FPSAC 2011, Reykjavik, Iceland DMTCS proc. **(subm.)**, by the authors, 1–12

Demazure crystals and the energy function

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Abstract. There is a close connection between Demazure crystals and tensor products of Kirillov–Reshetikhin crystals. For example, certain Demazure crystals are isomorphic as classical crystals to tensor products of Kirillov–Reshetikhin crystals via a canonically chosen isomorphism. Here we show that this isomorphism intertwines the natural affine grading on Demazure crystals with a combinatorially defined energy function. As a consequence, we obtain a formula of the Demazure character in terms of the energy function, which has applications to nonsymmetric Macdonald polynomials and q-deformed Whittaker functions.

Résumé. Les cristaux de Demazure et les produits tensoriels de cristaux Kirillov–Reshetikhin sont étroitement liés. Par exemple, certains cristaux de Demazure sont isomorphes, en tant que cristaux classiques, à des produits tensoriels de cristaux Kirillov–Reshetikhin via un isomorphisme que l'on peut choisir canoniquement. Ici, nous montrons que cet isomorphisme entremêle la graduation affine naturelle des cristaux de Demazure avec une fonction énergie définie combinatoirement. Comme conséquence, nous obtenons une formule pour le caractère de Demazure exprimée au moyen de la fonction énergie, avec des applications aux polynômes de Macdonald non symétriques et au fonctions de Whittaker q-déformées.

Keywords: Demazure crystals, affine crystals, nonsymmetric Macdonald polynomials, Whittaker functions

1 Introduction

Kashiwara's theory of crystal bases [20] provides a remarkable combinatorial tool for studying highest weight representations of symmetrizable Kac–Moody algebras and their quantized universal enveloping algebras $U_q(\mathfrak{g})$. Here we consider finite-dimensional representations of the derived algebras $U_q'(\mathfrak{g})$, where $\mathfrak{g}$ is an affine Kac-Moody algebra. These representations do not extend to representations of $U_q(\mathfrak{g})$, but one can nonetheless define the notion of a crystal basis. In this setting crystal bases do not always exist, but, at least in non-exceptional cases, there is an important class of such modules which do have crystal bases. These are tensor products of the so-called Kirillov–Reshetikhin modules $W^{r,k}$ [24], where r is a node in the classical Dynkin diagram and k is a positive integer.

The modules $W^{r,k}$ were first conjectured to admit crystal bases $B^{r,k}$ in [14, Conjecture 2.1], and moreover it was conjectured that these crystals are perfect whenever k is a multiple of a particular constant c_r (perfectness is a technical condition which allows one to use the finite crystal to construct highest

† Partially supported by the NSF grants DMS–0652641, DMS–0652652, and DMS–1001256.
‡ Partially supported by NSF grant DMS-0902649

weight crystals, see [17]). This conjecture has now been proven in all non-exceptional cases (see [32, 33] for a proof that the crystals exist, and [7, Theorem 1.2] for a proof that they are perfect). We call the crystal of such a module a Kirillov–Reshetikhin (KR) crystal.

The perfectness of KR crystals ensures that they are related to crystals of highest weight affine crystals via the construction in [17]. In [21], Kashiwara proposed that this relationship is connected to the theory of Demazure crystals [19, 29], by conjecturing that perfect KR crystals are isomorphic as classical crystals to the Demazure crystals (which are subcrystals of affine highest weight crystals). This was proven in most cases in [4, 5]. More general relations between Demazure crystals and tensor products of perfect KR crystals were investigated in [25, 26, 27, 8].

There is a natural grading $\deg$ on a highest weight affine crystal $B(\Lambda)$, where $\deg(b)$ records the number of f_0 in a string of f_i's that act on the highest weight element to give b (which is well-defined by weight considerations). Due to the ideas discussed above, it seems natural that this grading should transfer to a grading on a tensor product of KR crystals.

Gradings on tensor products of KR crystals have in fact been studied, and are usually referred to as "energy functions." These were first defined in full generality in [34] by studying a tensor category of graded simple crystals following conjectural definitions in [13]. A function D is defined as a sum involving local energy functions for each pair in the tensor product (see [34, Proposition 2.14]), as well as a term counting the 'intrinsic energy' of each single KR crystal. It was suggested that there is a simple global characterization of intrinsic energy on B, related to the affine grading on a corresponding highest weight crystal (see [35, Section 2.5], [13, Proof of Proposition 3.9]), but no precise statement was proven.

1.1 Demazure crystals and energy

In the present work, we restrict to non-exceptional type (i.e. all affine Kac–Moody algebras except $A_2^{(2)}$, $G_2^{(1)}$, $F_4^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$, $E_6^{(2)}$ and $D_4^{(3)}$), where KR crystals are known to exist. We define the intrinsic energy function E^{int} on a tensor product B of KR crystals by letting $E^{\text{int}}(b)$ record the minimal number of f_0 in a path from u to b, where u is a certain unique element (see Definition 4.1). We then recall the explicit construction of the function D from [34] (which we will refer to as the 'D-function'). One purpose of this note is to show that D and E^{int} agree, up to addition of a global constant.

