initialization of the stabilization process with a "wrong" value of q, i.e., q such that $x \in \mathcal{O} \setminus \Omega_q$. See also Example 4.7.

In the third step, we use the collection of local control Lyapunov functions V_q to show that the constructed feedback is stabilizing. The key property behind stability is that the constructed feedback guarantees each solution experiences a finite number of jumps. This simplifies the analysis considerably compared to the case where an infinite number of jumps may occur. (Tools for stability analysis in this latter situation exist in the literature, even for the case where the variable q takes values in a compact, not necessarily finite, set. For example, see the work on "multiple Lyapunov functions" Branicky (1998); DeCarlo et al. (2000).) For the case of a PCLF with an infinite number of patches, which is discussed later, solutions can experience an infinite number of jumps, although there is monotonicity in the evolution of the variable q. As Remark 4.6 will show, if the PCLF is strict, the functions V_q are used much like what is done with a standard control Lyapunov function. Robustness of stability is finally deduced from a generic robustness result of Prieur et al. (2007). The details are as follows.

Proof. (of Theorem 4.4) Given a PCLF, consider the hybrid feedback on $\mathcal{O} \times Q$ given by (6), and let u_q be any functions as in Lemma 4.5.

First, we note that the hybrid system (3) with the feedback (6) has favorable semicontinuity and closedness properties: for each $q \in Q$, sets C_q, D_q are relatively closed in $\mathcal{O}$, the function $x \mapsto f(x, u_q(x))$ is continuous (and hence it is outer semicontinuous, locally bounded, and has convex and nonempty values) on C_q, while the set-valued mapping G_q is outer semicontinuous, locally bounded, and has nonempty values that are subsets of Q; see (Prieur et al., 2007, Lemma 3.5). These properties guarantee that various results obtained in Goebel and Teel (2006) and Prieur et al. (2007) regarding sequential compactness of solutions to (3) and their "upper semicontinuous" dependence on initial conditions are applicable. We will rely on some of those below.

For a solution (x, q) to (3), let $j_0 := \max\{j \,|\, (0, j) \in \operatorname{dom}(x, q)\}$. (Such j_0 is finite by the definition of G_q and local finiteness of the covering $\{\Omega'_q\}_{q \in Q}$.) Then for all $(t, j) \in \operatorname{dom}(x, q)$ with $j \geq j_0$, the function $V_{q(t,j)}(x(t, j))$ is well defined, as then $x(t, j) \in \overline{\Omega_{q(t,j)}}$. Moreover, if $(t, j), (t', j) \in \operatorname{dom}(x, q)$, $j \geq j_0$, $t < t'$, then $q(t, j) = q(t', j)$, $x(t, j), x(t', j) \in C_{q(t,j)}$ and finally

$$V_{q(t',j)}(x(t', j)) < V_{q(t,j)}(x(t, j)) \tag{7}$$

by (a) of Lemma 4.5. If $(t, j + 1) \in \operatorname{dom}(x, q)$ for some $(t, j) \in \operatorname{dom}(x, q)$ such that $j \geq j_0$, then $x(t, j) \in C_{q(t,j)} \cap D_{q(t,j)}$ (since "before (t, j), $x(\cdot, j)$ was flowing") and so, by (b) of Lemma 4.5,

$$x(t, j) \in \left(\bigcup_{r>q} \partial\Omega'_r \right) \cap \Omega_q \subset \left(\bigcup_{r>q} \overline{\Omega'_r} \right) \cap \Omega_q .$$

Thus $q(t, j + 1) > q(t, j)$, i.e., "q is increasing during jumps", by the definition of G_q. By (iv) of Definition 4.1,

$$V_{q(t,j+1)}(x(t, j)) \leq (\overline{\gamma} \circ \underline{\gamma}^{-1})(V_{q(t,j)}(x(t, j))) \tag{8}$$

Thus, for all $(t, j) \in \operatorname{dom}(x, q)$, $j \geq j_0$, we have

$$V_{q(t,j)}(x(t, j)) \leq (\overline{\gamma} \circ \underline{\gamma}^{-1})^{j-j_0}(V_{q(0,j_0)}(x(0, j_0))) \,. \tag{9}$$

Let N be the number of elements in Q. Then $V_{q(t,j)}(x(t, j))$ is bounded above by $(\overline{\gamma} \circ \underline{\gamma}^{-1})^N(V_{q(0,j_0)}(x(0, j_0)))$ for any solution (x, q) and any $(t, j) \in \operatorname{dom}(x, q)$. Hence

$$\omega(x(t, j)) \leq \underline{\gamma}^{-1}(\overline{\gamma} \circ \underline{\gamma}^{-1})^N \overline{\gamma}(\omega(x(0, 0))). \tag{10}$$

Thus $\mathcal{A}$ is stable for (3) and $\omega(x(t, j))$ is bounded above for all solutions (x, q), all $(t, j) \in \operatorname{dom}(x, q)$.

Now suppose that (x, q) is a maximal solution to (3). As such, it is either complete, or eventually leaves any compact subset of $O \times Q$. Either way, let $J := \max\{j \,|\, (t, j) \in \operatorname{dom}(x, q) \text{ for some } t\}$. If (x, q) is complete, then $\sup_t(\operatorname{dom}(x, q)) = \infty$ (no solution can jump infinitely many times) and then

$$\nabla V_{q(t,J)} \cdot f\left(x(t, J), u_{q(t,J)}(x(t, J))\right) \leq -\alpha(\omega(x(t, J))) \tag{11}$$

for all t such that $(t, J) \in \operatorname{dom}(x, q)$. Standard arguments show that $\omega(x(t, J)) \to 0$ as $t \to \infty$. If (x, q) is not complete, then $x(t, J)$ must leave any compact subset of $\mathcal{O}$ as $t \to \sup_t(\operatorname{dom}(x, q))$, which by stability is only possible if $\omega(x(t, j)) \to 0$.

