Hybrid Feedback Control and Robust Stabilization of Nonlinear Systems

Christophe Prieur, Rafal Goebel, and Andrew R. Teel

Abstract—**In this paper, we show, for any nonlinear system that is asymptotically controllable to a compact set, that a logic-based, hybrid feedback can achieve asymptotic stabilization that is robust to small measurement noise, actuator error, and external disturbance. The construction of such a feedback hinges upon recasting a stabilizing patchy feedback in a hybrid framework by making it dynamic with a discrete state, while insisting on semicontinuity and closedness properties of the hybrid feedback and of the resulting closed-loop hybrid system. The robustness of stability is then shown as a generic property of hybrid systems having the said regularity properties. Auxiliary results give uniformity of convergence and of overshoots for hybrid systems, and give a $\mathcal{KL}$ characterization of asymptotic stability of compact sets.**

Index Terms—**Hybrid control, hybrid systems, nonlinear systems, patchy vector fields, robust stabilization.**

I. Introduction

A. Background

THE area of robust feedback stabilization for general nonlinear control systems has experienced important breakthroughs over the last decade, during which two main approaches for solving the problem have emerged. The first one is based on control-Lyapunov functions (CLFs), which are nonsmooth in general and require tools from nonsmooth analysis to develop the control laws and certify their stabilization and robustness properties. This approach was pioneered in [7] and continued in [36] and [6] with extensions to noncompact attractors considered in [19]. The other approach, proposed in [1], exploits a differential equation's continuous dependence on initial conditions and control inputs, and the fact that the effect of measurable controls can be approximated by piecewise constant controls. These properties are used to construct a feedback that is constant on given, bounded patches of the state space and is such that the closed-loop leads to a so-called "patchy vector field" (PVF). Asymptotic stability of the latter is verified not by a Lyapunov function, but from the properties inherited by the PVF from the "asymptotically controlling" measurable controls used to construct the feedback. A further alternative, time-varying continuous feedback, was proposed earlier in [10], [9], [27], and [29], mainly for nonlinear control systems without drift.

Both the CLF approach and the PVF approach apply to general, asymptotically controllable nonlinear systems. Moreover, they both guarantee robustness to small additive disturbances, including small additive actuator errors. Input-to-state stabilizability with respect to actuator errors using the CLF approach has been established in [25]. Since the feedback control laws are discontinuous in general (indeed, many asymptotically controllable nonlinear systems do not admit continuous feedback stabilizers), robustness to measurement noise cannot be guaranteed generically. The best results available for a continuous-time implementation of these feedback laws is robustness to small measurement noise that is also small in variation. See, for example, [2, Th. 3.4] where semiglobal, practical robustness is established.

To achieve robustness to small measurement noise without a condition on its variation, the main approach, both for CLF- and PVF-based designs, is to implement the feedback using a sample-and-hold mechanism. This method, or the closely related idea of considering only "sampling solutions," is used in [36], [6], [19], [25], and [3]. When using sample and hold, the sampling period should not be too large, in order to preserve the stabilizing properties of the feedback, yet it should not be too small, in order to guarantee robustness to measurement noise. An alternative approach has been developed in [30] as an extension of the PVF methodology: switching from one patch to another in a PVF is implemented using a hysteresis-based strategy, resulting in what is called a "hybrid patchy vector field." The results in [30] extend the ideas in [31] from nonholonomic systems in chained form to general asymptotically controllable nonlinear systems. The solution notion used in [30] is called an "Euler solution" and corresponds to the limit of sampling solutions of a corresponding differential equation with hysteresis.

The closed-loop control system generated in [30] is hybrid in nature, rather than just a differential equation with a discontinuous right-hand side. This is due to the presence of a discrete state variable, part of the hysteresis mechanism, that makes jumps at certain locations depending on its own value and the value of the control system state. The benefits of hybrid control have been recognized in the nonlinear control literature for some time; see [23], [17], and also [16], [28], [31],

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and [30], where robustness of different types of hybrid feedbacks to unknown parameters or unmodeled dynamics was addressed, and [32] and [33], where quasi-optimal robust stabilization was achieved by means of hybrid feedbacks. Nevertheless, a general theory of hybrid systems—those arising from the implementation of hybrid feedbacks, those describing dynamical systems with different modes of operation (say a thermostat), and those where "continuous" variables may jump (say a bouncing ball)—is not yet complete. Various analysis frameworks have been proposed—see, for example, [37], [26], [39], and [24]—without much stress on robustness analysis.

The framework outlined in [12] and studied in [13] is motivated by the pursuit of robustness in hybrid control systems. It stresses some basic regularity of the data of a hybrid system, suggests relying on "generalized solutions" [35] for robustness analysis, and shows that when the basic regularity is present, asymptotic stability for *compact sets* is robust in a semiglobal, practical sense, to small, persistent perturbations. A result on global robustness to small state-dependent perturbations was established in [5]. These facts reflect what has long been appreciated for continuous time systems (see [8] and [22]) where generalized solutions of Filippov [11] or Krasovskii [20] or limits of solutions under vanishing noise (see [15] and [14]) are used for similar purposes, and also in discrete-time systems [18]. In a sense, the results of [13] change the question of robust stabilization of a nonlinear system to the following: Can a stabilizing hybrid feedback be built to have the desired regularity structure?

B. Contribution

The current paper is also dedicated to a hysteresis-based implementation of patchy feedbacks for stabilization of asymptotically controllable nonlinear systems. We use hysteresis rather than sample-and-hold since the former leaves the sampling period free to assign for other considerations, including high closed-loop bandwidth, and can be more robust when disturbances force the system to operate somewhat near the boundary of a "patch." This point will be demonstrated via an example in Section II-E2.

The starting point for our hybrid feedback construction will be the stabilizing patchy feedbacks of [1]. A patchy feedback consists of a family of open sets and a corresponding family of constant control values. Based on a patchy feedback, we will build a hybrid feedback that will retain the stabilization property, but will also have the desirable robustness. At the most elementary level, the reason behind the hybrid feedback acquiring robustness to measurement noise is that such feedback keeps track, via a discrete variable, of where (i.e., in which patch) the current continuous variable is; moreover, this discrete variable will not change its evaluation of the current patch until a certain hysteresis threshold is reached. At a more technical level, the stabilization property will be retained by the hybrid feedback as the solutions it generates will match those generated by the patchy feedback of [1] for most initializations of the discrete variable; for other initializations, the trajectories will match after the first jump of the discrete variable. Robustness, besides the reason outlined previously, will then come from the regularity of data of the hybrid feedback.

Our contributions compared to those of [30] are the following: 1) the class of solutions we consider for the stated robustness properties is potentially larger than the class of solutions considered in [30], where only Euler solutions (limits of sampling solutions) are studied, 2) we show that a hybrid patchy feedback can be constructed directly from the patchy feedbacks of [1] rather than requiring a supplemental synthesis, 3) our construction is simpler: we only need two sets of patches here whereas in [30] seven sets were used, and 4) the hybrid feedback here has the regularity properties of [13], making it likely that robust asymptotic stability holds, as suggested by the results for asymptotically stable compact sets in [13]. Unfortunately, the patchy feedbacks of [1] do not necessarily have a finite number of patches (see the example in Section II-F) and, in turn, the overall closed-loop system (including the logic state) in our closed-loop does not have an asymptotically stable compact set.

This brings us to the second key result of this paper, which extends the robust stability result of [13]. We will show that *any stabilizing hybrid feedback, with a discrete logic variable, meeting some basic regularity assumptions is robust, even if the logic variable does not converge to a finite set*. This is, to our knowledge, the first general result addressing robustness of such hybrid feedbacks; in [17], [16], [28], [31], and [30], robustness was established for the particular hybrid feedbacks at hand. Preliminaries to this result will include translating some of the work of [13] to the setting of hybrid systems resulting from closing the loop with hybrid feedback having a discrete variable taking values in a possibly infinite index set.

C. Outline

In Section II, we present some examples to motivate hysteresis-based patchy feedback, as an alternative to sample-and-hold discontinuous feedback. Our main technical results begin in Section III, which introduces the basic concepts like that of a hybrid feedback and of a solution to a hybrid system, and states the main result of this paper, Theorem 3.7. Section IV states and proves our second key result: the general robustness property for a wide class of hybrid systems. Section V proves Theorem 3.7, and includes intermediate results, for example, on the existence of solutions to hybrid systems under time-dependent measurement noise.

II. Patchy Feedback: Discontinuous Versus Hybrid (and Sampling Versus Hysteresis)

Consider stabilizing the two point set $\mathcal{A} = \{0, 2\pi\}$ for the control system $\dot{x} = u$ defined on the line. We approach this problem from the point of view of patchy vector fields. We start with a discontinuous feedback and point out the lack of robustness to measurement noise. Then, we move to hybrid feedbacks in pursuit of robustness to measurement noise. First, we use sample and hold, then we use hysteresis. This example will serve as the basis for a more involved problem where the differences between sample-and-hold and hysteresis become more significant.

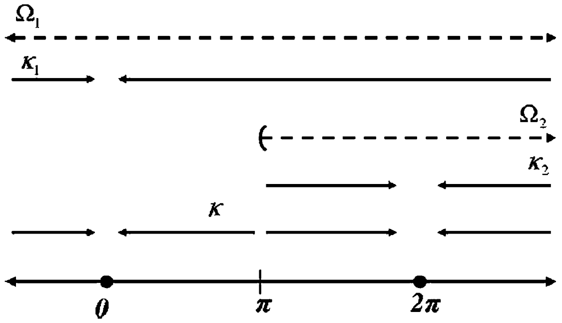

Fig. 1. Patchy feedback to (not robustly) globally asymptotically stabilize the two point set $\{0, 2\pi\}$.

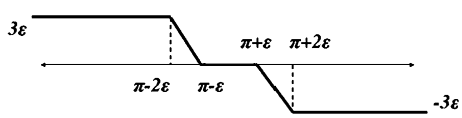

Fig. 2. Function that generates arbitrarily small, destabilizing measurement noise.

A. Nonhybrid Patchy Feedback: Lack of Robustness to Measurement Noise

Define the index set $\mathcal{Q} = \{1, 2\}$, two open patches that cover the state space $\Omega_1 = (-\infty, \infty), \Omega_2 = (\pi, \infty)$, and corresponding feedback control laws $\kappa_1(x) = -kx$ and $\kappa_2(x) = -k(x - 2\pi)$ where $k > 0$. In general, $\kappa_\alpha(x) = -kx + 2k(\alpha - 1)\pi$. The patches and associated flow are indicated at the top of Fig. 1. Then, define the patchy vector field (cf. Definition 5.1) by

$$\kappa(x) = \kappa_\alpha(x), \qquad \text{if} \quad x \in \Omega_\alpha \setminus \bigcup_{\beta > \alpha} \Omega_\beta. \tag{1}$$

In other words, $\kappa(x) = -kx$ if $x \le \pi$ and $\kappa(x) = -kx + 2k\pi$ if $x > \pi$. This function and the associated flow of the differential equation are depicted near the bottom of Fig. 1. Coincidentally, this is also the structure of the feedback that would be generated from the techniques presented in [7] from the CLF $V(x) := |x|$.

The feedback $u = \kappa(x)$ results in global exponential stabilization of $\mathcal{A}$ when considering Euler solutions (for more details, see [7]) but does not result in global stabilization if considering generalized solutions as in [11]. Equivalently (see [15] and [14]), the asymptotic stability is not robust to arbitrarily small measurement noise. This is seen by considering solutions to the feedback system $\dot{x} = \kappa(x + e_\varepsilon(x))$, where the function e_ε is plotted in Fig. 2. This system has the interval $[\pi - 2\varepsilon, \pi + 2\varepsilon]$ forward invariant.

B. Hybrid (Sample-and-Hold-Based) Patchy Feedback for Robustness to Measurement Noise

As observed in the work referenced in the Introduction, the asymptotic stability induced by the feedback in (1) is robust to sufficiently small measurement noise when "sampling solutions" are considered. The size of the tolerated measurement noise would be related to the sampling period used. Sampling solutions of an ordinary differential equations can also be viewed as (regular) solutions of a hybrid system. We will take that approach here to prepare for the general notion of solution to a hybrid system that will be formally defined in Section III-A. Consider the hybrid system

$$\begin{cases} \dot{x} = u \quad \dot{u} = 0 \quad \dot{\tau} = 1, & \tau \in [0, T] \\ x^+ = x \quad u^+ = \kappa(x) \quad \tau^+ = 0, & \tau = T \end{cases}. \tag{2}$$

A "solution" to this system is a function $(t, j) \mapsto (x(t, j), u(t, j), \tau(t, j))$, defined on the set $\bigcup_{j=0}^{\infty}([jT, (j+1)T], j)$ (cf. Definition 3.1), that, for fixed j, is continuously differentiable and satisfies the differential equation in (2) and makes the jumps indicated by the jump map in (2) that relates the solution at $((j+1)T, j+1)$ to the solution at $((j+1)T, j)$. Cf. Definition 3.2. By concatenating flows and jumps, we get

$$x((j+1)T, j+1)) = x(jT, j) + T\kappa(x(jT, j))$$

and then, from the form of κ, that the set $\mathcal{A}$ is exponentially stable as long as $T < 2/k$. It can also be verified that if the measurement noise has magnitude less than $Tk\pi$ then it is not possible for the control system to get stuck near $x = \pi$. (Note that $Tk\pi < 2\pi$.) Otherwise, measurement noise can keep the solution near $x = \pi$, i.e., forever away from the target set.