Our main tool is an enhancement of the relationship between KR crystals and Demazure crystals due to Fourier and Shimozono along with the first author. In [8, Theorem 4.4] it was shown that, under certain assumptions [8, Assumption 1], there is a unique embedding of the Demazure crystal into the KR crystal such that their classical crystal structure agrees and all zero edges in the Demazure crystal are taken to zero edges in the KR crystal (however, the KR crystal has more zero arrows). In most cases these assumptions follow from [6]. We deal with the remaining cases separately in Section 5, thereby firmly establishing the relationship between KR crystals and Demazure crystals in all non-exceptional types. We show that the resulting map intertwines the basic grading on the Demazure crystal with the D-function on the KR crystal, up to addition of a global constant. This in turn allows us to prove that E^{int} agrees with D up to a global constant, and in fact j interwines the basic grading with E^{int} exactly.

1.2 Applications

We furthermore discuss consequences of the relationship between grading and intrinsic energy in various contexts. First of all, it allows us to express the character of the Demazure modules in terms of the intrinsic energy (see Corollary 7.1). In addition, Ion [15] showed that, for the untwisted simply-laced affine root systems, the specialization of the nonsymmetric Macdonald polynomials $E_\lambda(x;q,t)$ at $t=0$ coincide with Demazure characters of level one affine integrable modules (see [36] for type A): $E_\lambda(x;q,0) = q^c\,\mathrm{ch}(V_{-\lambda}(\Lambda_0))$. Here c is a explicit exponent, while the affine Demazure character is computed by ignoring x^{Λ_0} and by setting $q := x^{-\delta}$. Note that if λ is dominant, then $E_\lambda(x;q,0) = P_\lambda(x;q,0)$ is the symmetric Macdonald polynomial.

As explained above, the Demazure module $V_{-\lambda}(\Lambda_0)$ can be realized as a tensor product of level one Kirillov–Reshetikhin crystals $B^{r_i,1}$. Since by Theorem 6.2 the intrinsic energy function intertwines with affine grading, this implies that the coefficients in the expansion of $P_\mu(x;q,0)$ in terms of the irreducible characters $\mathrm{ch}(V(\lambda))$ coincide with $X(\lambda; B^{r_1,1}\otimes B^{r_2,1}\otimes\cdots)$, where $V(\lambda)$ is the module of highest weight λ and $X(\lambda;B)$ is the one-dimensional configuration sum defined in terms of the intrinsic energy [13].

In addition, we discuss a relation between Demazure characters and q-deformed Whittaker functions for $\mathfrak{gl}_n$ [9, Theorem 3.2].

A long version of this paper containing all proofs will appear separately.

Acknowledgments

We thank Dan Bump, Cristian Lenart, Sergey Oblezin, and Masato Okado for enlightening discussions, and Nicolas Thiéry for his help with `Sage`. Most of the KR crystals have been implemented in the open-source mathematics system `Sage` (`sagemath.org`) by the first author.

2 Kac–Moody algebras and Crystals

Let $\mathfrak{g}$ be a Kac–Moody algebra. Let $\Gamma = (I,E)$ be its Dynkin diagram, where I is the set of vertices and E the set of edges. Let Δ denote the root system associated to $\mathfrak{g}$, and let P denote the weight lattice of $\mathfrak{g}$ and $P^\vee$ the coweight lattice. We denote by $\{\alpha_i \mid i\in I\}$ the set of simple roots and $\{\alpha_i^\vee \mid i \in I\}$ the set of simple coroots, with $Q = \bigoplus_{i\in I}\mathbb{Z}\alpha_i$ the root lattice and $Q^\vee = \bigoplus_{i\in I}\mathbb{Z}\alpha_i^\vee$ the coroot lattice.

Let $U_q(\mathfrak{g})$ be the corresponding quantum enveloping algebra, defined over $\mathbb{Q}(q)$. Let $\{E_i,F_i\}_{i\in I}$ be the standard elements in $U_q(\mathfrak{g})$ corresponding to the Chevalley generators of the derived algebra $\mathfrak{g}'$. We recall the triangular decomposition $U_q(\mathfrak{g}) \cong U_q(\mathfrak{g})^{<0}\otimes U_q(\mathfrak{g})^0 \otimes U_q(\mathfrak{g})^{>0}$, where $U_q(\mathfrak{g})^{<0}$ is the subalgebra generated by the F_i, $U_q(\mathfrak{g})^{>0}$ is the subalgebra generated by the E_i, and $U_q(\mathfrak{g})^0$ is the abelian group algebra generated by the usual elements K_w for $w\in P^\vee$, and the isomorphism is as vector spaces. Let $U_q'(\mathfrak{g})$ be the subalgebra generated by E_i, F_i and $K_i := K_{H_i}$ for $i\in I$.