The two paragraphs above showed that for (3), the set $\mathcal{A}$ is asymptotically stable with the basin of attraction equal to $\mathcal{O}$. According to (Prieur et al., 2007, Theorem 4.1), this asymptotic stability is robust, in the sense of Definition 3.3 with the change that F^ρ_q be defined by

$$\operatorname{con} \bigcup_{\xi \in (x+\rho(x)\mathbb{B}) \cap C_q} f(\xi, k_q(\xi)) + \rho(x)\mathbb{B}.$$

It is straightforward, from continuity of f and k_q's (it is enough for the k_q's to be locally bounded) and from local finiteness of the covering of O by C_q's, that for each compact $K \subset O$, each $\varepsilon > 0$, there exists $\delta > 0$ such that, for each $x \in K$, each $q \in Q$,

$$f(x, k_q((x + \delta\mathbb{B}) \cap C_q)) \subset \bigcup_{\xi \in (x+\varepsilon\mathbb{B}) \cap C_q} f(\xi, k_q(\xi)) + \varepsilon\mathbb{B}.$$

This in turn can be used to conclude that robustness as in (Prieur et al., 2007, Theorem 4.1) implies the robustness in the sense of Definition 3.3. This finishes the proof. $\square$

Remark 4.6 The proof of Theorem 4.4 simplifies somewhat if the PCLF is almost strict. Indeed, the arguments can be repeated up to the estimate (8). That estimate, by (vii) of Definition 4.2, can be replaced by

$$V_{q(t,j+1)}(x(t, j)) \leq V_{q(t,j)}(x(t, j)) \tag{12}$$

This shows that $V_{q(t,j)}(x(t, j))$ is nondecreasing along $\operatorname{dom}(x, q)$, and consequently, $V_{q(t,j)}(x(t, j)) \leq V_{q(0,j_0)}(x(0, j_0))$ for all $(t, j) \in \operatorname{dom}(x, q)$, $j \geq j_0$. Since $x(0, 0) = x(0, j_0)$, item (iv) of Definition 4.1 yields that $\underline{\gamma}(\omega(x(t, j))) \leq \overline{\gamma}(\omega(x(0, 0)))$ for all $(t, j) \in \operatorname{dom}(x, q)$.

This shows stability of $\mathcal{A}$ for (3). The remaining arguments can be repeated without change.

Example 4.7 (Artstein's circles, re-revisited) For the system (1), Example 4.3 exhibited a smooth PCLF. Theorem 4.4 implies the existence of a robust stabilizing hybrid feedback for (1), and recovers the result of Prieur (2000) from a more general context. (Note that (1) is affine, in fact linear, in u.) An example of such a hybrid feedback is as follows. The formula (6) leads to

$$C_1 = \{x = (r, \theta) \mid r > 0, -\delta' \le \theta \le \delta'\},$$

$$D_1 = \{x = (r, \theta) \mid r > 0, \delta' \le \theta \le 2\pi - \delta'\},$$

$$C_2 = \{x = (r, \theta) \mid r > 0, \delta \le \theta \le 2\pi - \delta\},$$

$$D_2 = \{x = (r, \theta) \mid r > 0, -\delta \le \theta \le \delta\},$$

$$G_1(x) = \{2\}, \ \forall x \in D_1, \quad G_2(x) = \{1\}, \ \forall x \in D_2,$$

where $\delta' = \pi - \beta'$, $\delta = \pi - \beta$ (so that $\delta < \delta'$, and $\delta, \delta' \in (\pi/4, \pi/2)$). We can set $k_1(x) = -1$ for all $x \in C_1$, $k_2(x) = 1$ for all $x \in C_2$. The behavior of the resulting closed loop hybrid system is as follows. Given an initial condition $q = 1$, $x \in C_1$, the continuous variable may flow clockwise (has to flow if $x \notin C_1 \cap D_1$) to 0. If $q = 1$, $x \in D_1 \setminus C_1$, the discrete variable switches to $q = 2$. After a switch from $q = 1$ to $q = 2$, only flow of the continuous variable is possible, counterclockwise, to 0. Behavior from initial conditions with $q = 2$ is similar, and in general, only one switch is possible. From this, one can deduce asymptotic stability. Regarding robustness, we only note that the presence of the discrete variable makes chattering impossible: small measurement noise does not affect q being either 1 or 2, and hence, it does not lead to repeated fast switching between control values.

We add that considering $\beta = \beta'$, and so $\Omega'_1 = \Omega_1$, $\Omega'_2 = \Omega_2$, leads to $C_1 = D_2$, $C_2 = D_1$, and the resulting hybrid feedback does not render 0 attractive. Indeed, in such a case, given any $x_0 \in C_1 \cap D_1 = D_1 \cap D_2$ and any $q_0 \in \{1, 2\}$, there exists a solution (x, q) with $\mathrm{dom}(x, q) = \{0\} \times \mathbb{N}$ [3] such that $x(0, j) = x_0$ for all $j \in \mathbb{N}$ while $q(0, j) = q_0$ for even j and $q(0, j) = 3 - q_0$ for odd j. Such a solution is maximal and x does not approach 0.

5 Infinite number of patches and a necessary and sufficient condition

5.1 Necessity

We will say that a family $\{\Omega_q\}_{q \in Q}$ is locally finite on $\mathcal{O}$ if for any compact $K \subset \mathcal{O}$, there are finitely many q's such that $K \cap \Omega_q \ne \emptyset$.