C. Hybrid (Hysteresis-Based) Patchy Feedback for Robustness to Measurement Noise: Pass I

To develop a hysteresis-based hybrid patchy feedback, we introduce "inflations" of the patches Ω_1 and Ω_2: Define $\Omega'_1 := \Omega_1 = (-\infty, \infty)$. Let $\lambda_2 \in (0, 1)$ and define $\Omega'_2 := (\lambda_2\pi, \infty) \supset (\pi, \infty) = \Omega_2$. These sets are plotted as dashed intervals in Fig. 3. Then, define [cf. (21)] $C_q = \overline{\Omega'_q} \setminus \bigcup_{\beta > q} \Omega_\beta, D_q = \bigcup_{\beta > q} \overline{\Omega_\beta} \cup (\mathbb{R} \setminus \Omega'_q)$, i.e.,

$$\begin{aligned} &C_1 = (-\infty, \pi] \quad D_1 = [\pi, \infty) \\ &D_2 = (-\infty, \lambda_2\pi] \quad C_2 = [\lambda_2\pi, \infty) \end{aligned} \tag{3}$$

and

$$G_q(x) = \begin{cases} \left\{\beta \in Q \,\middle|\, \begin{matrix} x \in \overline{\Omega_\beta} \\ \beta > q \end{matrix}\right\}, & x \in \bigcup_{\beta > q} \overline{\Omega_\beta} \cap \Omega'_q \\ \{\beta \in Q \,|\, x \in \overline{\Omega_\beta}\}, & x \in \mathbb{R} \setminus \Omega'_q \end{cases}$$

i.e.,

$$G_1(x) = \begin{cases} 2, & x \in [\pi, \infty) \\ \emptyset, & x \in (-\infty, \pi) \end{cases}$$
$$G_2(x) = \begin{cases} 1, & x \in (-\infty, \lambda_2\pi] \\ \emptyset, & x \in (\lambda_2\pi, \infty) \end{cases}.$$

Now, consider the hybrid system

$$\begin{cases} \dot{x} = \kappa_q(x), & x \in C_q \\ q^+ \in G_q(x), & x \in D_q \end{cases}.$$

Solutions (cf. Definitions 3.1 and 3.2) are again functions $(t, j) \mapsto (x(t, j), q(t, j))$ and these functions are such that q is constant as a function of t for fixed j, and x is constant in j for fixed t. The changes in x, which are allowed when $x \in C_q$, are governed by the differential equation $\dot{x} = \kappa_q(x)$ while the changes in q, which are allowed when $x \in D_q$ are governed by the update rule $q^+ \in G_q(x)$. The overall effect is to create a control law with hysteresis, as depicted near the bottom of Fig. 3. It can be verified that the set $\mathcal{A} = \{0, 2\pi\}$ is globally

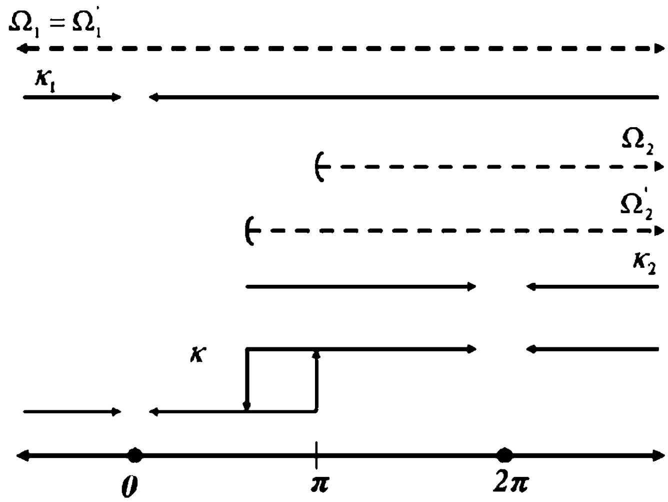

Fig. 3. Hybrid (hysteresis-based) patchy feedback to robustly globally asymptotically stabilize the two point set $\{0, 2\pi\}$.

exponentially stable for this system and that the solutions in the presence of measurement noise do not cause the variable q to oscillate unless the measurement error reaches the level $(1-\lambda_2)\pi$. In fact, this level can be made arbitrarily close to 2π (as is the case for the sample-and-hold solution) by making $C_1 = (-\infty, \lambda_1\pi]$ and $D_1 = [\lambda_1\pi, \infty)$ with $\lambda_1 \in [1, 2)$ and then noting that the noise level needed to create undesirable oscillations in q is $(\lambda_1 - \lambda_2)\pi$.

D. Hybrid (Hysteresis-Based) Patchy Feedback for Robustness to Measurement Noise: Pass II

The ideas discussed here correlate with the general derivation in Sections V-C and Sections V-D.

In the system of Section II-C, the solutions with sufficiently small measurement noise are well behaved when they exist. However, the existence of solutions is not guaranteed for all sufficiently small measurement noise in general. Existence problems occur for some arbitrarily small noise signals when an initial condition x_0 is at the boundary of C_1 or C_2. Then, the noise may cause the measurement to be in the flow set at the initial time but not in the flow set for any positive time. This makes it impossible to jump at the initial time and also makes it impossible to flow from the initial time. There are a couple of ways to guarantee existence of solutions. One is to sample the measurements. Here, there is no need to keep the sampling period away from zero for the sake of robustness, unlike the case without hysteresis. The other alternative is to endow the hysteresis element with a robust existence property. This can be done by inflating the flow sets so that the flow and jump sets overlap. Then, even under small measurement noise, x_0 is either in the jump set, or in the interior of the flow set, and solutions do exist.

In the context of the previous example, relative to (3), we take $C_1 = (-\infty, \lambda_{c1}\pi], \lambda_{c1} \in (1, 2)$, while leaving D_1 unchanged, and we take $C_2 = [\lambda_{c2}2\pi, \infty), \lambda_{c2} \in (0, \lambda_2)$, while leaving D_2 unchanged. This results in the "robust" hysteresis element shown in Fig. 4. The new system still has the set $\mathcal{A} = \{0, 2\pi\}$ globally exponentially stable, and is robust to small measurement noise, both in terms of existence and asymptotic behavior.

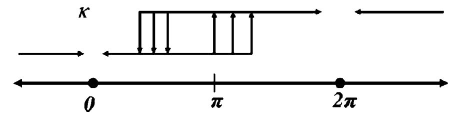

Fig. 4. Hybrid (hysteresis-based) patchy feedback to robustly globally asymptotically stabilize the two point set $\{0, 2\pi\}$. Robust behavior and robust existence are guaranteed.

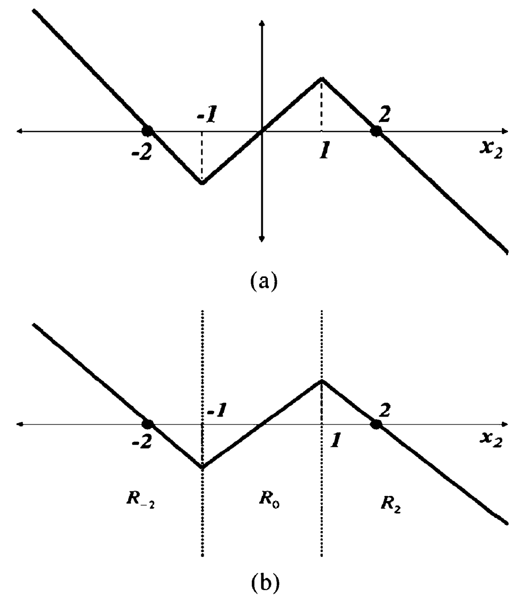

Fig. 5. Graphical description of the data in the disturbance attenuation problem. (a) Function "PWA" in (4). (b) Corresponding piecewise-affine regions.

E. Other Problems

1) Stabilization on the Circle and Artstein's Circles: The previous discussion applies also to the problem of stabilizing a point on the circle, by identifying 2π with 0, and also, to the family of circles given by Artstein's example. This is not done here due to space limitations.

2) Disturbance Attenuation Problem: For the planar system

$$\begin{aligned} \dot{x}_1 &= \mathrm{PWA}(x_2) + d \\ \dot{x}_2 &= u \end{aligned} \tag{4}$$

where the function PWA is plotted in Fig. 5(a) consider stabilizing the three point set $\mathcal{A} = \{(0,-2), (0,0), (0,2)\}$ and guaranteeing a zero steady-state output $y = x_1$ in the presence of constant disturbances d. We start with a discontinuous feedback that is piecewise affine in x_2 on the three regions $\mathcal{R}_q, q = -2, 0, 2$, indicated in Fig. 5(b). On the region $\mathcal{R}_q$, it is given as

$$\begin{aligned} \dot{\xi} &= x_1 \\ \kappa_q(x) &= -k_2(x_2 - q + \rho_q \mathrm{sat}(k_1 x_1 + k_\xi \xi)) \end{aligned} \tag{5}$$

where $\rho_0 = 1$ and $\rho_q = -1$ for $q \in \{-2, 2\}$ and $\mathrm{sat}(s) = \min\{1, \max\{-1, s\}\}$, i.e., the standard satura-

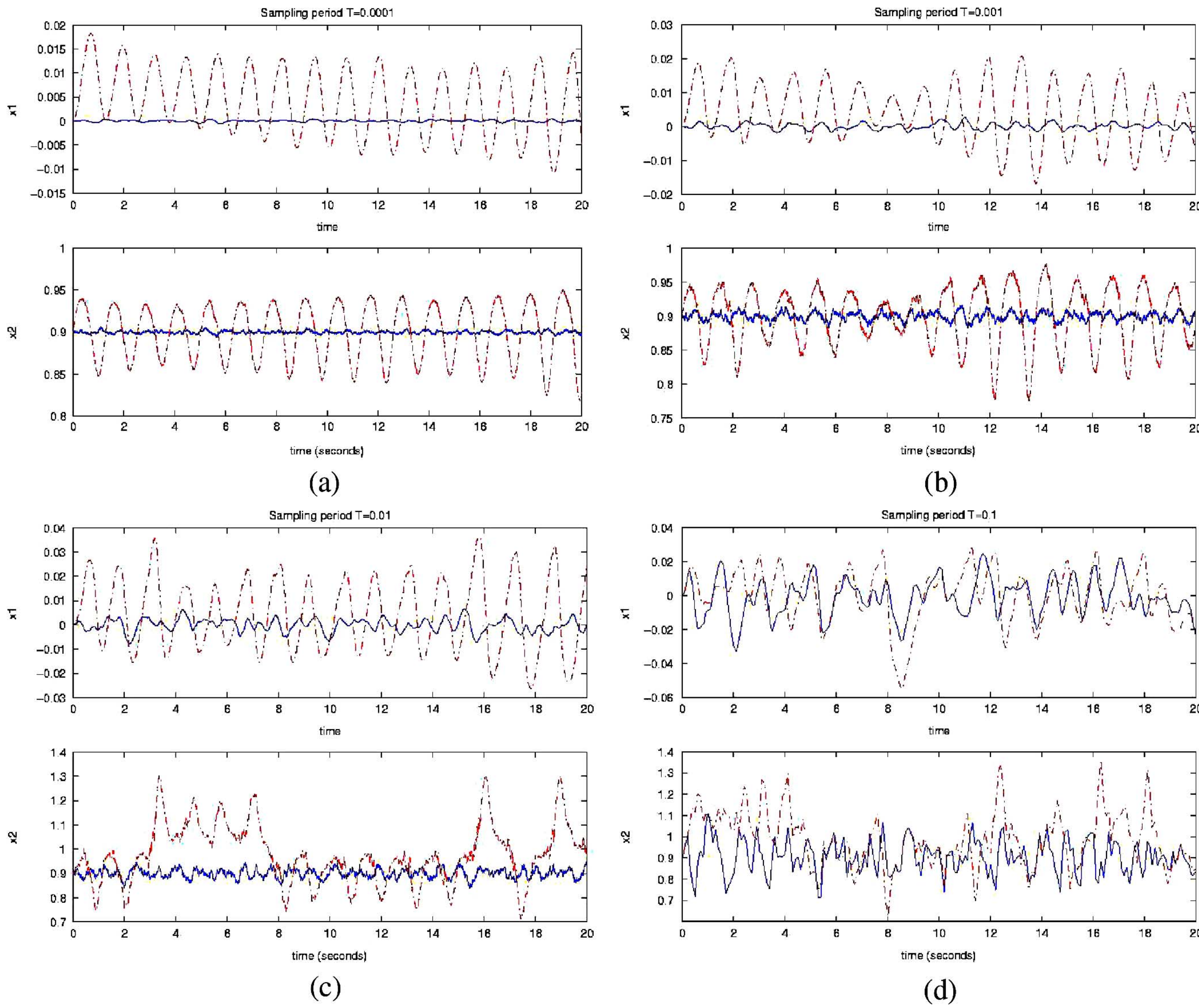

Fig. 6. Sample-and-hold comparison of PWA feedback (dashed–dotted curves) to hysteresis-based PWA feedback (solidmcurves) for the system (4) and (5). In each subplot, the upper plot is x_1 while the lower plot shows x_2. (a) Sampling period $T = 0.0001$ (b) Sampling period $T = 0.001$ (c) Sampling period $T = 0.01$ (d) Sampling period $T = 0.1$.

tion function. The motivation for the saturation in the feedback law is to make each region $\mathcal{R}_q$ forward invariant. The resulting closed-loop right-hand side is piecewise affine in (x, ξ). Unfortunately, there is no robustness to arbitrarily small measurement noise at the boundary of $\mathcal{R}_0$. Some robustness can be induced by sampling the feedback. Nevertheless, when the disturbance causes x_2 to operate near the boundary of $\mathcal{R}_0$, small measurement noise can still have a destabilizing effect. The dashed–dotted curves in Fig. 6(a)–(d) show the behavior of the closed-loop system with $k_2 = 5, k_1 = 10.1, k_\xi = 1, d = -0.9, x_1(0) = 0, x_2(0) = 0.9$, and $\xi(0) = -0.9$ with random noise on the measurement of x_2, generated uniformly over the interval $[-0.2, 0.2]$, using sample-and-hold with the sampling periods $T = 0.0001, 0.001, 0.01, 0.1$. An alternative response comes from implementing the feedback with hysteresis. The solid curves in Fig. 6(a)–(d) show the behavior using the same parameters, initial conditions, noise and sampling periods, and the control law given in (5) but with q implemented using hysteresis. Since sampling is not needed for robustness to measurement noise when using hysteresis, the fastest available sampling period can be used. None of the sample-and-hold implementations without hysteresis rival the responses with hysteresis for $T \leq 0.01$.