We are particularly interested in the case when $\mathfrak{g}$ is of affine type. We will use the following conventions: W, P and Λ_i denote the affine Weyl group, the affine weight lattice, and the affine fundamental weight corresponding to $i\in I$, respectively, while $\overline{W}, \overline{P}$ and ω_i denote the weight lattice, Weyl group and fundamental weights corresponding to the finite type Dynkin diagram $I\backslash\{0\}$.

2.1 Crystals for $U_q(\mathfrak{g})$

We refer the reader to [12] for a detailed explanation of crystals. For us, a crystal is a nonempty set B along with operators $e_i : B\to B\cup\{0\}$ and $f_i : B \to B\cup\{0\}$ for $i\in I$, which satisfy some conditions. The set B records certain combinatorial data associated to a representation V of a symmetrizable Kac–Moody algebra $\mathfrak{g}$, and the operators e_i and f_i correspond to the Chevalley generators E_i and F_i of $\mathfrak{g}$. Often the definition of a crystal includes three functions $\mathrm{wt}\,, \varphi, \varepsilon : B \to P$, where P is the weight lattice. In the case of crystals of integrable modules, these functions can be recovered (up to a global shift in a null direction in cases where the Cartan matrix is not invertible) from knowledge of the e_i and f_i. Explicitly, we define the weight of the highest weight element in the crystal $B(\lambda)$ of an irreducible highest weight module to be λ, and require that each operator f_i have weight $-\alpha_i$.

An important theorem of Kashiwara states that every integrable $U_q(\mathfrak{g})$-highest weight module $V(\lambda)$ has a crystal basis. We denote the resulting $U_q(\mathfrak{g})$ crystal by $B(\lambda)$.

2.2 $U_q'(\mathfrak{g})$ crystals

In the case when the Cartan matrix is not invertible, one can define an extended notion of $U_q'(\mathfrak{g})$ crystals that includes some cases which do not lift to $U_q(\mathfrak{g})$ crystals. These crystals are still directed graphs coming from crystal bases of $U_q'(\mathfrak{g})$ modules. See e.g. [22]. We define a weight function on such a crystal as follows: First set $\varepsilon_i(b) := \max\{m \mid e_i^m(b) \neq 0\}$ and $\varphi_i(b) := \max\{m \mid f_i^m(b) \neq 0\}$. These are always finite because our crystals B correspond to an integrable module. For each $b \in B$, define three elements in the weight lattice of $\mathfrak{g}$ by:

$$\varphi(b) := \sum_{i \in I} \varphi_i(b)\Lambda_i, \quad \varepsilon(b) := \sum_{i \in I} \varepsilon_i(b)\Lambda_i, \quad \text{and} \quad \mathrm{wt}\,(b) := \varphi(b) - \varepsilon(b).$$

Then $\mathrm{wt}\,(b)$ is the weight function. Notice that $\mathrm{wt}\,(b)$ is always in the space $P' := \mathrm{span}\{\Lambda_i \mid i \in I\}$. If the Cartan matrix of $\mathfrak{g}$ is not invertible, P' is a proper sublattice of weight space P.

Remark 2.1 *The simple roots α_i are not in general in the span of the fundamental weights, so in this case the weight of the operator f_i is not $-\alpha_i$. It is rather the projection of $-\alpha_i$ onto the space of the fundamental weights in the direction which sends the null root to* 0.

Remark 2.2 *It is straightforward to check that if the Cartan matrix of $\mathfrak{g}$ is invertible, so that $U_q(\mathfrak{g}) = U_q'(\mathfrak{g})$, the above notion of weight agrees with the notion of weight from Section 2.1.*

2.3 Extended affine Weyl group

Fix $\mathfrak{g}$ of affine type. Write the null root as $\delta = \sum_{i \in I} a_i \alpha_i$. Following [13], for each $i \in I \backslash \{0\}$, define $c_i = \max(1, a_i/a_i^\vee)$. It turns out that $c_i = 1$ in all cases except (1) $c_i = 2$ for $\mathfrak{g} = B_n^{(1)}$ and $i = n$, $\mathfrak{g} = C_n^{(1)}$ and $1 \leq i \leq n-1$, $\mathfrak{g} = F_4^{(1)}$ and $i = 3, 4$, and (2) $c_2 = 3$ for $\mathfrak{g} = G_2^{(1)}$. Here we use Kac's indexing of affine Dynkin diagrams from [16, Table Fin, Aff1 and Aff2]. Consider the sublattices of $\overline{P}$ given by

$$M = \bigoplus_{i \in I \backslash \{0\}} \mathbb{Z} c_i \alpha_i = \mathbb{Z}\overline{W} \cdot \theta / a_0 \quad \text{and} \quad \widetilde{M} = \bigoplus_{i \in I \backslash \{0\}} \mathbb{Z} c_i \omega_i.$$