[3] Such solutions to hybrid systems are sometimes called *instantaneously Zeno.*

Definition 5.1 *An almost strict patchy control Lyapunov function for (2) with the attractor $\mathcal{A}$ consists of a set Q and a collection of functions V_q and sets Ω_q, Ω'_q for each $q \in Q$, such that $Q \subset \mathbb{Z}$, conditions (ii)-(vii) of Definitions 4.1, 4.2 hold, and the family $\{\Omega_q\}_{q \in Q}$ is locally finite on $\mathcal{O}$. A strict patchy control Lyapunov function is strict if the inequality in condition (vii) is strict.*

We now show that a smooth patchy Lyapunov function exists for most asymptotically controllable to a compact set nonlinear systems. For completeness, we first recall that (2) is *asymptotically controllable* on $\widetilde{O}$ to $\mathcal{A}$ if:

- for each $x^0 \in \widetilde{O}$ there exists a measurable $u_{x^0} : [0, \infty) \to U$ such that the maximal trajectory x to (2) with u replaced by u_{x^0} is complete and such that $\lim_{t \to \infty} \mathrm{dist}_{\mathcal{A}}(x(t)) = 0$;
- for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x^0 \in \widetilde{O}$ with $\mathrm{dist}_{\mathcal{A}}(x) < \delta$ one can find u_{x^0} as in (a) so that the resulting trajectory x is such that $\mathrm{dist}_{\mathcal{A}}(x(t)) < \varepsilon$ for all $t \ge 0$.

Theorem 5.2 *Suppose f is smooth, U is compact, and (2) is asymptotically controllable on $\widetilde{O}$ to $\mathcal{A}$. Then there exists a strict patchy control Lyapunov function for (2).*

The proof will be based on the existence, for asymptotically controllable systems, of a stabilizing patchy feedback, as shown by Ancona and Bressan (1999).

Definition 5.3 *(Ancona and Bressan (1999)) A mapping $\mu : O \to U$ is a patchy feedback for (2) on O if there exist a set Q, and for each $q \in Q$, a set $\Omega_q \subset O$ and a control value $u_q \in U$ such that*

(a) for each $q \in Q$, the pair Ω_q, $f(\cdot, u_q)$ forms a patch, that is:
(a1) Ω_q is open, $\overline{\Omega_q} \subset O$, and the boundary of Ω_q is smooth;
(a2) $f(\cdot, u_q)$ is smooth on some neighborhood of $\overline{\Omega_q}$;
(a3) for any point $x \in \partial\Omega_q$

$$n_q(x) \cdot f(x, u_q) < 0, \tag{13}$$

where $n_q(x)$ is the unit (outward) normal vector to $\overline{\Omega_q}$ at x;
(b) Q is a totally ordered set;
(c) the sets $\{\Omega_q\}_{q \in Q}$ form a locally finite covering of O;

and μ can be written as $\mu(x) = u_q$ if $x \in \Omega_q \setminus \bigcup_{r > q} \Omega_r$, where $>$ is the ordering of Q.

A patchy feedback, in closed loop with (2), leads to a discontinuous vector field. Solutions to it are understood in the Caratheodory sense, and they have several desirable properties; see Ancona and Bressan (1999). The following result can be immediately deduced from (Ancona and Bressan, 1999, Theorem 1) and its proof.

Theorem 5.4 *Suppose that f is smooth, U is compact, and (2) is asymptotically controllable on $\widetilde{O}$ to $\mathcal{A}$. Then there exists a patchy feedback on O such that:*

(i) $Q \subset \mathbb{Z}$ is ordered by the standard inequality;
(ii) each Ω_q is bounded;
(iii) for each q, each complete solution $x(\cdot)$, $x(0) \in \overline{\Omega_q}$, to

$$\dot{x}(t) = f(x(t), u_q) \tag{14}$$

satisfies $x(t) \in \bigcup_{r>q} \Omega_r$ for some $t \geq 0$;
(iv) for any proper indicator ω of $\mathcal{A}$ with respect to $\widetilde{O}$ and $\delta > 0$ there exist $q_\delta, q^\delta \in Q$ such that $q > q_\delta$ implies $\sup \omega(\Omega_q) \leq \delta$ and $q^\delta > q$ implies $\inf \omega(\Omega_q) \geq \delta$.

We need to comment that the patchy feedbacks of Ancona and Bressan (1999) had the index set as a subset of $\mathbb{Z} \times \mathbb{N}$, ordered by the lexicographical ordering, and such that for each $z \in \mathbb{Z}$, $\{n \in \mathbb{N} \,|\, (z, n) \in Q\}$ was nonempty and finite. Any such set can be identified with a subset of integers, with the lexicographical order in the former corresponding to the standard order in the latter. We add that the last property above essentially means that patches close to $\mathcal{A}$ have large indices while patches away from $\mathcal{A}$ have small ones. Finally, in Ancona and Bressan (1999), the attractor $\mathcal{A}$ was the origin, and $O = \mathbb{R}^n$. The extension to the more general setting we have here is immediate, as one just relies on a proper indicator of $\mathcal{A}$ with respect to O rather than on the norm.

The four properties of the patchy feedback listed in Theorem 5.4 are enough to show that, in an appropriate sense, $\mathcal{A}$ is asymptotically stable on O for the closed-loop system. We will not need that here, and rather, after using the patchy feedback to build a hybrid feedback, we will show the stabilization property of the latter directly.