3) Obstacle Avoidance Problem: Another problem where (hybrid) patchy vector fields appear naturally is in the stabilization with obstacle avoidance problem for a planar kinematic vehicle modeled by $\dot{x} = u$ with x a vector in the plane. The obstacles are indicated by the closed, shaded regions in Fig. 7(a), while the target point is indicated by the dot in the middle of the figure. Five different patches have been identified and the flow directions, which are indicated in the figure, move the vehicle from low patch numbers to high patch numbers. Within the highest numbered patch (patch five) the flow is toward the target. The patchy vector field is transformed into a hybrid patchy vector field by introducing "robust" hysteresis elements as indicated in Fig. 7(b). Note that there are two distinct type of

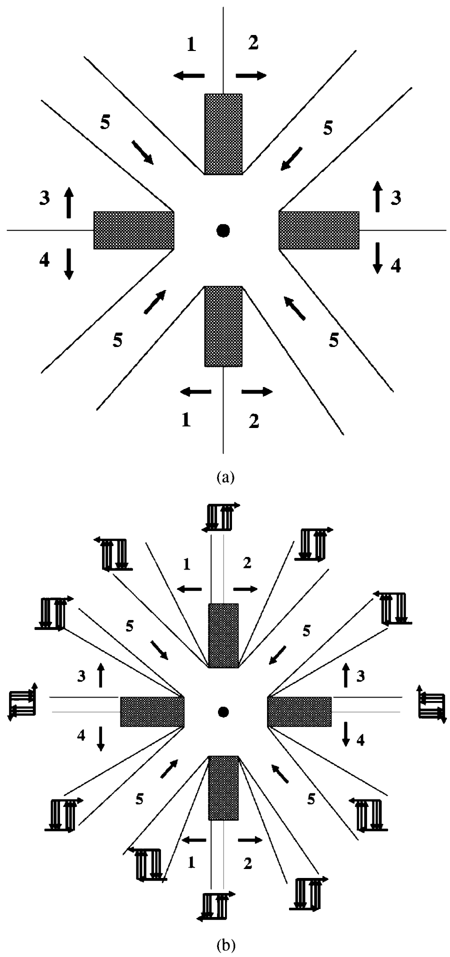

Fig. 7. Stabilization with robust obstacle avoidance. (a) Patchy feedback to (nonrobustly) stabilize the origin while avoiding obstacles. The patches are labeled 1–5. (b) Hybrid patchy feedback to robustly stabilize the origin while avoiding obstacles. Robust hysteresis elements, like in Fig. 4, are introduced at the patch boundaries.

hysteresis elements in the figure. The hysteresis at the boundary of patches one and two, and the hysteresis at the boundary of patches three and four are of the type described previously. They make robust the switching between competing flow directions. The hysteresis elements at the boundary of patch five enable switching between flows that are, for the most part, cooperative. They are introduced here to avoid the presence of discontinuities in the functions that define the closed-loop system. The resulting hybrid control system robustly steers the vehicle to the target while avoiding obstacles.

F. Infinite Number of Patches May be Required

Consider the scalar control system

$$\dot{x} = -x(|(x,u)|_S + \min\{0, |x| - 2|u|\}) \tag{6}$$

where

$$S := \{(x,u) : x = e^t \sin(t), u = e^t \cos(t), t \in \mathbb{R}\} \cup \{0,0\}.$$

It is asymptotically controllable to the origin as for each x there exists u such that $|u| \leq 0.5|x|$ and $|(x,u)|_S > 0$. However, there is no continuous stabilizer for this system, on any arbitrarily small neighborhood of 0. This is because, in order to have $xf(x,u) < 0$, it must be the case that $|u| \leq |x|$ since $|(x,u)|_S \leq |u| + |x|$. However then, for each $b > 0$, there exists $a > 0$ such that every path connecting a point $(b, u_b) \notin S$ with $|u_b| \leq 2b$ to a point $(a, u_a) \notin S$ with $|u_a| \leq 2a$ must necessarily pass through S and at such a point $xf(x,u) \geq 0$. For this reason, a patchy vector field to stabilize the origin of this system must have an infinite number of patches.

III. Prerequisites and the Main Result

Throughout this section, $\tilde{O} \subset \mathbb{R}^n$ is an open set, $\mathcal{A} \subset \tilde{O}$ is compact, and $O = \tilde{O} \backslash \mathcal{A}$. We will work with a nonlinear system defined on $\tilde{O}$ which is asymptotically controllable to $\mathcal{A}$. The set O will appear as the state space for the continuous variable of the hybrid systems we analyze.

A. Hybrid Systems and Their Solutions

Hybrid systems of our interest, in particular, those resulting from augmenting the state variable in a nonlinear system to include a discrete variable, and "closing the loop with the hybrid feedback," can be described informally as

$$\begin{cases} \dot{x} \in F_q(x), & x \in C_q \\ q^+ \in G_q(x), & x \in D_q \end{cases} \tag{7}$$

where $Q \subset \mathbb{Z}^{n_q}$ and for each $q \in Q$, F_q and G_q are (potentially set-valued) mappings and C_q and D_q are subsets of O. A hybrid system with the structure as in (7) will be denoted by $\mathcal{H}$.

Note that for $\mathcal{H}$, the continuous variable x only changes continuously and does not jump, while the discrete variable q only changes value via jumps. The sets C_q, respectively D_q, describe the sets in which the continuous variable of the hybrid system can flow, respectively, the set that enables the jump of the discrete variable. To make the definition of a solution to such a system precise, we recall some concepts from [13].

Definition 3.1: A subset $S \subset \mathbb{R}_{\geq o} \times \mathbb{N}$ is a *compact hybrid time domain* if $S = \bigcup_{j=0}^{J-1}([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \cdots \leq t_J$. S is a *hybrid time domain* if for all $(T, J) \in S$, $S \cap ([0,T] \times \{0, 1, \ldots J\})$ is a compact hybrid domain; equivalently, if S is a union of a finite or infinite sequence of intervals $[t_j, t_{j+1}] \times \{j\}$, with the last interval, if it exists, possibly of the form $[t_j, T)$ with T finite or $T = +\infty$.

In what follows, we will write $\sup_t(S)$ for the supremum of all t such that $(t,j) \in S$ for some j, and $\sup_j(S)$ for the supremum of all j such that $(t,j) \in S$ for some t.

Definition 3.2: A solution to the hybrid system (7) consists of a nonempty hybrid time domain S, a function $x : S \to O$ where $x(t,j)$ is locally absolutely continuous in t for a fixed j and constant in j for a fixed t over $(t,j) \in S$, and a function $q : S \to Q$ such that $q(t,j)$ is constant in t for a fixed j over $(t,j) \in S$, meeting the conditions: $x(0,0) \in C_{q(0,0)} \cup D_{q(0,0)}$ and

$S1$) for all $j \in \mathbb{N}$ and almost all t such that $(t,j) \in S$

$$\dot{x}(t,j) \in F_{q(t,j)}(x(t,j)) \quad x(t,j) \in C_{q(t,j)};$$

$S2$) for all $(t,j) \in S$ such that $(t,j+1) \in S$

$$q(t,j+1) \in G_{q(t,j)}(x(t,j)), \qquad x(t,j) \in D_{q(t,j)}.$$

Given a solution to (7), we will not mention the hybrid time domain explicitly, but will identify the solution by (x,q), and when needed, refer to the associated domain by $\mathrm{dom}(x,q)$. A solution to (7) is called *maximal* if it cannot be extended, and *complete* if its domain is unbounded. A complete solution (x,q) may have $\sup_t \mathrm{dom}(x,q) < \infty$, in which case the discrete variable q must jump infinitely many times.

As the asymptotically controllable to $\mathcal{A}$ nonlinear systems we consider are given on $\tilde{O}$ while the hybrid feedbacks, and resulting hybrid systems, will be defined on O (or rather, on $O \times Q$), we need to allow for solutions of the hybrid system to "reach $\mathcal{A}$" in finite time. Hence, we work with the following definitions of asymptotic stability.

Definition 3.3:

- The set $\mathcal{A}$ is *stable* for the hybrid system $\mathcal{H}$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that any solution (x,q) to (7) with $\mathrm{dist}_{\mathcal{A}}(x(0,0)) \le \delta$ satisfies $\mathrm{dist}_{\mathcal{A}}(x(t,j)) \le \varepsilon$ for all $(t,j) \in \mathrm{dom}(x,q)$.
- The set $\mathcal{A}$ is *attractive* for the hybrid system $\mathcal{H}$ if there exists $\delta > 0$ such that
  - — for any $(x_0,q_0) \in \mathcal{A} \times Q$ with $\mathrm{dist}_{\mathcal{A}}(x_0) \le \delta$ there exists a solution to $\mathcal{H}$ with $(x,q)(0,0) = (x_0,q_0)$;
  - — for any maximal solution (x,q) to $\mathcal{H}$ with $\mathrm{dist}_{\mathcal{A}}(x(0,0)) \le \delta$ we have $\mathrm{dist}_{\mathcal{A}}(x(t,j)) \to 0$ as $t \to \sup_t(\mathrm{dom}(x,q))$.
- If $\mathcal{A}$ is *asymptotically stable*, i.e., both stable and attractive, its *basin of attraction*, denoted $B_{\mathcal{A}}$, is the set of all $x_0 \in O$ such that for all $q_0 \in Q$, there exists a solution to $\mathcal{H}$ with $x(0,0) = x_0, q(0,0) = q_0$, and any such solution that is also maximal satisfies $\mathrm{dist}_{\mathcal{A}}(x(t,j)) \to 0$ as $t \to \sup_t(\mathrm{dom}(x,q))$.
- $\mathcal{A}$ is *(globally) asymptotically stable* on O if it is asymptotically stable and $B_{\mathcal{A}} = O$.

B. Main Result: Hybrid Feedback for an Asymptotically Controllable System

For the open set $\tilde{O}$ and the compact $\mathcal{A} \subset \tilde{O}$, consider a compact set of feasible controls $U \subset \mathbb{R}^m$, a smooth function $f : \tilde{O} \times U \to \mathbb{R}^n$, and a nonlinear system

$$\dot{x}(t) = f(x(t), u(t)), \qquad u(t) \in U, \qquad \text{for all } t \ge 0. \quad (8)$$

The system (8) is *asymptotically controllable* on $\tilde{O}$ to $\mathcal{A}$ if:

1) for each $x^0 \in \tilde{O}$, there exists a measurable $u_{x^0} : [0,\infty) \to \mathbb{R}^m$ with $u_{x^0}(t) \in U$ for almost all t such that the maximal solution x to (8) with u replaced by u_{x^0} is complete and such that $\lim_{t\to\infty} \mathrm{dist}_{\mathcal{A}}(x(t)) = 0$;
2) for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x^0 \in \tilde{O}, \mathrm{dist}_{\mathcal{A}}(x^0) < \delta$, there is u_{x^0} as in 1) so that the resulting solution x satisfies $\mathrm{dist}_{\mathcal{A}}(x(t)) < \varepsilon$ for all $t \ge 0$.

Definition 3.4: A hybrid feedback on O consists of the following:

- a totally ordered discrete set Q;
- for each $q \in Q$
  - — sets $C_q \subset O$ and $D_q \subset O$;
  - — a function $k_q : C_q \to U$;
  - — a set-valued mapping $g_q : D_q \rightrightarrows Q$.

The function k_q, in closed loop with (8), determines how the continuous variable flows, while g_q describes how the discrete variable jumps. We say that the *hybrid feedback renders $\mathcal{A}$ asymptotically stable on O for* (8) if $\mathcal{A}$ is asymptotically stable on O for the hybrid system $\mathcal{H}_{\mathrm{feed}}$ resulting from setting $F_q(x) = f(x, k_q(x))$ and $G_q(x) = g_q(x)$.

In what follows, *admissible measurement noise* and *admissible external disturbance* are functions ξ and ζ in $\mathcal{L}^\infty_{\mathrm{loc}}(O \times \mathbb{R}_{\ge o} \times \mathbb{N}; \mathbb{R}^n)$ that are continuous in $x \in O$ for each $(t,j) \in \mathbb{R}_{\ge o} \times \mathbb{N}$. The more standard noise and disturbance dependent only on x and t fits this framework (and is constant in j); we do allow for the noise and disturbance to change during jumps. As noted in [21, Remark 1.4], with the presence of ζ and the continuity of f in u, we can omit any explicit reference to actuator errors. Due to the discrete nature of the set Q, we do not consider noise or disturbances "for the discrete variable." Measurement noise and external disturbances lead to a time-varying hybrid system that can be represented by

$$\begin{cases} \dot{x} \in f(x, k_q(x+\xi)) + \zeta, & x+\xi \in C_q \\ q^+ \in g_q(x+\xi), & x+\xi \in D_q \end{cases}. \quad (9)$$

We will denote this system by $\mathcal{H}^{\xi,\zeta}_{\mathrm{feed}}$. Solutions to this system are defined as follows.

Definition 3.5: A solution to $\mathcal{H}^{\xi,\zeta}_{\mathrm{feed}}$ consists of a nonempty hybrid time domain S, a function $x : S \to O$ such that $x(t,j)$ is locally absolutely continuous in t for a fixed j and $(t,j) \in S$, and a function $q : S \to Q$ such that $q(t,j)$ is constant in t for a fixed j and $(t,j) \in S$ meeting the following conditions: $x(0,0) \in C_{q(0,0)} \cup D_{q(0,0)}$ and

S_p1) for all $j \in \mathbb{N}$ and almost all t such that $(t,j) \in S$

$$\begin{aligned} \dot{x}(t,j) &\in f\big(x(t), k_{q(t,j)}(x(t,j) \\ &\qquad + \xi(x(t,j),t,j)) + \zeta(x(t),t,j) \\ x(t,j) &\quad + \xi(x(t,j),t,j) \in C_{q(t,j)}; \end{aligned}$$

S_p2) for all $(t,j) \in S$ such that $(t,j+1) \in S$

$$\begin{aligned} q(t,j+1) &\in g_{q(t,j)}(x(t,j) + \xi(x(t,j),t,j)) \\ x(t,j) &\quad + \xi(x(t,j),t,j) \in D_{q(t,j)}. \end{aligned}$$

Given ξ and ζ, we will say that $\mathcal{A}$ is asymptotically stable on O for the system $\mathcal{H}^{\xi,\zeta}_{\mathrm{feed}}$ if the definition of asymptotic stability of $\mathcal{H}$ is met with solutions to $\mathcal{H}$ replaced by those to $\mathcal{H}^{\xi,\zeta}_{\mathrm{feed}}$. In the following, by an *admissible perturbation radius*, we mean any continuous function $\rho : O \to \mathbb{R}_{>o}$ such that $x + \rho(x)\mathbb{B} \subset O$ for all $x \in O$. ($\mathbb{B}$ is the closed unit ball in $\mathbb{R}^n$.)