Here the finite type Weyl group $\overline{W}$ acts on $\overline{P}$ by linearizing the rules $s_i \lambda = \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i$. Clearly $M \subset \widetilde{M}$ and the action of $\overline{W}$ on $\overline{P}$ restricts to actions on M and $\widetilde{M}$. Let $T(\widetilde{M})$ (resp. $T(M)$) be the subgroup of $T(\overline{P})$ generated by the translations t_λ by $\lambda \in \widetilde{M}$ (resp. $\lambda \in M$).

There is an isomorphism [16, Prop. 6.5] $W \cong \overline{W} \ltimes T(M)$ as subgroups of $\mathrm{Aut}(P)$, where W is the affine Weyl group. Under this isomorphism we have $s_0 = t_{\theta/a_0} s_\theta$, where θ is the highest root of $I \backslash \{0\}$. Define the extended affine Weyl group to be the subgroup of $\mathrm{Aut}(P)$ given by $\widetilde{W} = \overline{W} \ltimes T(\widetilde{M})$.

Let $C \subset P \otimes_{\mathbb{Z}} \mathbb{R}$ be the fundamental chamber, the set of elements λ such that $\langle \alpha_i^\vee, \lambda \rangle \geq 0$ for all $i \in I$. Let $\Sigma \subset \widetilde{W}$ be the subgroup of $\widetilde{W}$ consisting of those elements that send C into itself. Then $\widetilde{W} = W\Sigma$, and in particular every element $x \in \widetilde{W}$ can be written uniquely as $x = w\tau$ for some $w \in W$ and $\tau \in \Sigma$.

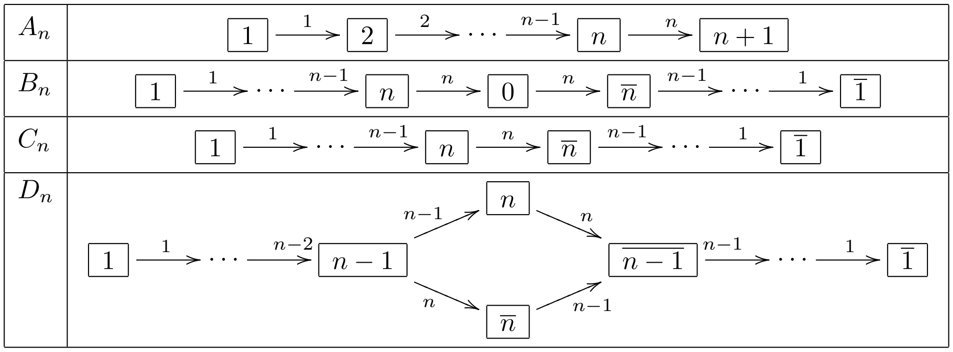

Fig. 1: Standard crystals $B(\omega_1)$

The usual affine Weyl group W is a normal subgroup of $\widetilde{W}$, so Σ acts on W by conjugation. Each $\tau \in \Sigma$ induces an automorphism (also denoted τ) of the affine Dynkin diagram Γ, which is characterized as the unique automorphism so that $\tau s_i \tau^{-1} = s_{\tau(i)}$ for each $i \in I$.

Remark 2.3 *When $\mathfrak{g}$ is of untwisted type, $M \cong Q^\vee$, $\widetilde{M} \cong P^\vee$, with the isomorphism given by $c_i\omega_i = \nu(\omega_i^\vee)$, and $c_i\alpha_i = \nu(\alpha_i^\vee)$ for $i \in I\backslash\{0\}$.*

2.4 Demazure modules and crystals

Let λ be a dominant integral weight for $\mathfrak{g}$. Define $W^\lambda := \{w \in W \mid w\lambda = \lambda\}$. Fix $\mu \in W\lambda$, and recall that the μ weight space in $V(\lambda)$ is one-dimensional. Let u_μ be a non-zero element of the μ weight space in $V(\lambda)$. Write $\mu = w\lambda$ where w is the shortest element in the coset wW^λ.

Define the Demazure module

$$V_w(\lambda) := U_q(\mathfrak{g})^{>0} \cdot u_{w(\lambda)}.$$

It is known that $V_w(\lambda)$ has a crystal base $B_w(\lambda)$ (see [19]). Define the set

$$f_w(b) := \{\, f_{i_N}^{m_N} \cdots f_{i_1}^{m_1}(b) \mid m_k \in \mathbb{Z}_{\geq 0}\}, \tag{2.1}$$

where $w = s_{i_N} \cdots s_{i_1}$ is any fixed reduced decomposition of w. By [19, Proposition 3.2.3], we know that, as sets, $B_w(\lambda) = f_w(u_\lambda)$.