Lemma 5.5 *Under the assumptions of Theorem 5.4, for each $q \in Q$ there exists a compact set $K_q \subset \Omega_q \cap \bigcup_{r>q} \Omega_r$ and a nonnegative function W_q that is smooth on a neighborhood of $\overline{\Omega_q}$ and such that*

$$\nabla W_q(x) \cdot f(x, u_q) < 0 \quad \textit{for all } x \in \overline{\Omega_q} \setminus K_q, \tag{15}$$

and in particular, for all $x \in \overline{\Omega_q} \setminus \bigcup_{r>q} \Omega_q$.

The proof of this lemma is in the Appendix.

Proof.(of Theorem 5.2) Let K_q be as in Lemma 5.5. For each $q \in Q$, one can find an open set Ω'_q such that $\overline{\Omega'_q} \subset \Omega_q$, $K_q \subset \Omega'_q$, $K_p \subset \Omega'_q$ if $p \in Q$ is such that $K_p \subset \Omega_q$, $\{\Omega'_q\}_{q \in Q}$ is a covering of O (necessarily locally finite), and finally such that

$$\nabla W_q(x) \cdot f(x, u_q) < 0 \quad \text{for all } x \in \overline{\Omega_q} \setminus \bigcup_{r>q} \Omega'_q. \tag{16}$$

For each $q \in Q$, pick any $c_q > \sup_{x \in \Omega_q} W_q(x)$ so that $W_q(x) \in [0, c_q]$ when $x \in \Omega_q$, and for each $q \in Q$ let $b_q = 2^{-q}$ and $a_q = 2^{-q}/c_q$. Then the functions $V_q(x) := a_q W_q(x) + b_q$ are positive and such that $\inf_{x \in \Omega_q} V_q(x) > \sup_{x \in \Omega_r} V_r(x)$ whenever $r > q$, this verifies condition (vii) of Definition 4.2.

We now check the remaining conditions to show that V_q's above yield a strict PCLF as in Definition 5.1. By (i) in Theorem 5.4, Q is totally ordered. For each $q \in Q$, $\overline{\Omega'_q} \subset \Omega_q$ by construction, while Ω_q's form a locally finite covering of O by condition (c) in Definition 5.3. Also, for each $q \in Q$, V_q is smooth on a neighborhood of $\overline{\Omega_q}$ by Lemma 5.5. Let ω be any proper indicator of $\mathcal{A}$ on $\widetilde{O}$. For (iv) of Definition 4.1, we can consider

$$\underline{\beta}(r) = \inf\{V_q(x) \,|\, \omega(x) \geq r, x \in \Omega_q\},$$

$$\overline{\beta}(r) = \sup\{V_q(x) \,|\, \omega(x) \leq r, x \in \Omega_q\},$$

so that $\underline{\beta}(\omega(x)) \leq V_q(x) \leq \overline{\beta}(\omega(x))$ if $x \in \Omega_q$. By the very definitions, both functions are nondecreasing, and by local finiteness of $\{\Omega_q\}_{q \in Q}$, positive (and finite) for $r > 0$. (They need not be continuous though.) By (i) and (iv) of Theorem 5.4, and by the choice of a_q, b_q's above, both functions tend to 0 as $r \to 0$ and to ∞ if $r \to \infty$. Finally, one can pick $\mathcal{K}_\infty$ functions $\underline{\gamma} \leq \underline{\beta}$ and $\overline{\gamma} \geq \overline{\gamma}$; these satisfy (iv) of Definition 4.1.

For conditions (v) and (vi) of Definition 4.1, we have

$$\nabla V_q(x) \cdot f(x, u_q) < 0 \quad \text{for all } x \in \overline{\Omega_q} \setminus \bigcup_{r>q} \Omega'_r \tag{17}$$

by Lemma 5.5 and since $a_q > 0$. Continuity of ∇V_q and $x \mapsto f(x, u_q)$ on $\overline{\Omega_q}$, implies that there exists a constant $\alpha_1^q > 0$ such that

$$\nabla V_q(x) \cdot f(x, u_q) < -\alpha_1^q \quad \text{for all } x \in \overline{\Omega_q} \setminus \bigcup_{r>q} \Omega'_r. \tag{18}$$

Similarly, for each $q \in Q$ we have, by (a3) of Definition 5.3, continuity of $n_q(\cdot)$ and $f(\cdot, u_q)$, and compactness of $\partial\Omega_q$, that there exists a constant $\alpha_2^q > 0$ so that

$$n_q(x) \cdot f(x, u_q) \leq -\alpha_2^q \quad \text{for all } x \in \partial\Omega_q. \tag{19}$$

Now, by local finiteness of the covering of O by Ω_q's, we can find a continuous function $\alpha : (0, \infty) \to (0, \infty)$ such that $\alpha(\omega(x)) \leq \min\{\alpha_1^q, \alpha_2^q\}$ if $x \in \Omega_q$. This verifies (v) and (vi) of Definition 4.1. ◻

We note that Theorem 5.2 was proved directly, by taking a stabilizing patchy feedback as a starting point, but it did involve one use of a converse Lyapunov result of Cai et al. (2007) (used to construct W_q, a smooth Lyapunov function for a differential equation where all solutions reach a certain set in finite time, but then possibly leave it). An alternative approach to Theorem 5.2 is to apply the converse Lyapunov theorems in Cai et al. (2007) to the hybrid closed-loop system developed in Prieur et al. (2007) (again, based on a patchy feedback of Ancona and Bressan (1999)) with a scaling, like that used in Artstein (1983), to guarantee forward completeness.