Definition 3.6: A hybrid feedback on O renders $\mathcal{A}$ asymptotically stable on O for (8), robustly to measurement noise, actuator errors, and external disturbances if there exists an admissible perturbation radius $\delta : O \to \mathbb{R}_{>o}$ such that for all admissible measurement noise ξ and admissible external disturbance ζ satisfying

$$\begin{aligned} \|\xi(x,t,j)\| &\le \delta(x) \\ \|\zeta(x,t,j)\| &\le \delta(x) \end{aligned}, \qquad \text{for all } x \in O, (t,j) \in \mathbb{R}_{\ge o} \times \mathbb{N} \quad (10)$$

$\mathcal{A}$ is asymptotically stable on O for the hybrid system $\mathcal{H}^{\xi,\zeta}_{\text{feed}}$.

We are now ready to state the promised result.

Theorem 3.7: If (8) is asymptotically controllable on $\tilde{O}$ to $\mathcal{A}$, then there exists a hybrid feedback on O with $Q \subset \mathbb{Z}$ with the standard ordering that renders $\mathcal{A}$ asymptotically stable on O for the system (8), robustly to measurement noise, actuator errors, and external disturbances.

IV. Key Structural Property of Hybrid Systems—Robustness

We will now show that asymptotic stability for hybrid systems (7) whose data has some elementary regularity property is robust (see Definition 4.1) to autonomous perturbations.

Definition 4.1: The set $\mathcal{A}$ is asymptotically stable on O for the system $\mathcal{H}$, robustly to autonomous perturbations, if there exists an admissible perturbation radius ρ such that, for the system $\mathcal{H}^\rho$ given by

$$\begin{cases} \dot{x} \in F_q^\rho(x), & \dot{q} = 0, \quad x \in C_q^\rho \\ q^+ \in G_q^\rho(x), & x^+ = x, \quad x \in D_q^\rho \end{cases} \tag{11}$$

with the data

$$\begin{aligned} F_q^\rho(x) &:= \text{con}F_q((x+\rho(x)\mathbb{B}) \cap C_q) + \rho(x)\mathbb{B} \\ G_q^\rho(x) &:= G_q((x+\rho(x)\mathbb{B}) \cap D_q) \\ C_q^\rho &:= \{x \in O \,|\, (x+\rho(x)\mathbb{B}) \cap C_q \neq \emptyset\} \\ D_q^\rho &:= \{x \in O \,|\, (x+\rho(x)\mathbb{B}) \cap D_q \neq \emptyset\} \end{aligned} \tag{12}$$

the set $\mathcal{A}$ is asymptotically stable on O.

About $\mathcal{H}$, we will assume the following:

$A0$) $\tilde{O} \subset \mathbb{R}^n$, $\mathcal{A} \subset \tilde{O}$ is compact, $O = \tilde{O} \setminus \mathcal{A}$, and $Q \subset \mathbb{Z}^{n_q}$;

and that for all $q \in Q$

A_q1) C_q and D_q are relatively closed subsets of O;

A_q2) $F_q : O \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded, and $F_q(x)$ is nonempty and convex for all $x \in C_q$;

A_q) $G_q : O \rightrightarrows Q$ is outer semicontinuous and locally bounded, and $G_q(x)$ is nonempty for all $x \in D_q$.

The (set-valued) mapping $F_q : O \rightrightarrows \mathbb{R}^n$ is *outer semicontinuous* (osc) if for any convergent sequences $x_i \to x \in O$ and $y_i \in F_q(x_i)$ with $y_i \to y$, one has $y \in F(x)$; similarly for G_q. The mapping F_q is *locally bounded* if for each compact $K \subset O$, there exists a compact $K' \subset \mathbb{R}^n$ so that $F(x) \subset K'$ for all $x \in K$; similarly for G_q. (An osc and locally bounded set-valued mapping has compact values and is "upper semicontinuous." For details, see [34, ch. 5].) The properties of the data of $\mathcal{H}$ stated previously reflect those suggested by [12] and explored in [13], for the purposes of guaranteeing a key property of the set of solutions to $\mathcal{H}$: that an appropriately understood limit of a sequence of solutions is still a solution, and this holds also if the sequence is generated under (vanishing) errors. This property in turn can be used to show another important for robustness analysis fact: solutions generated under perturbations are close, in an appropriate sense (related to graphical convergence), to solutions to the nominal system.

Here, we will also need the additional assumptions.

$A4$) The family $\{C_q\}_{q \in Q}$ forms a locally finite covering of O.

$A5$) The mappings $G_q : O \rightrightarrows Q$ are locally bounded in x uniformly in q.

$A6$) For all $q \in Q$, $C_q \cup D_q = O$.

The family $\{C_q\}_{q \in Q}$ is a locally finite covering of O if $O = \bigcup_{q \in Q} C_q$ and for any compact $K \subset O$, finitely many qs intersect K. Local boundedness of G_q's uniform in q means that for each compact set $K \subset O$ there exists a compact $K' \subset Q$ such that $G_q(K) \subset K'$ for all $q \in Q$.

Remark 4.2: The significance of $A6$) is that, in presence of A_q1), A_q2), and A_q3) for all $q \in Q$, it guarantees that from any initial condition in $O \times Q$, there exists a (nontrivial) solution to $\mathcal{H}$. Then, as a direct consequence of [13, Prop. 2.1], we can say the following about maximal solutions: any maximal solution (x,q) to (7) is either complete, or such that $\|x(t,j)\| \to \infty$ or $x(t,j) \to \text{bdry}O$ as $t \to \sup_t \text{dom}(x,q)$ (and $j = \sup_j \text{dom}(x,q)$).

We can now state the main result of the section.

Theorem 4.3: Consider the hybrid system $\mathcal{H}$ and assume that $A0$), $A4$), $A5$), and $A6$), and for all $q \in Q$, A_q1), A_q2), and A_q3) hold. If $\mathcal{A}$ is asymptotically stable on O for (7), then it is robustly asymptotically stable.

Robustness results of [13] do not apply directly to the hybrid systems discussed here, partly because we consider convergence of the continuous variable only (which in a bigger framework corresponds to a noncompact attractor), and partly because the attractor is not in the state space (this feature is inherited by our hybrid feedback from the patchy feedback of [1]). Similarly, the results of [5], while addressing the robustness of a noncompact attractor, lead to perturbation bounds depending on the full state of the system, not just on x.

A. Preliminary Results

Throughout this section, we pose the assumption of Theorem 4.3.

Lemma 4.4: Let ρ be any admissible perturbation radius. Then, $\mathcal{H}^\rho$ defined by (11) and (12) satisfies $A4$), $A5$), and $A6$), and for all $q \in Q$, A_q1), A_q2), and A_q3).

Proof: That the conditions A_q1), A_q2), and A_q3), for each $q \in Q$, are satisfied by $\mathcal{H}^\rho$, was shown in [4, Prop. 3.1]. To see $A4$) and $A5$), pick a compact $K \subset O$, note that $K_\rho := \{x + \rho(x)\mathbb{B} \,|\, x \in K\}$ is a compact subset of O, and that $K \cap C_q^\rho \neq \emptyset$ if and only if $K_\rho \cap C_q \neq \emptyset$. Since $\{C_q\}_{q \in Q}$ forms a locally finite cover of O, so does $\{C_q^\rho\}_{q \in Q}$ (this also relies on the fact that $C_q \subset C_q^\rho$). Now, note that $G_q^\rho(K) = G_q(K_\rho)$, and by the local boundedness in x of G_q's uniform in q, there exists a compact $K' \subset O$ so that $G_q(K_\rho) \subset K'$. Thus, the mappings G_q^ρ are locally bounded in x, uniformly in q.

Finally, $C_q \subset C_q^\rho$, $D_q \subset D_q^\rho$, and $C_q \cup D_q = O$ imply $C_q^\rho \cup D_q^\rho = O$, and $A6$) holds for $\mathcal{H}^\rho$. ■

Proposition 4.5: Suppose that $\mathcal{H}$ has no instantaneous Zeno solutions. Then, there exists an admissible perturbation radius ρ such that $\mathcal{H}^\rho$ in (11) and (12) has no instantaneous Zeno solutions.

Proof: Let σ be any admissible perturbation radius. Pick a compact $K \subset O$. Then, there exists $\rho_K > 0$ such that $\mathcal{H}^{\rho_K \sigma}$ has no instantaneous Zeno solutions (x,q) with $x(0,0) \in K$. Indeed, if this was false, there would exist a sequence (x_i, q_i) of instantaneous Zeno solutions to $\mathcal{H}^{\sigma/i}$ with $x_i(0,0) \in K$.

Since G_q are locally bounded in x uniformly in q, there exists a compact $Q_K \subset Q$ such that $q_i(0,j) \in Q_K$ for all $j = 1, 2, \ldots$. Thus, the sequence of (instantaneous Zeno) solutions (x_i', q_i') to $\mathcal{H}^{\sigma/i}$ given by $(x_i', q_i')(0,j) = (x_i, q_i)(0, 1+j)$ for $j = 0, 1, \ldots$ is uniformly bounded. We can thus pick a subsequence converging uniformly on each "hybrid interval" $\{0\} \times [0, 1, \ldots, J]$. The limit of such a subsequence is a solution to $\mathcal{H}$; see [13, Th. 5.1 and 5.4].[1] This limit is also instantaneous Zeno, which violates the assumption.

Now, pick any increasing sequence of compact sets $K_i \subset O$ with $K_0 = \emptyset$ such that $\bigcup_{i=1}^{\infty} K_i = O$. For each $i = 1, 2, \ldots$, let $\rho_{K_i} > 0$ be such that $\mathcal{H}^{\rho_{K_i}\sigma}$ has no instantaneous Zeno solutions on K_i. Find any admissible perturbation radius ρ such that, for $i = 1, 2, \ldots, \rho(x) \leq \rho_{K_i}\sigma(x)$ for all $x \in K_i \setminus K_{i-1}$. Then, $\mathcal{H}^{\rho}$ has no instantaneous Zeno solutions. ■

We call a sequence $\{(x_i, q_i)\}_{i=1}^{\infty}$ of solutions to (7) *locally eventually bounded* if for any $m > 0$ there exists $i_0 > 0$ and a compact $K \subset O \times Q$ such that for all $i > i_0$, all $(t,j) \in \mathrm{dom}(x_i, q_i)$ with $t + j \leq m$ we have $(x_i(t,j), q_i(t,j)) \in K$. Note that $\{(x_i, q_i)\}_{i=1}^{\infty}$ is not locally eventually bounded, in particular, when for some $m > 0$ there exist $(t_i, j_i) \in \mathrm{dom}(x_i, q_i)$ with $t_i + j_i \leq m$ such that the sequence $x_i(t_i, q_i)$ eventually leaves any compact subset of O.

Lemma 4.6: Fix $x^0 \in O$. Let $(x_i, q_i), i = 1, 2, \ldots$, be solutions to (7) such that

$$\lim_{i \to \infty} x_i(0,0) = x^0$$

and let $X \subset O$ be a relatively closed set such that $x_i(t,j) \in X$ for all $(t,j) \in \mathrm{dom}(x_i, q_i), i = 1, 2, \ldots$. If any of the following conditions holds:

1) each (x_i, q_i) is maximal;
2) the sequence of (x_i, q_i)s is not locally eventually bounded;
3) the sequence of (x_i, q_i)s is locally eventually bounded and no subsequence of (x_i, q_i)s has uniformly bounded domains $\mathrm{dom}(x_i, q_i)$;

then there exists a maximal solution (x, q) to (7) such that $x(0,0) = x^0$ and $x(t,j) \in X$ for all $(t,j) \in \mathrm{dom}(x, q)$.

Proof: For $i = 1, 2, \ldots$, if (x_i, q_i) is such that $(0,1) \in \mathrm{dom}(x_i, q_i)$ (i.e., if (x_i, q_i) jumps from its initial condition), define $(x_i'(t,j), q_i'(t,j)) := (x_i(t, j+1), q_i(t, j+1))$ and note that (x_i', q_i') is also a maximal solution to (7). If (x_i, q_i) is such that for some $\varepsilon_i > 0, (\varepsilon_i, 0) \in \mathrm{dom}(x_i, q_i)$, set $(x_i', q_i') = (x_i, q_i)$. By assumptions $A4)$ and $A5)$, there exists a compact subset $K_Q \subset Q$ such that for all $i = 1, 2, \ldots, q_i'(0,0) \in K_Q$. Hence, we can pick a subsequence of (x_i', q_i')s, which we do not relabel, such that $x_i'(0,0) \to x^0$ as $i \to \infty$ while for some $q^0 \in Q, q_i'(0,0) = q^0$ for $i = 1, 2, \ldots$. Note that $x_i'(t,j) \in S$ for all $(t,j) \in \mathrm{dom}(x_i', q_i')$.

If 3) holds, or if (x_i, q_i)s are locally eventually bounded and maximal, then the same assumptions hold for the sequence of (x_i', q_i')s. By [13, Th. 4.4 and Lemma 4.5], there exists a complete solution (x, q) to (7) with the desired properties (such a solution is obtained as a graphical limit of a graphically convergent subsequence of (x_i', q_i')s). If (x_i', q_i')s are not locally eventually bounded, then via the arguments in the proof of [13, Th. 4.6], one can obtain a maximal (but not complete) solution (x, q) to (7) (such a solution is the graphical limit of a graphically convergent subsequence of truncations of (x_i', q_i')s). In either case, $x(t,j) \in X$ for all $(t,j) \in \mathrm{dom}(x, q)$ follows directly from the definition of graphical convergence. ■

For the purposes of the previous lemma, assumption $A6)$ is not crucial; it can be replaced by a viability condition

VC) for all $q \in Q, x \in C_q \setminus D_q$, there exists a neighborhood U of x such that for all $x' \in U \cap C_q, T_{C_q}(x') \cap F_q(x') \neq \emptyset$

where $T_{C_q}(x')$ is the tangent cone to C_q at x'; see [34, ch. 6]. Condition VC) implies existence of solutions from all points, and that the maximal and not complete solutions "blow up."