For $\mathfrak{g}$ affine, we extend this definition to give a Demazure module and crystal $B_w(\lambda)$ for each $w \in \widetilde{W}$ as follows. We may express w uniquely as $w = z\tau$ where $z \in W$ and $\tau \in \Sigma$. We define the Demazure module to be $V_w(\lambda) := V_z(\tau(\lambda))$. Its crystal graph is denoted $B_w(\lambda) = B_z(\tau\lambda)$.

2.5 Non-exceptional finite type crystals

We call the set of symbols that show up in the boxes of the standard crystal of type $X_n = A_n, B_n, C_n, D_n$ the type X_n alphabet. Impose a partial order $\prec$ on this alphabet by saying $x \prec y$ iff x is to the left of y in the presentation of the standard crystals in Figure 1 (in type D_n, the symbols n and $\overline{n}$ are incomparable).

Definition 2.4 *Fix $\mathfrak{g}$ of type X_n, for $X = A, B, C, D$. Fix a dominant integral weight γ for $\mathfrak{g} = X_n$. Write $\gamma = m_1\omega_1 + m_2\omega_2 + \cdots + m_{n-1}\omega_{n-1} + m_n\omega_n$. Define a generalized partition $\Lambda(\gamma)$ associated to γ, which is defined case by case as follows:*

- *If $X = A, C$, Λ has m_i columns of each height i for each $1 \le i \le n$;*
- *If $X = B$, Λ has m_i columns of height i for each $1 \le i \le n-1$, and $m_n/2$ columns of height n;*
- *If $X = D$, Λ has m_i columns of each height i for each $1 \le i \le n-2$, $\min(m_{n-1}, m_n)$ columns of height $n-1$, and $|m_n - m_{n-1}|/2$ columns of height n.*

In cases where the above formulas involve a fractional number of columns at some height, we denote this by putting a single column of half width. Notice that this can only happen for columns of height n in $\Lambda(\gamma)$, and at worst we get a single column of width $1/2$.

In [23], the highest weight crystals of types A_n, B_n, C_n, D_n were constructed in terms of tableaux, now known as Kashiwara–Nakashima (KN) tableaux. An element in the highest weight crystal $B(\gamma)$, where γ is a non-spin dominant weight (m_n even for B_n and $m_{n-1} = m_n = 0$ for D_n), is realized inside the tensor product $B(\omega_1)^{\otimes|\Lambda(\gamma)|}$.

3 Kirillov–Reshetikhin modules and their crystals

Let $\mathfrak{g}$ be an affine Kac-Mody algebra with index set I. The Kirillov–Reshetikhin modules were first introduced for the Yangian of $\mathfrak{g}'$ in [24], and developed for $U'_q(\mathfrak{g})$ in [3]. One can characterize the KR module $W^{r,s}$ for $U'_q(\mathfrak{g})$, where $r \in I \setminus \{0\}$ and $s \ge 1$, as the irreducible representations of $U'_q(\mathfrak{g})$ whose Drinfeld polynomials are given by $P_i(u) = (1 - q_i^{1-s}u)(1 - q_i^{3-s}u) \cdots (1 - q_i^{s-1}u)$ if $i = r$ and 1 otherwise. Here $q_i = q^{(\alpha_i|\alpha_i)/2}$.

Theorem 3.1 *[33, 7] In all non-exceptional types, $W^{r,s}$ has a crystal base $B^{r,s}$. Furthermore, if s is a multiple of c_r then the resulting crystals are perfect, where $c_r = 2$ for type $B_n^{(1)}$ and $r = n$, and for type $C_n^{(1)}$ and $r < n$, and $c_r = 1$ in all other non-exceptonal cases.* □

Work of Chari [1] shows that every $B^{r,s}$ decomposes as a classical crystal as

$$B^{r,s} \cong \bigoplus_{\lambda} B(\lambda), \tag{3.1}$$

where the sum is over various classical highest weights λ. Explicitly, the λ which occur in the decomposition (3.1) are obtained from $s\omega_r$ by removing $\diamond$'s from $\Lambda(\lambda)$ (and furthermore, all occur with multiplicity 1 in the decomposition). Here (see for example [34, Eq. (6.27)], [13])

$$\diamond = \begin{cases} \emptyset & \text{for type } A_n^{(1)} \text{ and } 1 \le r \le n \\ & \text{for types } C_n^{(1)}, D_{n+1}^{(2)} \text{ and } r = n \\ & \text{for type } D_n^{(1)} \text{ and } r = n-1, n \\ \text{vertical domino} & \text{for type } D_n^{(1)} \text{ and } 1 \le r \le n-2 \\ & \text{for types } B_n^{(1)},\, A_{2n-1}^{(2)} \text{ and } 1 \le r \le n \\ \text{horizontal domino} & \text{for types } C_n^{(1)},\, D_{n+1}^{(2)} \text{ and } 1 \le r < n \\ \text{box} & \text{for type } A_{2n}^{(2)} \text{ and } 1 \le r \le n. \end{cases} \tag{3.2}$$