5.2 Sufficiency

We now state the most general definition of a patchy control Lyapunov function. It allows for infinite number of patches and does not insist on strictness or almost strictness, i.e., it allows for $V_q(x)$ to increase at jumps. Briefly, the meaning behind PCLF according to the definition below is that it is a collection of patches and functions that leads to asymptotic stability of (3), subject to picking control functions k_q in the hybrid feedback (6) to satisfy Lemma 4.5. For this to hold, some conditions on the arrangement of patches need to be met; we give an example of such conditions in Proposition 5.7 below.

Definition 5.6 *A patchy control Lyapunov function for (2) with the attractor $\mathcal{A}$ consists of a set Q and a collection of functions V_q and sets Ω_q, Ω'_q for each $q \in Q$, such that $Q \subset \mathbb{Z}$, conditions (ii)-(vi) of Definition 4.1 hold, and the hybrid feedback given by (6) and by any functions k_q satisfying the conditions of Lemma 4.5 renders $\mathcal{A}$ asymptotically stable.*

Note that the definition does not require the k_q's to be constructed as in the proof of Lemma 4.5; they can be any functions having the right properties.

Theorem 4.4 and its proof showed that a PCLF with finitely many patches is a patchy control Lyapunov function (as in Definition 5.6) subject to a convexity assumption. Below, we state that an almost strict smooth PCLF is indeed a PCLF in the sense of the definition above. The proof is postponed until the Appendix.

Proposition 5.7 *Suppose that a set Q and a collection of functions V_q and sets Ω_q, Ω'_q for each $q \in Q$ is such that $Q \subset \mathbb{Z}$, conditions (ii)-(vi) of Definition 4.1 hold and, for each $N \in \mathbb{N} \cup \{\infty\}$,*

(pa$_1$) for each $\varepsilon > 0$ there exists $\delta > 0$ such that, for any increasing sequence $\{q_n\}_{0\leq n\leq N}$ in Q, and for any sequence $\{x_n\}_{0\leq n\leq N}$ in O satisfying $x_0 \in \Omega_{q_0}$, $V_{q_0}(x_0) \leq \delta$ and, for all $0 \leq n \leq N$,

$$x_{n+1} \in \Omega_{q_n} \cap \partial\Omega'_{q_{n+1}} \ , \ V_{q_n}(x_{n+1}) \leq V_{q_n}(x_n) \ ,$$

we have $V_{q_n}(x_n) \leq \varepsilon$, for all $0 \leq n \leq N$;

(pa$_2$) for any increasing sequence $\{q_n\}_{0\leq n\leq N}$ in Q, and for any sequence $\{x_n\}_{0\leq n\leq N}$ in O satisfying $x_0 \in \Omega_{q_0}$ and, for all $0 \leq n \leq N$,

$$x_{n+1} \in \Omega_{q_n} \cap \partial\Omega'_{q_{n+1}} \ , \ V_{q_n}(x_{n+1}) \leq V_{q_n}(x_n) \ ,$$

there exists $M > 0$ such that $V_{q_n}(x_n) \leq M$, for all $0 \leq n \leq N$.

Then this collection is a PCLF for (2).

Corollary 5.8 *An almost strict smooth patchy control Lyapunov function for (2) in the sense of Definition 5.1 is a smooth patchy control Lyapunov function for (2) in the sense of Definition 5.6.*

Proof. If Q is finite there is nothing to show. In the opposite case, we will show that (pa$_1$) and (pa$_2$) of Proposition 5.7 are met. Let $\varepsilon > 0$, $N \in \mathbb{N} \cup \{\infty\}$, $\{q_n\}_{0\leq n\leq N}$ be an increasing sequence in Q and $\{x_n\}_{0\leq n\leq N}$ be a sequence in O satisfying $x_0 \in \Omega_{q_0}$, $V_{q_0}(x_0) \leq \varepsilon$, and, for all $0 \leq n \leq N$, $x_{n+1} \in \Omega_{q_n} \cap \Omega_{q_{n+1}}$ and $V_{q_n}(x_{n+1}) \leq V_{q_n}(x_n)$ With $V_{q_{n+1}}(x_{n+1}) < V_{q_n}(x_{n+1})$, we get $V_{q_n}(x_n) \leq \varepsilon$, for all $0 \leq n \leq N$. Let $\{q_n\}_{0\leq n\leq N}$ be an increasing sequence in Q, and $\{x_n\}_{0\leq n\leq N}$ a sequence in O satisfying $x_0 \in \Omega_{q_0}$, and, for all $0 \leq n \leq N$, $x_{n+1} \in \Omega_{q_n} \cap \Omega_{q_{n+1}}$ and $V_{q_n}(x_{n+1}) \leq V_{q_n}(x_n)$. With $V_{q_{n+1}}(x_{n+1}) < V_{q_n}(x_{n+1})$, by denoting $M = V_{q_0}(x_0)$, we have $V_{q_n}(x_n) \leq M$, for all $0 \leq n \leq N$. $\square$

6 Illustration

Throughout the paper, we used Artstein's circles to illustrate the definitions and the results. Here we consider another classical example, the Brockett integrator:

$$\dot{x}_1 = u_1 \ , \ \dot{x}_2 = u_2 \ , \ \dot{x}_3 = x_1 u_2 - x_2 u_1 \ . \qquad (20)$$

The necessary condition (Brockett (1983)) for the stabilization, of the origin, by means of a continuous feedback or robust stabilization by locally bounded feedback (Ryan (1994)) fail for this system. We will show the existence of a hybrid stabilizing feedback by applying Theorem 4.4. (This partially recovers the result of Prieur and Trélat (2005), see also Prieur and Trélat (2006), where an optimization criterion was also considered.) To do that, we will display a PCLF with two patches, based on a simplified version of the hybrid controller of Hespanha and Morse (1999) (see also Goebel et al. (2004)).