A *proper indicator* of $\mathcal{A}$ with respect to $\tilde{O}$ is any continuous function $\omega : \tilde{O} \to \mathbb{R}_{\geq o}$ such that $\omega(x) = 0$ if and only if $x \in \mathcal{A}$, and $\omega(x) \to \infty$ if $x \to (\mathrm{bdry}\tilde{O})$ or $\|x\| \to \infty$. Given any proper indicator of $\mathcal{A}$ with respect to $\tilde{O}$, stability can be equivalently defined by the following: for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any solution (x, q) to $\mathcal{H}$ with $\omega(x(0,0)) \leq \delta$ we have $\omega(x(t,j)) \leq \varepsilon$ for all $(t,j) \in \mathrm{dom}(x, q)$. Similarly, indicators can be used to capture convergence to $\mathcal{A}$.

Lemma 4.7: Let $\mathcal{A}$ be asymptotically stable for (7). Then, $B_{\mathcal{A}}$, the basin of attraction of $\mathcal{A}$, is open. Also, $B_{\mathcal{A}} \cup \mathcal{A}$ is open, and for any proper indicator $\omega : B_{\mathcal{A}} \cup \mathcal{A} \to \mathbb{R}_{\geq o}$ of $\mathcal{A}$ with respect to $B_{\mathcal{A}} \cup \mathcal{A}$, we have the following:

1) Uniform convergence—for any $m, \varepsilon > 0$, there exists $T > 0$ such that any solution (x, q) to $\mathcal{H}$ with $\omega(x(0,0)) \leq m$ satisfies $\omega(x(t,j)) \leq \varepsilon$ for all $(t,j) \in \mathrm{dom}(x, q)$ with $t \geq T$.
2) Uniform overshoots—for any $m > 0$, there exists $M > 0$ such that any solution (x, q) to $\mathcal{H}$ with $\omega(x(0,0)) \leq m$ satisfies $\omega(x(t,j)) \leq M$ for all $(t,j) \in \mathrm{dom}(x, q)$.

We note that in 1), with given m, ε and appropriately chosen T, there may exist solutions (x, q) with $\omega(x(0,0)) \leq m$ for which no point $(t,j) \in \mathrm{dom}(x, q)$ is such that $t \geq T$. Such solutions "reach" $\mathcal{A}$ in finite time less or equal to T.

Proof: If $B_{\mathcal{A}}$ is not open, then for some $x^0 \in B_{\mathcal{A}}$ there exists a sequence of maximal solutions (x_i, q_i) to (7) with $\lim_{i\to\infty} x_i(0,0) = x^0$ such that $x_i(t,j) \not\to \mathcal{A}$ as $(t,j) \to \sup \mathrm{dom}(x_i, q_i)$. By attractivity of $\mathcal{A}$, there exists $\delta > 0$ such that $x_i(t,j) \notin \mathcal{A} + \delta\mathbb{B}$ for all $(t,j) \in \mathrm{dom}(x_i, q_i)$. By Lemma 4.6, there exists a maximal solution (x, q) to (7) with $x(0,0) = x^0$ and such that $x(t,j) \notin \mathcal{A} + \delta/2\mathbb{B}$ for all $(t,j) \in \mathrm{dom}(x, q)$. This is a contradiction with $x^0 \in B_{\mathcal{A}}$.

Suppose 1) fails. Then, for some $m > 0, \varepsilon > 0$, there exists a sequence of maximal solutions $\{(x_i, q_i)\}_{i \geq 1}$ to the hybrid system, with $\omega(x_i(0,0)) \leq m$ and $\omega(x_i(t_i, j_i)) \geq \varepsilon$ for some $(t_i, j_i) \in \mathrm{dom}(x, q)$ with $t_i \geq i$. By stability of $\mathcal{A}$, for some $\delta > 0$, $\omega(x_i(t,j)) \geq \delta$ for all $(t,j) \in \mathrm{dom}(t,j)$ with $t \leq t_i$. As $\{y \,|\, \omega(y) \leq m\}$ is compact, without loss of generality, we can assume that $x_i(0,0)$s converge (to some point in $B_{\mathcal{A}}$). The sequence of truncations of (x_i, q_i)s to $(t,j) \in \mathrm{dom}(x_i, q_i)$ with $(t,j) \prec (t_i, j_i)$ is such that no subsequence has uniformly bounded domains (since $t_i \geq i$). Applying Lemma 4.6 to these truncations gives a maximal solution (x, q) to (7) with $x(0,0) \in B_{\mathcal{A}}$ and such that for all $(t,j) \in \mathrm{dom}(x, q)$, $\omega(x(t,j)) \geq \delta$. This is a contradiction.

[1] In [13], graphical convergence is used. Here, as the solutions in question are instantaneously Zeno, graphical convergence reduces to uniform convergence on compact sets.

Similarly, if 2) is false, then for some $m > 0$, there exists a sequence of maximal solutions $\{(x_i, q_i)\}_{i\geq 1}$ to the hybrid system, with $x_i(0,0)$ convergent to some point in $B_{\mathcal{A}}$, and such that $\omega(x_i(t_i, j_i)) \geq i$ for some $(t_i, j_i) \in \mathrm{dom}(x, q)$. By part 2), there exists $T > 0$ such that $t_i \leq T$. Now, let (x_i', q_i') be truncations of (x_i, q_i) to $(t, j) \in \mathrm{dom}(x_i, q_i), (t, j) \preceq (t_i, j_i)$. By stability of $\mathcal{A}$, there exists $\delta > 0$ so that $\omega(x_i(t, j)) \geq \delta$ for all $(t, j) \in \mathrm{dom}(x_i', q_i')$. The sequence of (x_i', q_i')s is either locally eventually bounded, in which case $\omega(x_i(t_i, j_i)) \geq i$ implies that domains of (x_i', q_i')s are not uniformly bounded, or is not locally eventually bounded. In either case, Lemma 4.6 yields a maximal solution (x, q) to (7) with $x(0,0) \in B_{\mathcal{A}}$ and such that for all $(t, j) \in \mathrm{dom}(x, q)$, $\omega(x(t, j)) \geq \delta$. This is a contradiction. ■

Lemma 4.8: Let σ be an admissible perturbation radius and $\gamma_i \in (0, 1)$ be such that $\gamma_i \to 0$. The conclusions of Lemma 4.6 hold if the assumption that $(x_i, q_i), i = 1, 2, \ldots$, are solutions to (7) is replaced by the following: (x_i, q_i) is a solution to $\mathcal{H}^{\gamma_i \sigma}$ [i.e., to (11) with $\rho = \gamma_i \sigma$].

Proof: The proof is similar to that of Lemma 4.6, with [13, Th. 5.1 and Corollary 5.2] applied to a sequence of hybrid system replacing Theorem 4.4, Lemma 4.5, and Theorem 4.6 that were used in proving Lemma 4.6. ■

A function $\beta : \mathbb{R}_{\geq o} \times \mathbb{R}_{\geq o} \to \mathbb{R}_{\geq o}$ is a *$\mathcal{KL}$ function* if it is continuous, $r \mapsto \beta(r, t)$ is 0 at 0 and nondecreasing for each t, and $t \mapsto \beta(r, t)$ is nonincreasing and goes to 0 as $t \to \infty$ for each r.

Proposition 4.9: Let $\mathcal{A}$ be (globally) asymptotically stable on O, and let ω be a proper indicator of $\mathcal{A}$ with respect to $\tilde{O}$. Then, there exists a $\mathcal{KL}$ function $\beta : \mathbb{R}_{\geq o} \times \mathbb{R}_{\geq o} \to \mathbb{R}_{\geq o}$ such that for any solution (x, q) to the hybrid system (7)

$$\omega(x(t, j)) \leq \beta(\omega(x(0,0)), t), \qquad \text{for all } (t, j) \in \mathrm{dom}(x, q). \tag{13}$$

Furthermore, for each such $\mathcal{KL}$ function, and any admissible perturbation radius σ, any $m > 0, \varepsilon > 0$, there exists $\gamma^* > 0$ such that for all $\gamma \in [0, \gamma^*]$, all solutions (x_ρ, q_ρ) to the hybrid system $\mathcal{H}^{\gamma\sigma}$ (i.e., to (12) given by $\rho = \gamma\sigma$) that satisfy $\omega(x_\rho(0,0)) \leq m$ are such that

$$\omega(x_\rho(t, j)) \leq \beta(\omega(x_\rho(0,0)), t) + \varepsilon, \quad \text{for all } (t, j) \in \mathrm{dom}(x_\rho, q_\rho). \tag{14}$$

Proof: Define $\alpha : \mathbb{R}_{\geq o} \times \mathbb{R}_{\geq o} \to \mathbb{R}_{\geq o}$ by

$$\alpha(r, s) := \max\{0, \sup\{\omega(x(t, j)) \,|\, \omega(x(0,0)) \leq r, t \geq s\}\}$$

with the supremum taken over all solutions (x, q) to (7) and all $(t, j) \in \mathrm{dom}(x, q)$. This function has the properties required of $\mathcal{KL}$ functions except continuity. This can be justified using Lemma 4.7, following the arguments given for [13, Th. 6.5]. There does exist $\beta : \mathbb{R}_{\geq o} \times \mathbb{R}_{\geq o} \to \mathbb{R}_{\geq o}$ that has these properties and is continuous, and moreover, $\alpha(r, s) \leq \beta(r, s)$ for all $r, s \in \mathbb{R}_{\geq o}$; see, for example, [38, Remark 3]. β has the properties required by the first conclusion of the theorem.

To verify the second part, fix $\varepsilon > 0, m > \varepsilon$. Pick $T > 0$ so that $\beta(m, t) \leq \varepsilon/2$ when $t \geq T$. We claim that there exists $\gamma^* > 0$ such that for all $\gamma \in [0, \gamma^*]$, all solutions (x_ρ, q_ρ) to (12) with $\rho = \gamma\sigma$ such that $\omega(x_\delta(0,0)) \leq m$ satisfy

$$\omega(x_\rho(t, j)) \leq \beta(\omega(x_\rho(0,0)), t) + \varepsilon/2 \tag{15}$$

for all $(t, j) \in \mathrm{dom}(x_\rho, q_\rho)$ with $t \leq 2T$. If this is true, then in particular, $\omega(x_\rho(t, j)) \leq \varepsilon$ for all $(t, j) \in \mathrm{dom}(x_\rho, q_\rho)$ such that $t \in [T, 2T]$. Using this fact recursively and relying on $m > \varepsilon$ shows that $\omega(x_\delta(t, j)) \leq \varepsilon$ when $t \geq T$. Combined with (15), this shows (14).

To see that the claim holds, suppose otherwise: that there exists a sequence $\{(x_i, q_i)\}_{i\geq 1}$ of complete solutions to $\mathcal{H}^{i^{-1}\sigma}$ with $\omega(x_i(0,0)) \leq m$ (without loss of generality, we can assume that $x_i(0,0)$s converge), and points $(t_i, j_i) \in \mathrm{dom}(x_i, q_i)$ with $t_i \leq 2T$ so that (15) does not hold, i.e.,

$$\omega(x_i(t_i, j_i)) > \beta(\omega(x_i(0,0)), t_i) + \varepsilon/2.$$

Unless j_is are uniformly bounded and (x_i, q_i)s are locally eventually bounded, Lemma 4.8 can be invoked to show the existence of a maximal trajectory (x, q) to (7) with $\omega(x(t, j)) \geq \varepsilon/2$ for all $(t, j) \in \mathrm{dom}(x, q)$. If j_is are uniformly bounded and (x_i, q_i)s are locally eventually bounded, extracting (via [13, Th. 4.4]) a convergent subsequence of (x_i, q_i)s and then a further subsequence such that (t_i, j_i)s converge yields a solution (x, q) to (7) for which (13) is violated by x at $\lim(t_i, j_i)$. (The final argument uses continuity of β and ω.) This is a contradiction. ■

B. *Proof of Theorem 4.3*

Let ω be any proper indicator of $\mathcal{A}$ with respect to $\tilde{O}$. Let σ be any admissible perturbation radius such that $\mathcal{H}^\sigma$ has no instantaneous Zeno solutions (such σ exists by Proposition 4.9). Let β be as in Proposition 4.9. Pick any family of positive numbers $\{r_n\}_{n\in\mathbb{Z}}$ such that $r_{n+1} \geq 4\beta(r_n, 0, 0)$, so in particular, $r_{n+1} \geq 4r_n, \lim_{n\to-\infty} r_n = 0, \lim_{n\to\infty} r_n = \infty$. For each $n \in \mathbb{Z}$, by Proposition 4.9, there exist $\delta_n \in (0, 1)$ and $\tau_n > 0$ such that each solution (x, q) to $\mathcal{H}^{\delta_n \sigma}$ with $\omega(x(0,0)) \leq r_n$ satisfies

$$\omega(x(t, j)) \leq \beta(\omega(x(0,0)), t) + r_{n-1}/2 \quad \forall (t, j) \in \mathrm{dom}(x, q) \tag{16}$$

and hence, also

$$\omega(x(t, j)) \leq r_{n+1}/2 \qquad \forall (t, j) \in \mathrm{dom}(x, q) \tag{17}$$

and, finally

$$\omega(x(t, j)) \leq r_{n-1} \qquad \forall (t, j) \in \mathrm{dom}(x, q), \quad t \geq \tau_n. \tag{18}$$

Now, find any continuous function $\delta : O \to [0, \infty)$ that is positive on O and such that

$$\delta(x) \leq \frac{1}{2}\min\{\delta_{n-1}, \delta_n\}\sigma(x) \text{ if } r_{n-1} \leq \omega(x) \leq r_n. \tag{19}$$

Note that δ is an admissible perturbation radius. Furthermore, by $A4$) and since $\delta(x) < (1/2)\sigma(x)$, the covering of O by sets C_q^δ is locally finite.

Note that for each $n \in \mathbb{Z}$, each solution (x, q) to $\mathcal{H}^\delta$ with $\omega(x(0,0)) \leq r_n$ satisfies (17), and hence, $\mathcal{A}$ is stable for $\mathcal{H}^\delta$. To see that (17) is satisfied by (x, q) as described, note that if $r_{n-1} \leq \omega(x(t,j))$ for all $(t,j) \in \mathrm{dom}(x,q) \cap [0,T] \times \{0,1,\ldots,J\}$, then by the choice of $\delta, (x,q)$ is a solution to $\mathcal{H}^{\delta_n \sigma}$ and $\omega(x(t,j)) \leq r_{n+1}/2$ for all $(t,j) \in \mathrm{dom}(x,q) \cap [0,T] \times \{0,1,\ldots,J\}$ by (17). If $\omega(x(T,J)) < r_{n-1}$ for some $(T,J) \in \mathrm{dom}(x,q)$, then one can consider $(x',q')(t,j) := (x,q)(T+t, J+j)$, which is also a solution to $\mathcal{H}^\delta$, and for which $\omega(x'(0,0)) \leq r_{n-1}$.