By [28, Proposition 3.8], a tensor product $B = B^{r_1,s_1} \otimes \cdots \otimes B^{r_N,s_N}$ of KR-crystals is connected. We refer to such a B as a *composite KR-crystal*. As in [17], if the factors are all perfect KR crystals of the same level ℓ, then $B = B^{r_1,\ell c_{r_1}} \otimes \cdots \otimes B^{r_N,\ell c_{r_N}}$ is also perfect of level ℓ. We refer to such a perfect crystal as a *composite KR-crystal of level* ℓ.

Explicit combinatorial models for KR crystals $B^{r,s}$ for the non-exceptional types were constructed in [6]. Here we just state a lemma which is important in the proof of the correct definition of the energy functions. The proof requires careful analysis in each case, and makes heavy use of [37, Lemma 5.1], which leads to a hands on description of the action of e_0 on X_{n-2} highest weight vectors in these crystals, where X_n is the underlying classical type.

Lemma 3.2 *Let $B^{r,s}$ be a KR crystal of non-exceptional type. Fix $b \in B^{r,s}$, and assume that b (resp. $e_0(b)$) lies in the classical component $B(\gamma)$ (resp. $B(\gamma')$) of* (3.1). *Then $\varepsilon_0(b) \leq \lceil s/c_r \rceil$ for $\diamond = \emptyset$ and otherwise:*

(i) $\Lambda(\gamma')$ *is either equal to* $\Lambda(\gamma)$*, or else is obtained from* $\Lambda(\gamma)$ *by adding or removing* $\diamond$ *as in* (3.2).

(ii) If $\varepsilon_0(b) > \lceil s/c_r \rceil$*, then* $\Lambda(\gamma')$ *is obtained from* $\Lambda(\gamma)$ *by removing a* $\diamond$.

4 Energy functions

Here we define two a priori different energy functions on tensor products of KR crystals. The function E^{int} is defined by a fairly natural "global" condition, and D is defined by summing up combinatorially defined "local" contributions. It was suggested (but not proven) in [35, Section 2.5] that these two functions in fact agree. This will be proven in Theorem 6.2 below.

4.1 The function E^{int}

The following is essentially the definition of a ground state path from [17].

Definition 4.1 *Let $B = B^{r_N,\ell c_{r_N}} \otimes \cdots \otimes B^{r_1,\ell c_{r_1}}$ be a composite level ℓ KR crystal. Define $u_B = u_B^N \otimes \cdots \otimes u_B^1$ to be the unique element of B such that (1) $\varepsilon(u_B^1) = \ell\Lambda_0$ and (2) for each $1 \leq j < N$, $\varepsilon(u_B^{j+1}) = \varphi(u_B^j)$. This is well-defined by the definition of a perfect crystal. The element u_B is called the ground state path of B.*

Definition 4.2 *Let B be a composite KR crystal of level ℓ and consider u_B as in Definition 4.1. Define the intrinsic energy $E^{\text{int}}(b)$ for $b \in B$ to be the minimal number of f_0 in a string $f_{i_N} \cdots f_{i_1}$ such that $f_{i_N} \cdots f_{i_1}(u_B) = b$.*

4.2 The D function

Definition 4.3 *The D-function on $B^{r,s}$ is the function defined as follows:*

(i) $D_{B^{r,s}} : B^{r,s} \to \mathbb{Z}$ *is constant on all classical components.*

(ii) On the component $B(\lambda)$*,* $D_{B^{r,s}}$ *records the maximum number of* $\diamond$ *that can be removed from* $\Lambda(\lambda)$ *such that the result is still a (generalized) partition, where* $\diamond$ *is as in* (3.2).

In those cases when $\diamond = \emptyset$, this is interpreted as saying that $D_{B^{r,s}}$ is the constant function 0.

Let B_1, B_2 be two affine crystals with generators u_1 and u_2, respectively, such that $B_1 \otimes B_2$ is connected. By [28, Proposition 3.8], this holds for any two KR crystals. The *combinatorial R-matrix* [17, Section 4] is the unique crystal isomorphism $\sigma : B_2 \otimes B_1 \to B_1 \otimes B_2$ such that $\sigma(u_2 \otimes u_1) = u_1 \otimes u_2$.