Let $f(x,u)$ be the right-hand side of the system (20), where $x = (x_1, x_2, x_3)$ and $u = (u_1, u_2)$. Set $r = \sqrt{x_1^2 + x_2^2}$, and pick $\rho > 1$ and $0 < \varepsilon < 1$ such that $\sqrt{\rho}\rho + \varepsilon\sqrt{\rho} < 2$. With $\widetilde{O} = \mathbb{R}^3$ and $\mathcal{A}$ being the origin, let $Q = \{1, 2\}$ (so (i) of Definition 4.1 is verified); let $\Omega'_1 = \Omega_1 = \mathbb{R}^3 \setminus \{0\}$ and $V_1 : \mathbb{R}^3 \to \mathbb{R}$ be given by

$$V_1(x) = (\rho + \varepsilon)\sqrt{|x_3|} - x_1.$$

Furthermore, let

$$\Omega_2' = \{x \mid r^2 > \rho|x_3|\}\,, \quad \Omega_2 = \{x \mid r^2 > |x_3|\}\,,$$

and $V_2 : \mathbb{R}^3 \to \mathbb{R}$ be given by

$$V_2(x) = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right) = \frac{1}{2}\left(r^2 + x_3^2\right).$$

The families $\{\Omega_1, \Omega_2\}$, $\{\Omega_1', \Omega_2'\}$ consist of nonempty and open sets, they cover $\mathcal{O} = \mathbb{R}^3 \setminus \{0\}$, and the remaining conditions of (ii) of Definition 4.1 are easy to check. The function V_1 is smooth on a (relative to $\mathbb{R}^3 \setminus \{0\}$) neighborhood of $\overline{\Omega_1 \setminus \Omega_2'}$, and so (iii) of Definition 4.1 is verified. For all $x \in \Omega_1 \setminus \Omega_2'$, we have

$$\frac{\varepsilon\sqrt{|x_3|}}{2} + \frac{\varepsilon r}{2\sqrt{\rho}} \le V_1(x) \le (\sqrt{\rho} + \varepsilon)\sqrt{|x_3|} + r\,.$$

This verifies (iv) of Definition 4.1. Now take $u_{1,x} = (1, 0)$ for all $x \in \mathbb{R}^3$. Observe that [4], for all $x \in \Omega_1 \setminus \Omega_2'$,

$$\nabla V_1(x) \cdot f(x, u_{1,x}) \le \frac{\sqrt{\rho}\rho + \varepsilon\sqrt{\rho}}{2} - 1\,.$$

The choice of ρ and ε verifies (v) of Definition 4.1 for $q = 1$. Take $u_{2,x} = \left(-x_1 + 4\frac{x_2 x_3}{r^2}, -x_2 - 4\frac{x_1 x_3}{r^2}\right)$, for all $x \in \Omega_2$. Then, for all $x \in \Omega_2$, $\nabla V_2(x) \cdot f(x, u_{2,x}) = -r^2 - 4x_3^2$, what verifies (v) of Definition 4.1 for $q = 2$. Since $\partial\Omega_1 \setminus \Omega_2'$ is empty, (vi) of Definition 4.1 is verified for $q = 1$. The unit normal vector $n_2(x)$ to $\overline{\Omega_2}$ at $x \in \partial\Omega_2 \cap \mathcal{O}$ is $n_2(x) = \frac{1}{\sqrt{4r^2+1}}(-2x_1, -2x_2, sgn(x_3))$. Since $|x_3| = r^2$ and $|x|^2 = |x_3| + |x_3|^2$ for all $x \in \partial\Omega_2$,

$$n_2(x) \cdot f(x, u_{2,x}) = \frac{2r^2 - 4|x_3|}{\sqrt{4r^2+1}} \le 1 - \sqrt{1 + 4|x|^2}\,.$$

This verifies (vi) of Definition 4.1 for $q = 2$.

Thus, the set Q, the open sets $\{\Omega_q\}_{q\in Q}$ and $\{\Omega_q'\}_{q\in Q}$, and the family of functions $\{V_q\}_{q\in Q}$ constitutes a PCLF, with finite number of patches, for (20). Since (20) is affine with respect to u, the convexity assumption in Theorem 4.4 holds. Thus, with $u_q : C_q \to \mathbb{R}^2$ given by $u_q(x) = u_{q,x}$ for all $x \in C_q$, the hybrid controller (6) renders $\mathcal{A} = \{0\}$ asymptotically stable on $\mathbb{R}^3 \setminus \{0\}$.

It is also possible to scale V_1 to obtain a strict PCLF. Indeed, finding a continuously differentiable class-$\mathcal{K}_\infty$ function γ with $\gamma'(s) > 0$ for $s > 0$ such that, for all $x \in \Omega_1 \setminus \partial\Omega_2'$, $V_1(x) \ge \gamma\left(\frac{|x|^2}{2}\right)$, and replacing V_1 by $W_1(x) = \gamma^{-1}(V_1(x))$ leads to a strict PCLF. This highly technical step is not necessary to guarantee the existence of a robustly stabilizing hybrid feedback.

[4] Here and in what follows, $sgn(x_3)$ is the sign of $x_3 \neq 0$.

7 Conclusion

This work introduces and studies the concept of a smooth patchy control Lyapunov function (PCLF). The first main result states that, under a mild convexity assumption, the existence of such PCLF implies the existence of a stabilizing hybrid feedback. Moreover, via Prieur et al. (2007), this hybrid controller is robust with respect to measurement noise and external disturbances. The second main result states that a smooth PCLF exists for any asymptotically controllable system. This result relies on the existence, for such system, of a stabilizing patchy feedback, as shown in Ancona and Bressan (1999). This second result is also related to the converse Lyapunov theorems of Cai et al. (2007, 2008).