To see that $\mathcal{A}$ is (globally) attractive, pick any maximal solution (x,q) to H^δ and (without loss of generality) assume $\omega(x(0,0)) \leq r_n$. If (x,q) is not complete, then by Remark 4.2 and Lemma 4.4 it must "blow up" to the boundary of O, that is either $\omega(x(t,J)) \to \infty$ or $\omega(x(t,J)) \to 0$ as $t \to \sup_t \mathrm{dom}(x,q)$, where $J = \sup_j \mathrm{dom}(x,q)$. By (17), it must be that $\omega(x(t,J)) \to 0$, and hence, x approaches $\mathcal{A}$. If (x,q) is complete and $\sup_t \mathrm{dom}(x,q) = \infty$, then there exists $t' \leq \tau_n$ such that $\omega(x(t',j)) \leq r_{n-1}$ with j such that $(t',j) \in \mathrm{dom}(x,q)$. Indeed, either $\omega(x(t,j)) > r_{n-1}$ for all $t \in [0,\tau_n)$, in which case (x,q) solves $\mathcal{H}^{\delta_n \sigma}$, and hence, $\omega(x(\tau_n,j)) = r_{n-1}$, or there exists $t \in [0,\tau_n)$ with $\omega(x(t,j)) \leq r_{n-1}$. This argument can be repeated to show that there exist times for which $\omega(x(t,j))$ is arbitrarily small, and thus x approaches $\mathcal{A}$. Finally, suppose that (x,q) is complete with $\sup_t \mathrm{dom}(x,q)$ finite and that x does not approach $\mathcal{A}$. By stability, this means that there exists a compact set $K \subset O$ with $x(t,j) \in K$ for all $(t,j) \in \mathrm{dom}(x,q)$. For $i = 0,1,\ldots$, let t_i be any times such that $(t_i, i) \in \mathrm{dom}(x,q)$, and consider a sequence (of solutions to $\mathcal{H}^\rho$) given by $(x_i,q_i)(t,j) = (x,q)(t+t_i, j+i)$. Note that $\sup_t \mathrm{dom}(x_i,q_i)$ converges to 0 as $i \to \infty$. As G_q^ρ are locally bounded in x uniformly in q, there exists a compact set $Q_K \subset Q$ such that $q_i(t,j) \in Q_K$ for all $(t,j) \in \mathrm{dom}(x_i,q_i)$. Hence, the sequence of (x_i,q_i) is uniformly bounded, and some subsequence of it converges (graphically) by [13, Th. 4.4], and in the limit, yields a solution to $\mathcal{H}^\rho$ that is instantaneously Zeno. This is impossible by the choice of σ and ρ.

V. Proof of Theorem 3.7

We now prove Theorem 3.7. In Section V-A, we recall the concept of a patchy vector field introduced by [1] and show how a hybrid system with the data having the regularity discussed in Section IV can be built from a patchy vector field, so that, when no perturbations are present, solutions to the two objects have the same asymptotic behavior and the transient behavior is not much different. Section V-B shows that if a patchy vector field results from a patchy feedback that stabilizes a nonlinear system, then the related hybrid system inherits the asymptotic stability. Section V-C addresses the existence of solutions to hybrid systems under time-varying perturbations. Section V-D connects the dots, exhibiting a hybrid feedback that robustly stabilizes the nonlinear system. Theorem 4.3 is used there to claim the needed robustness.

A. Patchy Vector Fields and Hybrid Systems

Essentially, a patchy vector field is a family of indexed open sets (patches) covering the state space, and a corresponding family of continuous vector fields. At each point, the flow is governed by the vector field corresponding to the patch of the highest index that contains the point.

Definition 5.1 [1]: A mapping $\phi : O \to \mathbb{R}^n$ is a *patchy vector field* on O if there exist a set $\mathcal{Q}$, and for each $\alpha \in \mathcal{Q}$, sets $\Omega_\alpha \subset O, O_\alpha \subset O$ and a function f_α such that

1) for each $\alpha \in \mathcal{Q}$, the triple $\Omega_\alpha, O_\alpha, f_\alpha$ forms a patch, that is
   a) Ω_α, O_α are open, $\overline{\Omega_\alpha} \subset O_\alpha$, and the boundary of Ω_α is smooth;
   b) $f_\alpha : O_\alpha \to \mathbb{R}^n$ is smooth;
   c) for any point $x \in \mathrm{bdry}\Omega_\alpha$
   $$\langle f_\alpha(x), n_\alpha(x) \rangle < 0$$
   where $n_\alpha(x)$ is the outer normal to $\overline{\Omega_\alpha}$ at x;
2) $\mathcal{Q}$ is a totally ordered set;
3) the sets Ω_α form a locally finite covering of O;

and ϕ can be written in the form

$$\phi(x) = f_\alpha(x), \qquad \text{if } x \in \Omega_\alpha \setminus \bigcup_{\beta \succ \alpha} \Omega_\beta$$

where $\succ$ is the ordering of $\mathcal{Q}$.

For a differential equation with a right-hand side that is a patchy vector field, solutions are understood in the Caratheodory sense, i.e., as locally absolutely continuous functions that almost everywhere satisfy $\dot{x}(t) = \phi(x(t))$, and more specifically, $\dot{x}(t) = f_{\alpha^*(x(t))}(x(t))$, where

$$\alpha^*(x) = \max\{\alpha \in \mathcal{Q} \,|\, x \in \Omega_\alpha\}. \tag{20}$$

We now construct a hybrid system related to a PVF. A preliminary result is needed.

Lemma 5.2: Let Ω_α, O_α, and f_α form a patch, and suppose that Ω_α is bounded. Then, there exist an open set Ω'_α so that $\overline{\Omega_\alpha} \subset \Omega'_\alpha \subset O_\alpha$ and a constant $T_\alpha > 0$ so that any maximal solution to $\dot{x}(t) = f_\alpha(x(t))$ with $x(0) \in \overline{\Omega'_\alpha}$ is complete and such that $x(t) \in \Omega'_\alpha$ for all $t > 0$ and $x(t) \in \Omega_\alpha$ for all $t > T_\alpha$.

Proof: It is easy to check that there exists (an arbitrarily small) $T_\alpha > 0$ such that the set of all $y \in \mathbb{R}^n$ for which all maximal solutions to $\dot{x} = f_\alpha(x)$ with $x(0) = y$ are such that $x(t) \in \Omega_\alpha$ for all $t \geq T_\alpha$ has the properties desired of Ω'_α. ■

We add that, in Lemma 5.2, Ω'_α can be chosen as a subset of any given neighborhood of Ω_α.

Consider a patchy vector field for which the patches Ω_α are bounded (from now on, we refer to such a patchy vector field as $\mathcal{PVF}$). Let $Q = \mathcal{Q}$, and for each $q \in Q$, find $c_q > 0$ so that $\overline{\Omega_q + c_q \mathbb{B}} \subset O_q$ (this is possible as $\overline{\Omega_q} \subset O_q$) and thus the family $\{\Omega_q + c_q\mathbb{B}\}_{q \in Q}$ is a locally finite covering of O (this is possible as $\{\Omega_q\}_{q \in Q}$ is a locally finite covering of O and Ω_qs are bounded). For each $q \in Q$, use Lemma 5.2 to find $\Omega'_q \subset \Omega_q + c_q\mathbb{B}$ and $T_q > 0$. Now, let $\mathcal{H}_{\mathcal{PVF}}$ be the hybrid system defined on $O \times Q$ as follows. For each $q \in Q$, let

$$C_q = \overline{\Omega'_q} \setminus \bigcup_{\beta \succ q} \Omega_\beta \quad F_q(x) = f_q(x)$$
$$D_q = \bigcup_{\beta \succ q} \overline{\Omega_\beta} \cup (O \setminus \Omega'_q),$$

$$G_q(x) = \begin{cases} \left\{ \beta \in Q \,|\, \begin{matrix} x \in \overline{\Omega_\beta} \\ \beta \succ q \end{matrix} \right\}, & x \in \bigcup_{\beta \succ q} \overline{\Omega_\beta} \cap \Omega'_q \\ \{\beta \in Q \,|\, x \in \overline{\Omega_\beta}\}, & x \in O \setminus \Omega'_q \end{cases} \quad (21)$$

Lemma 5.3: The sets and mappings defined in (21) are such that the assumptions $A4), A5)$, and $A6)$, and for all $q \in Q, A_q1), A_q2)$, and $A_q3)$ are satisfied.

Proof: Fix q. First, note that

$$C_q \cup D_q = \left(\overline{\Omega'_q} \setminus \bigcup_{\beta \succ q} \Omega_\beta \right) \cup \left(\bigcup_{\beta \succ q} \overline{\Omega_\beta} \cup (O \setminus \Omega'_q) \right)$$
$$\supset \overline{\Omega'_q} \cup (O \setminus \Omega'_q) \supset O.$$

The set C_q is closed (not just relatively closed in O) as it is an intersection of closed sets: $C_q = \overline{\Omega'_q} \cap (O \setminus \bigcup_{\beta \succ q} \Omega_\beta)$. Since, by construction, the covering of O by $\overline{\Omega_\beta}$s is locally finite, the set $\bigcup_{\beta \succ q} \overline{\Omega_\beta}$ is closed. This immediately implies that D_q is relatively closed. The function f_q is continuous on a neighborhood of C_q (i.e., on O_q), hence it is outer semicontinuous and locally bounded as a set-valued mapping, and its values are nonempty and convex on C_q (as they are just single points). The mapping G_q is outer semicontinuous and locally bounded not just on D_q but also on O. Indeed, it is locally bounded since the covering of O by $\overline{\Omega_\beta}$s is locally finite and $G_q(x) \subset \{\beta \in Q \,|\, x \in \overline{\Omega_\beta}\}$; in fact, this implies that the local boundedness is uniform in q. To see it is outer semicontinuous, note that its graph can be written as

$$\left(\bigcup_{\beta \succ q} (\overline{\Omega_\beta} \times \{\beta\}) \right) \cup \left(\bigcup_{\beta \in Q} ((O \setminus \Omega'_q) \cap \overline{\Omega_\beta}) \times \{\beta\} \right)$$

and this set is closed. Now, invoke [34. Th. 5.7]. Finally, note that for each $x \in O$ there exists $q \in Q$ so that $x \in C_q$, in fact $x \in C_{q^*(x)}$; this shows that C_qs cover O. This covering is locally finite, as $C_q \subset \Omega'_q$, and the covering by the latter sets is locally finite by construction. ■

We now compare solutions of $\mathcal{PVF}$ to those of $\mathcal{H}_{\mathcal{PVF}}$. Roughly, Lemma 5.4 says that in absence of perturbations, to any solution x to the patchy vector field $\mathcal{PVF}$ corresponds a solution (x, q) to the hybrid system $\mathcal{H}_{\mathcal{PVF}}$, while for any solution (x, q) to the hybrid system, after the first jump the continuous variable x is a solution to $\mathcal{PVF}$. In presence of perturbations, this is no longer the case: $\mathcal{PVF}$ is far more sensitive to perturbations than the corresponding $\mathcal{H}_{\mathcal{PVF}}$.

Lemma 5.4:

1) Let $x : [0, T] \to O$ be a solution to $\mathcal{PVF}$. Let $t_1 < t_2 < \cdots < t_{J-1}$ be the sequence of discontinuities of $t \to \alpha^*(x(t))$ [recall (20)], and let $t_0 = 0, t_J = T$. Then, (x, q) given on the hybrid time domain
$$\bigcup_{j=1}^{J} [t_{j-1}, t_j] \times \{j\}$$
by $(x, q)(t, j) = (x(t), \alpha^*(x(t_j)))$ for $t \in [t_{j-1}, t_j]$ is a solution to $\mathcal{H}_{\mathcal{PVF}}$.

2) Let (x, q) be a solution to $\mathcal{H}_{\mathcal{PVF}}$ with compact $\mathrm{dom}(x, q)$. Let $[t_{j-1}, t_j], j = 1, 2, \ldots, J$ be the sequence of all nontrivial (i.e., with $t_{j-1} < t_j$) intervals such that for all $j, [t_{j-1}, t_j] \times \{i_j\} \in \mathrm{dom} y$ for some i_j. (It may happen that this sequence is empty.) Then, the function $x' : [t_1, t_J] \to O$ given by $x'(t) = x(t, i_j)$ for $t \in [t_{j-1}, t_j]$ is a solution to $\mathcal{PVF}$.

Proof: We show 1) first. For almost all $t \in (t_{j-1}, t_j)$ (and so for almost all $t \in [t_{j-1}, t_j]$), we have $\dot{x}(t) = f_{\alpha^*(x(t))}(x(t))$ by the very definition of the patchy vector field; while by construction of C_qs, we have $x(t) \in C_{\alpha^*(x(t))}$, for $j = 1, 2, \ldots, J$. This shows that the condition $S1)$ of the definition of a solution to a hybrid system is satisfied by the described (x, q). To see that $S2)$ is also satisfied, note that $x(t_j) \in \Omega_{\alpha^*(x(t_j))}$ while $x(t) \in \Omega_q$ for $t \in (t_j, t_j + 1)$, for some $q \neq \alpha^*(x(t_j))$ (in fact, for $q = \alpha^*(x(t))$). Since $\Omega_{\alpha^*(x(t))}$ is open, by the definition of α^* it must be the case that $q \succ \alpha^*(x(t_j))$. This, and continuity of x, implies that $x(t_j) \in \overline{\Omega_q}$, and consequently, $x(t_j) \in D_{\alpha^*(x(t_j))}$ as well as $q \in G_{\alpha^*(x(t_j))}(x(t_j))$.