As in [17], [34, Theorem 2.4], there is a function $H = H_{B_2,B_1} : B_2 \otimes B_1 \to \mathbb{Z}$, unique up to global additive constant, such that, for all $b_2 \in B_2$ and $b_1 \in B_1$,

$$H(e_i(b_2 \otimes b_1)) = H(b_2 \otimes b_1) + \begin{cases} -1 & \text{if } i = 0 \text{ and LL,} \\ 1 & \text{if } i = 0 \text{ and RR,} \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Here LL (resp. RR) indicates that e_0 acts on the left (resp. right) tensor factor in both $b_2 \otimes b_1$ and $\sigma(b_2 \otimes b_1)$. When B_1 and B_2 are KR crystals, we normalize H_{B_2,B_1} by requiring $H_{B_2,B_1}(u_{B_2} \otimes u_{B_1}) = 0$, where u_{B_1} and u_{B_2} are as in Definition 4.1.

Definition 4.4 *For $B = B^{r_N,s_N} \otimes \cdots \otimes B^{r_1,s_1}$, set $D_j := D_{B^{r_j,s_j}} \sigma_1 \sigma_2 \cdots \sigma_{j-1}$ and $H_{j,i} := H_i \sigma_{i+1} \sigma_{i+2} \cdots \sigma_{j-1}$, where σ_j and H_j act on the j-th and $(j+1)$-st tensor factors and $D_{B^{r_j,s_j}}$ is the D-function for B^{r_j,s_j} as given in Definition 4.3 acting on the rightmost factor. The D-function $D_B : B \to \mathbb{Z}$ is defined as*

$$D_B := \sum_{N \geq j > i \geq 1} H_{j,i} + \sum_{j=1}^{N} D_j. \tag{4.2}$$

5 Perfect KR crystals and Demazure crystals

In [8] a precise relationship between KR crystals and Demazure crystals was established, under a few additional assumptions on the KR crystals. In most cases, those assumptions have now been shown to hold, mainly through the results of [7] showing that the relevant KR crystals are perfect. In a couple of special cases (type $A_{2n}^{(2)}$ and exceptional nodes in type $D_n^{(1)}$) the assumptions from [8] need to be proven separately or slightly modified, which we do in the long version of this paper. Thus we establish the following:

Theorem 5.1 *Let $B = B^{r_N,\ell c_{r_N}} \otimes \cdots \otimes B^{r_1,\ell c_{r_1}}$ be a level ℓ composite KR crystal. For each k, write $t_{-c_{r_k}\omega_{r_k}} \in T(\widetilde{M}) \subset \widetilde{W}$ as $t_{-c_{r_k}\omega_{r_k}} = v_{r_k}\tau_{r_k}$ where $v_{r_k} \in W$ and $\tau_{r_k} \in \Sigma$. Let $\tau = \tau_{r_N} \cdots \tau_{r_1}$ and $\lambda = c_{r_1}\omega_{\theta(r_1)} + \cdots + c_{r_N}\omega_{\theta(r_N)}$, where θ is the finite type diagram automorphism such that $\omega_{\theta(i)} = -w_0(\omega_i)$. Then there is a unique isomorphism of affine crystals*

$$j : B(\ell\Lambda_{\tau(0)}) \to B \otimes B(\ell\Lambda_0). \tag{5.1}$$

This satisfies $j(u_{\ell\Lambda_{\tau(0)}}) = u_B \otimes u_{\ell\Lambda_0}$, where u_B is the distinguished element from Definition 4.1, and

$$j\left(B_{t_{-\lambda}}(\ell\Lambda_0)\right) = B \otimes u_{\ell\Lambda_0}, \tag{5.2}$$

where $B_{t_{-\lambda}}(\ell\Lambda_0)$ is the Demazure crystal.

6 The affine grading via the energy function

We show that the two energy functions D and E^{int} from Section 4 agree up to addition of a simple overall constant, and that furthermore the map j from Theorem 5.1 intertwines the affine degree map with E^{int}. We now state these results precisely.

Definition 6.1 *Let* $\deg : B_{c_{r_1}w_{r_1}+\cdots+c_{r_N}w_{r_N}}(\ell\Lambda_0) \to \mathbb{Z}_{\geq 0}$ *be the affine degree map, defined by* $\deg(u_{\ell\Lambda_0}) = 0$*, and giving each* f_i *degree* $\delta_{i,0}$.

Theorem 6.2 *Fix a composite level* ℓ *KR-crystal* $B = B^{r_N,\ell c_{r_N}} \otimes \cdots \otimes B^{r_1,\ell c_{r_1}}$*. Let*

$$\tilde{j} : B_{c_{r_1}w_{r_1}+\cdots+c_{r_N}w_{r_N}}(\ell\Lambda_0) \to B$$

be the restriction of the map from Theorem 5.1 to $B_{c_{r_1}w_{r_1}+\cdots+c_{r_N}w_{r_N}}(\ell\Lambda_0)$*, where* $B \otimes u_{\ell\Lambda_0}$ *is identified with* B*. Then for any* $b \in B$*,* $\deg(b) = D(\tilde{j}(b)) - D(\tilde{j}(u_B)) = E^{int}(b)$.