8 Appendix

Proof.(of Lemma 4.5) Fix $q \in Q$. Let S be the set $\left(\partial\Omega_q \setminus \bigcup_{r>q}\Omega_r'\right) \cap \mathcal{O}$. Note that S is relatively closed in $\mathcal{O}$, and let $N : \mathcal{O} \to \mathbb{R}^n$ be a continuous extension of n_q from S (so N is continuous and $N(x) = n_q(x)$ for $x \in S$). For each $\bar{x} \in S$ we have $n_q(\bar{x}) \cdot f(\bar{x}, u_{q,\bar{x}}) \le -\alpha(\omega(\bar{x}))$, and by continuity of N, f, α, and ω, there exists an open neighborhood $O_{\bar{x}} \subset \mathcal{O}$ such that

$$N(x) \cdot f(x, u_{q,\bar{x}}) < -\alpha(\omega(x))/2 \tag{21}$$

for all $x \in O_{\bar{x}}$. Let $O_S = \bigcup_{\bar{x}\in S} O_{\bar{x}}$.

Let $P_S = \mathcal{O} \setminus O_S$, note that P_S is closed. By Urysohn's Lemma, there exists a continuous $\phi : \mathcal{O} \to [0, 1]$ such that $\phi(x) = 1$ if $x \in S$, $\phi(x) = 0$ if $x \in P_S$. Consider the function $\Phi : O \times U \to \mathbb{R}$ given by

$$\Phi(x, u) = \phi(x)N(x) \cdot f(x, u) + \phi(x) - 1.$$

Note that Φ is continuous, and for $x \in S$, $\Phi(x, u) = n_q(x) \cdot f(x, u)$. For all $x \in \mathcal{O}$ there exists $u \in U$ so that

$$\Phi(x, u) < -\phi(x)\alpha(\omega(x))/2. \tag{22}$$

Indeed, if $x \in O_S$, and hence $x \in O_{\bar{x}}$ for some $\bar{x} \in S$, then u can be chosen as $u_{q,\bar{x}}$ thanks to (21) and the fact that $\phi(x) \in [0, 1]$. If $x \notin O_S$, that is $x \in P_S$, then $\phi(x) = 0$ and thus $\Phi(x, u) = -1$ while $-\phi(x)\alpha(\omega(x))/2 = 0$, and any choice of $u \in U$ satisfies (22).

By assumption, and by continuity of ∇V_q, f, α, and ω, for each $x \in \left(\overline{\Omega_q \setminus \bigcup_{r>q}\Omega_r'}\right) \cap \mathcal{O}$ there exists $u \in U$ such that

$$\nabla V_q(x) \cdot f(x, u) < -\alpha(\omega(x))/2. \tag{23}$$

On $\left(\overline{\Omega_q \setminus \bigcup_{r>q} \Omega'_r}\right) \cap \mathcal{O}$, consider a set-valued mapping Ψ given by

$$\Psi(x) = \left\{ u \in U \;\middle|\; \begin{array}{c} \Phi(x,u) \leq -\phi(x)\alpha(\omega(x))/2 \\ \nabla V_q(x) \cdot f(x,u) \leq -\alpha(\omega(x))/2 \end{array} \right\}$$

For all $x \in \left(\overline{\Omega_q \setminus \bigcup_{r>q} \Omega'_r}\right) \cap \mathcal{O}$, $\Psi(x)$ is nonempty, as the discussion above implies. By the second assumption in Theorem 4.4, $\Psi(x)$ is convex. Furthermore, Ψ is continuous (see Example 5.10 in Rockafellar and Wets (1998)), and, in particular, closed-valued. Thus, there exists a continuous selection $u_q : \left(\overline{\Omega_q \setminus \bigcup_{r>q} \Omega'_r}\right) \cap \mathcal{O} \to U$ from Ψ (that is, a continuous function with $u_q(x) \in \Psi(x)$); see Theorem 5.58 in Rockafellar and Wets (1998). Such a selection can be realized by choosing $u \in \Psi(x)$ of minimal norm; see Example 5.57 in Rockafellar and Wets (1998) or Theorem 3.11 in Freeman and Kokotović (1996). This selection meets the requested conditions. $\square$

Proof.(of Lemma 5.5) Fix $q \in Q$. By (iii) of Theorem 5.4, any maximal solution $x(\cdot)$ to (14) with $x(0) \in \overline{\Omega_q}$ is such that for some $t \geq 0$, $x(t) \in \bigcup_{r>q} \Omega_r$. In fact, there exists a compact set $K_q \subset \Omega_q \cap \bigcup_{r>q} \Omega_r$, a time $T_q > 0$ such that any maximal solution $x(\cdot)$ to (14) with $x(0) \in \overline{\Omega_q}$ is such that for some $t \in [0, T_q]$, $x(t) \in K_q$. Otherwise, there is an increasing sequence of compact sets $K_q^n \subset K_q^{n+1} \subset \Omega_q \cap \bigcup_{r>q} \Omega_r$ so that $\bigcup_{n=1}^{\infty} K^{n+1} = \Omega_q \cap \bigcup_{r>q} \Omega_r$, an increasing sequence of times $T_q^n > T_q^{n+1} + 1$, and a sequence of maximal solutions $x^n(\cdot)$ to (14) with $x^n(0) \in \overline{\Omega_q}$ such that $x^n(t) \notin K_q^n$ for all $t \in [0, T_q^n]$. There exists a subsequence of $x^n(\cdot)$'s that converges uniformly on compact intervals to a solution, say $x(\cdot)$, of (14), that is complete, $x(0) \in \overline{\Omega_q}$, and $x(t) \notin \Omega_q \cap \bigcup_{r>q} \Omega_r$ for all $t \in [0, \infty)$. This contradicts (iii).