We now show 2). By the definition of a solution to a hybrid system, for all $j = 1, 2, \ldots, J$ and almost all $t \in [t_{j-1}, t_j]$, we have $\dot{x}(t, i_j) = f_{q(t,i_j)}(x(t, i_j))$ and $x(t, i_j) \in C_{q(t,i_j)}$. We will claim that for $j = 2, 3, \ldots, J$ and all $t \in (t_{j-1}, t_j], q(t, i_j) = \alpha^*(x(t, i_j))$; the claim is sufficient to guarantee that x' is a solution to $\mathcal{PVF}$ on $[t_1, t_J]$. To see that the claim is true, note that $q(t, i_j) = \alpha^*(x(t, i_j))$ holds when $x(t, i_j) \in \Omega_{q(t,i_j)} \setminus \bigcup_{\beta \succ q(t,i_j)} \Omega_\beta$. The latter is the case for all $t \in (t_{j-1}, t_j]$ if we know that $x(t, i_j) \in \overline{\Omega_{q(t,i_j)}} \setminus \bigcup_{\beta \succ q(t,i_j)} \Omega_\beta$ for all $t \in [t_{j-1}, t_j]$, and in fact, if we only know that $x(t_{j-1}, i_j) \in \overline{\Omega_{q(t_{j-1},i_j)}} \setminus \bigcup_{\beta \succ q(t_{j-1},i_j)} \Omega_\beta$; this is justified by the properties of patches (patches are flow invariant and, in fact, solutions from the boundary of a patch flow into the interior immediately). Now, note that for $j = 2, 3, \ldots, J, q(t_{j-1}, i_j) \in G_r(x(t_{j-1}, i_{j-1}))$ for some $r \in Q$, which by the definition of G implies that $x(t_{j-1}, i_j) \in \overline{\Omega_{q(t_{j-1},i_j)}}$. Since $[t_{j-1}, t_j] \times \{i_j\} \in \mathrm{dom}(x, q)$, and thus $x(t, i_j) \in C_{q(t,i_j)}$ for $t \in [t_{j-1}, i_j], x(t, i_j) \notin \bigcup_{\beta \succ q(t,i_j)} \Omega_\beta$. In particular, $x(t_{j-1}, i_j) \notin \bigcup_{\beta \succ q(t_{j-1},i_j)} \Omega_\beta$. This finishes the proof. ■

B. Stabilization via Patchy Feedback

Definition 5.5 [1]: A mapping $u : O \to U$ is a *patchy feedback* if there exists a patchy vector field $\phi : O \to \mathbb{R}^n$ on O (given by the index set $\mathcal{Q}$, and sets Ω_α, O_α and a function f_α for each $\alpha \in \mathcal{Q}$) and control values $u_\alpha \in U$ such that

$$u(x) = u_\alpha, \qquad \text{if } x \in \Omega_\alpha \setminus \bigcup_{\beta \succ \alpha} \Omega_\beta$$

and ϕ can be written in the form

$$\phi(x) = f(x, u(x)), \qquad \text{if } x \in \Omega_\alpha \setminus \bigcup_{\beta \succ \alpha} \Omega_\beta.$$

In our construction of the stabilizing hybrid feedback we will take advantage of a patchy feedback that, for the system (8), renders $\mathcal{A}$ globally asymptotically stable on O. (This asymptotic stability entails stability in the standard sense and convergence of solutions x of the patchy vector field to $\mathcal{A}$ as t approaches the supremum of the domain of x; solutions are not necessarily complete. See [1].) The result below extends [1, Th. 1] to compact attractors $\mathcal{A}$ and a general open domain O; [1] considered

$\mathcal{A} = \{0\}$ and $O = \mathbb{R}^n$. The proof is essentially the same: one replaces the norm by any proper indicator of $\mathcal{A}$ with respect to O. Also, in [1], $Q \subset \mathbb{Z}^2$ is ordered lexicographically and is such that for each $m \in \mathbb{Z}, (m,n) \in Q$ for finitely many (or none) $n \in \mathbb{Z}$. Such Q can be identified with a subset of $\mathbb{Z}$ with the standard order.

Theorem 5.6: For any asymptotically controllable on $\tilde{O}$ to $\mathcal{A}$ nonlinear system (8), there exists a patchy feedback on O that renders $\mathcal{A}$ asymptotically stable on O for (8). The patchy feedback can be chosen so that the index set $\mathcal{Q} \subset \mathbb{Z}$ is ordered by the standard inequality and for each $\alpha \in \mathcal{Q}, \overline{\Omega_\alpha}$ is a compact subset of O.

We note that the patchy feedback of [1] is robust to external disturbances. For the purposes of semiglobal practical stabilization, it is also robust to time-dependent measurement error with (appropriately small) total variation; see [2]. We do not need to rely on such robustness properties of the patchy feedbacks. As long as the patchy feedback is stabilizing, far stronger robustness properties will be acquired once the feedback is recast in the hybrid framework.

Henceforth, let $\mathcal{SPVF}$ denote an asymptotically stable patchy vector field resulting from the application of a stabilizing patchy feedback as in Theorem 5.6 to (8). Let $\mathcal{H}_{\mathcal{S}}\mathcal{PVF}$ be the hybrid system resulting from the application to $\mathcal{SPVF}$ of the construction described below Lemma 5.2.

Lemma 5.7: For the hybrid system $\mathcal{H}_{\mathcal{S}}\mathcal{PVF}, \mathcal{A}$ is asymptotically stable on O.

Proof: The key to this result is Lemma 5.4. Regarding stability of $\mathcal{H}_{\mathcal{S}}\mathcal{PVF}$, given any $\delta > 0$ pick $\epsilon' > 0$ so that $\mathcal{A} + \epsilon'\mathbb{B} \subset \tilde{O}$ and solutions y to $\mathcal{SPVF}$ with $\mathrm{dist}_{\mathcal{A}}(y(0)) \le \epsilon'$ satisfy $\mathrm{dist}_{\mathcal{A}}(y(t)) < \delta$ for all $t \in \mathrm{dom} y$. Now, find $\epsilon > 0$ (with $\epsilon < \epsilon'$) so that $\overline{\Omega'_q} \cap (\mathcal{A} + \epsilon\mathbb{B}) \neq \emptyset$ implies $\overline{\Omega'_q} \subset (\mathcal{A} + \epsilon'\mathbb{B})$. Then, any solution (x,q) to $\mathcal{H}_{\mathcal{S}}\mathcal{PVF}$ with $\mathrm{dist}_{\mathcal{A}}(x(0,0)) \le \epsilon$ implies $\mathrm{dist}_{\mathcal{A}}(x(t,j)) < \delta$ for all $(t,j) \in \mathrm{dom}(x,q)$. Indeed, by the choice of ϵ with relation to $\epsilon', \mathrm{dist}_{\mathcal{A}}(x(t,0)) \le \epsilon'$ for all t with $(t,0) \in \mathrm{dom}(x,q)$, i.e., for all times before the first jump of (x,q). After the jump, x can be identified with a solution y to $\mathcal{SPVF}$, thanks to Lemma 5.4. Hence, $\mathrm{dist}_{\mathcal{A}}(x(t,j)) < \delta$ for all $(t,j) \in \mathrm{dom}(x,q)$.

Regarding attractivity, first note that by Lemma 5.3 and Remark 4.2 nontrivial solutions to $\mathcal{H}_{\mathcal{S}}\mathcal{PVF}$ exist (and maximal ones (x,q) are complete or such that x "blows up"). Let (x,q) be a maximal solution to $\mathcal{H}_{\mathcal{S}}\mathcal{PVF}$. Suppose that $x(t,j)$ does not approach $\mathcal{A}$ as $(t,j) \to \sup \mathrm{dom}(x,q)$, which by stability means that for some $\epsilon > 0, \mathrm{dist}_{\mathcal{A}}(x(t,j)) \ge \epsilon$ for all $(t,j) \in \mathrm{dom}(x,q)$. A maximal solution to $\mathcal{H}_{\mathcal{S}}\mathcal{PVF}$ cannot be such that x remains in one of the compact sets C_q, hence (x,q) jumps, and after that jump, x can be identified with a solution to $\mathcal{SPVF}$. By attractivity of $\mathcal{A}$ for $\mathcal{SPVF}$, and as $\mathrm{dist}_{\mathcal{A}}(x(t,j)) \ge \epsilon$, we can conclude that there exists a compact set $K \subset O$ such that $x(t,j) \in K$ for all $(t,j) \in \mathrm{dom}(x,q)$, and moreover, that $\sup_t \mathrm{dom}(x,q) < \infty$. Hence, for (x,q) to be maximal, it must be complete and in fact, it must be that $\sup_j \mathrm{dom}(x,q) = \infty$. However, as the covering of O by sets C_q is locally finite, K intersects only finitely many C_qs, and the discrete variable of (x,q) can take on only finitely many values. However, after the first jump, that discrete variable is increasing, which contradicts $\sup_j \mathrm{dom}(x,q) = \infty$. Hence, there do not exist maximal solutions to $\mathcal{H}_{\mathcal{S}}\mathcal{PVF}$ for which the continuous variable does not converge to $\mathcal{A}$. ∎

C. Robustness to Time-Varying Perturbations

Consider the system $\mathcal{H}$ and let ρ be an admissible perturbation radius so that for $\mathcal{H}^\rho, \mathcal{A}$ is asymptotically stable on O. Suppose that ξ and ζ are admissible measurement noise and external disturbance, such that (10) hold. Then, any solution to the (time-varying) system $\mathcal{H}$ in presence of ξ and ζ is also a solution to $\mathcal{H}^\rho$. However, the very existence of solutions to the hybrid system in presence of ξ and ζ can be problematic.[2]

Example 5.8: Consider a hybrid system (7) on $O = \mathbb{R}$ and $Q = \{0\}$ given by $C = (-\infty, 0], D = [0, \infty), F(x) = 0, G(x) = 0$, measurement noise (independent of j) given by $\xi(x,t) = \xi(t)$ with $\xi(0) = -1, \xi(t) = 1$ for all $t > 0$ (similar, but arbitrarily small noise could be used here) and any external disturbance. Then, there does not exist a solution to $\mathcal{H}^{\xi,\zeta}_{\mathrm{feed}}$ with $x(0,0) = 0$. Indeed, the solution cannot jump at the hybrid time $(0,0)$ (i.e., $(0,0), (0,1) \in \mathrm{dom}(x,q)$ cannot happen) since $x(0,0) + \xi(x(0,0),0) = -1 \notin D$. Similarly, the solution cannot flow at the hybrid time $(0,0)$ (i.e., $[0,\varepsilon) \times \{0\} \in \mathrm{dom}(x,q)$ cannot happen for $\varepsilon > 0$) because we would have $x(t,0) + \xi(x(t,0),t) = 1 \notin C$ for all small $t > 0$.

It turns out that if the sets C_q and D_q "overlap," the existence can be guaranteed, at least for the system $\mathcal{H}_{\mathrm{feed}}$ coming from the application of a hybrid feedback as in Definition 3.4 to the nonlinear system (8). In the following, by $\mathrm{int} S$ we mean the interior of a set S.

Lemma 5.9: Consider the hybrid system $\mathcal{H}_{\mathrm{feed}}$ and assume that $A4)$ and for all $q \in Q, A_q1), A_q2)$, and $A_q3)$, hold and k_q is continuous. Suppose furthermore that, for all $q \in Q$, each $x \in O$ is such that either $x \in \mathrm{int} C_q$ or $x \in \mathrm{int} D_q$. Then, there exists an admissible perturbation radius ρ such that for any admissible measurement noise and external disturbance ξ and ζ for which (10) holds, solutions to $\mathcal{H}$ in presence of ξ and ζ exist for any initial point $(x_0, q_0) \in O \times Q$.

Proof: Suppose that ρ is an admissible perturbation radius with the property that for all $q \in Q$, either $x + \rho(x)\mathbb{B} \subset \mathrm{int} C_q$ or $x + \rho(x)\mathbb{B} \subset \mathrm{int} D_q$. For such ρ, and for any ξ and ζ for which (10) holds, solutions to $\mathcal{H}^{\xi,\zeta}_{\mathrm{feed}}$ exist for any initial point $(x_0, q_0) \in \vee \times Q$. Indeed, if $x_0 + \rho(x_0)\mathbb{B} \subset \mathrm{int} D_q$ then $x_0 + \xi(x_0, 0) \in D_{q_0}$, and hence, there exists a solution to $\mathcal{H}^{\xi,\zeta}_{\mathrm{feed}}$ with $(0,1)$ in its domain (a solution that jumps from the initial condition). If $x_0 + \rho(x_0)\mathbb{B} \subset \mathrm{int} C_q$, then by continuity of $\rho, z + \rho(z)\mathbb{B} \subset C_q$ for all z in some neighborhood of x_0. The function $(x,t) \mapsto f(x, k_q(x + \xi(t)) + \zeta(t)$ is continuous in x and measurable in t, and thus the differential equation $\dot{x}(t) = f(x(t), k_q(x(t) + \xi(t)) + \zeta(t)$ has a solution from x_0; moreover, such a solution satisfies $x(t) + \xi(t) \in C_q$ for small $t \ge 0$. Consequently, there exists a solution to $\mathcal{H}^{\xi,\zeta}_{\mathrm{feed}}$ with $[0,\varepsilon) \times \{0\}$ it its domain, for some $\varepsilon > 0$ (a solution that flows from the initial condition).

[2]While we have not formally defined solutions to $\mathcal{H}$ in (7) in the presence of ξ and ζ, they should be understood similarly to the solutions described in Definition 3.5.