Let O_q be any neighborhood of $\overline{\Omega_q}$ on which $f(\cdot, u_q)$ is smooth with the property that any maximal solution $x(\cdot)$ to (14) with $x(0) \in O_q$ is such that $x(t) \in \Omega_q$ for $t > 1$. (Such O_q exists by compactness of $\overline{\Omega_q}$ and the inward pointing condition (13).) In the terminology of Cai et al. (2008), the set K_q is pre-asymptotically stable for the "hybrid" system with state z, state space O_q, continuous dynamics $\dot{z}(t,j) = f(z(t,j), u_q)$ if $z(t,j) \in O_q \setminus \bigcup_{r>q} \Omega_r$ and no discrete dynamics. Hence (Cai et al., 2008, Corollary 3.4, Theorem 3.14) yield the existence of a smooth Lyapunov function verifying the pre-asymptotic stability of K_q. In particular, such Lyapunov function has the properties requested of W_q in the lemma. $\square$

Proof.(of Proposition 5.7) Given a PCLF, consider the hybrid feedback (6). Let u_q be any functions as in Lemma 4.5. As in the proof of Theorem 4.4, one can argue that the hybrid system (3) with the feedback given by (6) has the desired semicontinuity and closedness properties. Also, the arguments leading to (7) and (9) are still valid.

We now show stability. If Q is finite, and N is the number of its elements, (10) yields stability. Suppose Q is infinite, and pick $\varepsilon > 0$. Let δ be as in assumption (pa$_1$). Let (x_0, q_0) in $\mathbb{R}^n \times Q$ be such that $\omega(x_0) \leq \overline{\gamma}^{-1}(\delta)$ and let (x,q) be a solution to (3) with the initial condition (x_0, q_0). From (iv) of Definition 4.1, we have $V_{q(0,j_0)}(x(0,j_0)) \leq \delta$. Let $j_0 = \max\{j \,|\, (0,j) \in \mathrm{dom}(x,q)\}$, and let $\{t_n\}$ be the (finite or infinite) nondecreasing sequence of jump times after the j_0-th one, i.e., let t_n be such that $(t_n, j_0 + n) \in \mathrm{dom}(x,q)$ as well as $(t_n, j_0 + n + 1) \in \mathrm{dom}(x,q)$. The sequence $\{q_n\}$ defined by $q'_n = q(t_n, j_0 + n)$ is increasing, while the sequence $\{x_n\}$ defined by $x_n = x(t_n, j_0 + n)$ satisfies $x_0 \in \Omega_{q_0}$, $V_{q_0}(x_0) \leq \delta$, $x_{n+1} \in \Omega_{q_n} \cap \partial\Omega'_{q_{n+1}}$, and $V_{q_n}(x_{n+1}) \leq V_{q_n}(x_n)$. This and Assumption (pa$_1$) implies that $V_{q_n}(x_n) \leq \varepsilon$ for all n, and thus, with (7), we get $V_{q(t,j)}(x(t,j)) \leq \varepsilon$, for all $(t,j) \in \mathrm{dom}(x,q)$.

Now suppose that (x,q) is a maximal solution to (3). As such, it is either complete, or eventually leaves any compact subset of $\mathcal{O} \times Q$. If the solution jumps finitely many times, i.e. $J := \max\{j \,|\, (t,j) \in \mathrm{dom}(x,q) \;\text{ for some } t\}$ is finite, and is not complete, then $x(t,J)$ must leave any compact subset of $\mathcal{O}$ as $t \to \sup\{t \,|\, (t,j) \in \mathrm{dom}(x,q)\text{for some } j\}$. Thus $\omega(x(t,j)) \to 0$. If J is finite and the solution is complete, then (11) and standard arguments show that $\omega(x(t,J)) \to 0$ as $t \to \infty$. Finally, if (x,q) jumps infinitely many times (which implies completeness), then by local finiteness of $\{\Omega'_q\}_{q \in Q}$, and the fact that $q(t,j)$ is increasing after the first jump (and so $x(t,j)$ does not return to a set Ω'_q after leaving it), we must have $x(t,j)$ eventually leaving any compact subset of $\mathcal{O}$. By assumption (pa$_2$), $V_{q(t_n, j_0+n)}(x(t_n, j_0+n))$ are bounded (with the sequence of t_n's as in the previous paragraph and by taking x_n in (pa$_2$) to be $x(t_n, j_0+n)$). Thus $\omega(x(t,j))$ remains bounded over $(t,j) \in \mathrm{dom}(x,q)$ and thus it must be the case that $\omega(x(t,j)) \to 0$.

Thus, for (3), the set $\mathcal{A}$ is (globally) asymptotically stable. Robustness of the said asymptotic stability follows from (Prieur et al., 2007, Theorem 4.3), thanks to the already mentioned closedness and semicontinuity properties of the data, the local finiteness of $\{C_q\}_{q \in Q}$, the local boundedness of mappings G_q in x that is uniform in q, and finally, the fact that $C_q \cup D_q = \mathcal{O}$ for all $q \in Q$. $\square$

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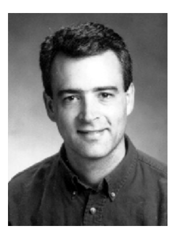

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