It is left to show that there exists an admissible perturbation radius with the property that for all $q \in Q$, either $x + \rho(x)\mathbb{B} \subset \text{int}C_q$ or $x + \rho(x)\mathbb{B} \subset \text{int}D_q$. For $x \in O$ define

$$r(x) = \inf_{q\in Q} \max\{\text{dist}_{O\setminus C_q}(x), \text{dist}_{O\setminus D_q}(x)\}.$$

Pick any compact $K \subset O$ and $\varepsilon_K > 0$ so that $K+\varepsilon_K\mathbb{B} \subset O$. Using assumption $A4$), the set $Q_{K,\varepsilon_K} = \{q \in Q \,|\, C_q \cap (K + \varepsilon_K\mathbb{B}) \neq \emptyset\}$ is finite. For $q \notin Q_{K,\varepsilon_K}$ and $x \in K$, $\text{dist}_{O\setminus D_q}(x) \geq \text{dist}_{C_q}(x) \geq \varepsilon_K$. Consequently, for $x \in K$

$$r(x) \geq \min\left\{\varepsilon_K, \min_{q\in Q_{K,\varepsilon_K}} \max\{\text{dist}_{O\setminus C_q}(x), \text{dist}_{O\setminus D_q}(x)\}\right\}.$$

For any $q \in Q$, $\max\{\text{dist}_{O\setminus C_q}(x), \text{dist}_{O\setminus D_q}(x)\}$ is positive, and depends continuously on x. As the set Q_{K,ε_K} is finite, the previous inequality implies that r is bounded below on K by a positive continuous function. This is sufficient to guarantee that there exists an admissible perturbation radius ρ so that $\rho(x) \leq r(x)/2$ for all $x \in O$. Such ρ has the desired properties. ■

D. Construction of the Stabilizing Hybrid Feedback

Let $u : O \to U$ be a patchy feedback that asymptotically stabilizes (8), as given by Theorem 5.6. Let the associated patchy vector field $\mathcal{SPVF}$ be given by the index set $\mathcal{Q} \subset \mathbb{Z}$, sets Ω_α, O_α, and control values $u_\alpha \in U$ (leading to f_α for each $\alpha \in \mathcal{Q}$); recall Definition 5.5. Set $Q := \mathcal{Q}$, and for $q \in Q$, let Ω'_q be as constructed below Lemma 5.2.[3] Then, define

$$C'_q = \overline{\Omega'_q} \setminus \bigcup_{\beta \succ q} \Omega_\beta \quad D_q = \bigcup_{\beta\succ q} \overline{\Omega_\beta} \cup (O \setminus \Omega'_q) \tag{22}$$

the mappings $k_q : O \to U$ by

$$k_q(x) = u_q, \qquad \text{for all } x \in O \tag{23}$$

and the mapping $G_q : O \to Q$ as in (21). Let $\mathcal{H}$ stand for the hybrid system (7) given by sets $C'_q, D_q, F_q(x) := f(x, k_q(x))$, and $G_q(x) = g_q(x)$. Let ρ be an admissible perturbation radius resulting from the application of Theorem 4.3 to $\mathcal{H}$; in particular, we thus have that $\mathcal{A}$ is asymptotically stable on O for $\mathcal{H}^\rho$.

In the following, C'^{ρ}_q and $C'^{\rho'}_q$ are "inflations" of C'_q as in (12), and $(C'^{\rho'}_q)^{\rho'}$ is the "inflation" of $C'^{\rho'}_q$.

Lemma 5.10: Given sets C'_q and the admissible perturbation radius ρ as previously, there exists an admissible perturbation radius ρ' bounded above by ρ such that, for all $q \in Q$, $(C'^{\rho'}_q)^{\rho'} \subset C'^{\rho}_q$.

Proof: Without loss of generality one can assume that $\rho(x) \leq 1$ for all $x \in O$; otherwise, one can consider $x \mapsto \min\{1, \rho(x)\}$. By the very definition of the "inflations," it is enough to find an admissible perturbation radius ρ' such that for all $x \in O$ and for all $y \in x + \rho(x)\mathbb{B}$ one has $\rho'(y) \leq \rho(x)/2$ (note that this entails $\rho'(x) \leq \rho(x)/2$). Indeed, say that for such a $\rho', x \in (C'^{\rho'}_q)^{\rho'}$. This implies that $x+\rho'(x)\mathbb{B}\cap C'^{\rho'}_q \neq \emptyset$, and hence that there exists $y \in x+\rho'(x)\mathbb{B}$ so that $y + \rho'(y)\mathbb{B} \cap C_q \neq \emptyset$. Since $\rho'(x) \leq \rho(x)/2$ and $\rho'(y) \leq \rho(x)/2$, this yields $x + \rho(x)\mathbb{B} \cap C'_q \neq \emptyset$, and so $x \in C'^{\rho}_q$. Finally, an example of a desired ρ' is

$$\rho'(x) = \inf_{z\in O} \left\{\rho''(z)/2 + \text{dist}_{\rho''(z)\mathbb{B}}(x - z)\right\}.$$

■

We ensure that for each $q \in Q$, each $x \in Q$ is either in $\text{int}C_q$ or $\text{int}D_q$ (or both) by setting

$$C_q = C'^{\rho'}_q. \tag{24}$$

Note that as k_q is defined on O (and is nonempty, etc., there), not just on C'_q, it is certainly well defined on C_q. Let ρ_e be the admissible perturbation radius as guaranteed by Lemma 5.9 and let $\delta(x) := \min\{\rho'(x), \rho_e(x)\}$.

Consider the hybrid feedback given by Q, the sets C_q in (24), the sets D_q in (22), the mapping k_q as in (23), and the mapping g_q as in (21). Let ξ and ζ be admissible measurement noise and admissible external disturbance so that (10) holds. By Lemma 5.9, solutions to $\mathcal{H}^{\xi,\zeta}_{\text{feed}}$ exist for any initial point in $O \times Q$. By construction, and since $\delta(x) \leq \rho(x)$, all solutions to $\mathcal{H}^{\xi,\zeta}_{\text{feed}}$ are solutions to $\mathcal{H}^\rho$. As for the latter system, $\mathcal{A}$ is asymptotically stable on O, so is the case for $\mathcal{H}^{\xi,\zeta}_{\text{feed}}$. This finishes the proof.

VI. Conclusion

The capabilities of hybrid feedback control have been recognized for some time now. We confirm them further, by showing that any asymptotically controllable to a compact set nonlinear system can be robustly stabilized by logic-based hybrid feedback. The basis for our construction is the observation that certain discontinuous feedbacks, after being recast as hybrid feedbacks, can be given some fundamental regularity properties that are impossible to achieve in continuous time. These regularity properties—outer semicontinuity of the flow and the jump mappings, and closedness of the flow and the jump sets—have previously been shown to enable general invariance principles for hybrid systems and to guarantee the existence of smooth Lyapunov functions for robustly stable hybrid systems. Here, they turn out to be sufficient to give a stabilizing hybrid feedback some nominal robustness to measurement errors and external disturbances.

References

[1] F. Ancona and A. Bressan, "Patchy vector fields and asymptotic stabilization," *ESAIM: Control Optim. Calc. Var.*, vol. 4, pp. 445–471, 1999.

[2] F. Ancona and A. Bressan, "Flow stability of patchy vector fields and robust feedback stabilization," *SIAM J. Control Optim.*, vol. 41, no. 5, pp. 1455–1476, 2003.

[3] F. Ancona and A. Bressan, , M. S. de Queiroz, M. Malisoff, and P. Wolenski, Eds., "Stabilization by patchy feedbacks and robustness properties," in *Lecture Notes in Control and Information Sciences*. Berlin, Germany: Springer-Verlag, 2004, vol. 301, Optimal Control, Stabilization and Nonsmooth Analysis, pp. 185–199.

[4] C. Cai, A. R. Teel, and R. Goebel, "Smooth Lyapunov functions for hybrid systems. Part I: Existence is equivalent to robustness," *IEEE Trans. Autom. Control*, vol. 52, no. 7, pp. 1264–1277, Jul. 2007.

[5] C. Cai, A. R. Teel, and R. Goebel, "Converse Lyapunov theorems and robust asymptotic stability for hybrid systems," in *Proc. 24th Amer. Control Conf.*, 2005, pp. 12–17.

[3]Even though $Q \subset \mathbb{Z}$ and $\succ$ is just the standard inequality $>$, we keep the general notation to indicate that the arguments can be easily repeated for more general index sets.

[6] F. H. Clarke, Y. S. Ledyaev, L. Rifford, and R. J. Stern, “Feedback stabilization and Lyapunov functions,” *SIAM J. Control Optim.*, vol. 39, no. 1, pp. 25–48, 2000.

[7] F. H. Clarke, Y. S. Ledyaev, E. D. Sontag, and A. I. Subbotin, “Asymptotic controllability implies feedback stabilization,” *IEEE Trans. Autom. Control*, vol. 42, no. 10, pp. 1394–1407, Oct. 1997.

[8] F. H. Clarke, Y. S. Ledyaev, and R. J. Stern, “Asymptotic stability and smooth Lyapunov functions,” *J. Differ. Equ.*, vol. 149, no. 1, pp. 69–114, 1998.

[9] R. T. Closkey and R. M. Murray, “Exponential stabilization of driftless nonlinear control systems using homogeneous feedback,” *IEEE Trans. Autom. Control*, vol. 42, no. 5, pp. 614–628, May 1997.

[10] J.-M. Coron, “Global asymptotic stabilization for controllable systems without drift,” *Math. Control Signals Syst.*, vol. 5, pp. 295–312, 1992.

[11] A. F. Filippov, *Diffrential Equations with Discontinuous Righthand Sides*. Norwell, MA: Kluwer, 1988.

[12] R. Goebel, J. Hespanha, A. R. Teel, C. Cai, and R. Sanfelice, “Hybrid systems: Generalized solutions and robust stability,” in *Proc. IFAC Symp. Nonliear Control Systems*, Stuttgart, Germany, 2004, pp. 1–12.

[13] R. Goebel and A. R. Teel, “Solutions to hybrid inclusions via set and graphical convergence with stability theory applications,” *Automatica*, vol. 42, pp. 573–587, 2006.

[14] O. Hájek, “Discontinuous differential equations, I,” *J. Differ. Equ.*, vol. 32, pp. 149–170, 1979.

[15] H. Hermes, “Discontinuous vector fields and feedback control,” in *Differential Equations and Dynamical Systems*. New York: Academic, 1967, pp. 155–165.

[16] J. P. Hespanha, D. Liberzon, and A. S. Morse, “Towards the supervisory control of uncertain nonholonomic systems,” in *Proc. 18th Amer. Control Conf.*, San Diego, CA, 1999, pp. 3520–3524.

[17] J. P. Hespanha and A. S. Morse, “Stabilization of nonholonomic integrators via logic-based switching,” *Automatica*, vol. 35, no. 3, pp. 385–393, 1999.

[18] C. M. Kellett and A. R. Teel, “Smooth Lyapunov functions and robustness of stability for difference inclusions,” *Syst. Control Lett.*, vol. 52, pp. 395–405, 2004.

[19] C. M. Kellett and A. R. Teel, “Weak converse Lyapunov theorems and control-Lyapunov functions,” *SIAM J. Control Optim.*, vol. 42, no. 6, pp. 1934–1959, 2004.

[20] N. N. Krasovskii, *Game-Theoretic Problems of Capture* (in Russian). Moscow, Russia: Nauka, 1970.

[21] Y. S. Ledyaev and E. D. Sontag, “A remark on robust stabilization of general asymptotically controllable systems,” in *Proc. Conf. Inf. Sci. Syst.*, Baltimore, MD, 1997, pp. 246–251.

[22] Y. S. Ledyaev and E. D. Sontag, “A Lyapunov characterization of robust stabilization,” *Nonlinear Anal.*, vol. 37, pp. 813–840, 1999.

[23] P. Lucibello and G. Oriolo, “Stabilization via iterative state steering with application to chained-form systems,” in *Proc 35th IEEE Conf. Decision Control*, Kobe, Japan, 1995, pp. 2614–2619.

[24] J. Lygeros, K. H. Johansson, S. N. Simić, J. Zhang, and S. S. Sastry, “Dynamical properties of hybrid automata,” *IEEE Trans. Autom. Control*, vol. 48, no. 1, pp. 2–17, Jan. 2003.

[25] M. Malisoff, L. Rifford, and E. D. Sontag, “Global asymptotic controllability implies input-to-state stabilization,” *SIAM J. Control Optim.*, vol. 42, no. 6, pp. 2221–2238, 2004.

[26] A. N. Michel and B. Hu, “Towards a stability theory of general hybrid dynamical systems,” *Automatica*, vol. 35, no. 3, pp. 371–384, 1999.

[27] P. Morin and C. Samson, “Exponential stabilization of nonlinear driftless systems with robustness to unmodeled dynamics,” *ESAIM Control Optim. Calc. Var.*, vol. 4, pp. 1–35, 1999.

[28] P. Morin and C. Samson, “Robust stabilization of driftless systems with hybrid open-loop/feedback control,” in *Proc. 19th Amer. Control Conf.*, Chicago, IL, 2000, pp. 3929–3933.

[29] C. Samson and P. Morin, “Practical stabilization of driftless systems on Lie groups: The transverse function approach,” *IEEE Trans. Autom. Control*, vol. 48, no. 9, pp. 1496–1508, Sep. 2003.

[30] C. Prieur, “Asymptotic controllability and robust asymptotic stabilizability,” *SIAM J. Control Optim.*, vol. 43, pp. 1888–1912, 2005.

[31] C. Prieur and A. Astolfi, “Robust stabilization of chained systems via hybrid control,” *IEEE Trans. Autom. Control*, vol. 48, no. 10, pp. 1768–1772, Oct. 2003.

[32] C. Prieur and E. Trélat, “Robust optimal stabilization of the brockett integrator via a hybrid feedback,” *Math. Control Signals Syst.*, vol. 17, pp. 201–216, 2005.

[33] C. Prieur and E. Trélat, “Quasi-optimal robust stabilization of control systems,” *SIAM J. Control Optim.*, vol. 45, no. 5, pp. 1875–1897, 2006.

[34] R. T. Rockafellar and R. J.-B. Wets, *Variational Analysis*. New York: Springer-Verlag, 1998.

[35] R. G. Sanfelice, R. Goebel, and A. R. Teel, “A feedback control motivation for generalized solutions to hybrid systems,” in *Lecture Notes in Computer Science*, ser. 3927, J. P. Hespanha and A. Tiwari, Eds. Berlin, Germany: Springer-Verlag, 2006, pp. 522–536.

[36] E. D. Sontag, “Clocks and insensitivity to small measurement errors,” *ESAIM Control Optim. Calc. Var.*, vol. 4, pp. 537–557, 1999.

[37] L. Tavernini, “Differential automata and their discrete simulators,” *Nonlinear Anal.*, vol. 11, no. 6, pp. 665–683, 1987.

[38] A. R. Teel and L. Praly, “A smooth Lyapunov function from a class-$\mathcal{KL}$ estimate involving two positive semidefinite functions,” *ESAIM Control Optim. Calc. Var.*, vol. 5, pp. 313–367, 2000.

[39] A. van der Schaft and H. Schumacher, *Lecture Notes in Control and Information Science*. Berlin, Germany: Springer-Verlag, 2000, vol. 251, An Introduction to Hybrid Dynamical Systems.

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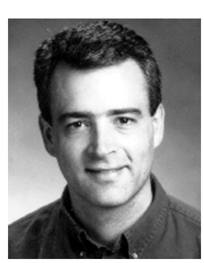

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