These results are proven using the following lemma, which in turn follows from Lemma 3.2.

Lemma 6.3 *Let* $B = B^{r_N,\ell c_{r_N}} \otimes \cdots \otimes B^{r_1,\ell c_{r_1}}$ *be a composite level* ℓ *KR crystal, and fix* $b = b_N \otimes \cdots \otimes b_1 \in B$*. If* $e_0(b) \neq 0$ *then* $D(e_0(b)) \geq D(b) - 1$*, and if* $\varepsilon_0(b) > \ell$ *then this is an equality.*

7 Applications

In this section we show how the relation between the affine grading in the Demazure crystal and the energy function can be used to derive a formula for the Demazure character using the energy function, and discuss how they are related to nonsymmetric Macdonald polynomials and Whittaker functions.

7.1 Demazure characters

By definition the Demazure character is $\mathrm{ch}V_w(\lambda) = \sum_\mu \dim(V_w(\lambda))_\mu e^\mu$, where $(V_w(\lambda))_\mu$ is the μ weight space of the Demazure module $V_w(\lambda)$.

Kashiwara [19] proved a conjecture of Littelmann [30] that the Demazure character has a simple expression in terms of the Demazure crystal $B_w(\lambda)$ given by

$$\mathrm{ch}V_w(\lambda) = \sum_{b \in B_w(\lambda)} e^{\mathrm{wt}\,(b)}. \tag{7.1}$$

It follows immediately that:

Corollary 7.1 *Let* $B = B^{r_N,\ell c_{r_N}} \otimes \cdots \otimes B^{r_1,\ell c_{r_1}}$ *be a* $U_q'(\mathfrak{g})$*-composite level-*ℓ *KR crystal and* $\lambda = c_{r_1}\omega_{\theta(r_1)} + \cdots + c_{r_N}\omega_{\theta(r_N)}$ *with* θ *as in Theorem 5.1. Then*

$$\mathrm{ch}V_{t_{-\lambda}}(\ell\Lambda_0) = e^{\ell\Lambda_0} \sum_{b \in B} e^{wt(b) - \delta E^{int}(b)} = e^{\ell\Lambda_0} \sum_{b \in B} e^{\mathrm{wt}^{\mathrm{aff}}(b)}, \tag{7.2}$$

where $t_{-\lambda} = v\tau$*,* $\mathrm{wt}^{\mathrm{aff}}(b) = \mathrm{wt}\,(b) - \delta E^{int}(b)$ *and* $\mathrm{wt}\,(b)$ *is the* $U_q'(\mathfrak{g})$*-weight of* b*.*

7.2 Nonsymmetric Macdonald polynomials

Recall that P is the weight lattice for $\mathfrak{g}$. Let X be the ambient space for the weight lattice. For example for GL_n, $X = \mathbb{Z}^n$ and $\alpha_i = e_i - e_{i+1}$ where e_i is the ith unit vector in $\mathbb{Z}^n$. Then we can identify $\mathbb{Q}(q,t)X$ with the Laurent polynomial ring $\mathbb{Q}(q,t)[x_1^{\pm 1}, \dots, x_n^{\pm}]$.

Cherednik's inner product [2] on $\mathbb{Q}(q,t)X$ is defined by $\langle f, g\rangle_{q,t} = [x^0](f\overline{g}\Delta_1)$, where $\bar{\cdot}$ is the involution $\overline{q} = q^{-1}$, $\overline{t} = t^{-1}$, $\overline{x}_i = x_i^{-1}$, $\Delta_1 = \Delta/([x^0]\Delta)$, and $[x^0]$ denotes the constant term in the expression. Furthermore,

$$\Delta = \prod_{\alpha \in R_+^{\mathrm{aff}}} \frac{1 - x^\alpha}{1 - t x^\alpha},$$

where R_+^{aff} is the set of positive affine real roots.

The nonsymmetric Macdonald polynomials $E_\lambda(x;q,t) \in \mathbb{Q}(q,t)X$ for $\lambda \in P$ were first introduced by Opdam [31] in the differential setting and Cherednik [2] in general. Here we use the conventions of Haglund, Haiman, Loehr [10, 11]. The nonsymmetric Macdonald polynomials are uniquely characterized by (i) (Triangularity): $E_\lambda \in x^\lambda + \mathbb{Q}(q,t)\{x^\mu \mid \mu < \lambda\}$ and (ii) (Orthogonality): $\langle E_\lambda, E_\mu\rangle_{q,t} = 0$ for $\lambda \neq \mu$. Here $<$ is Bruhat ordering on X where we identify X with the set of minimal coset representatives in $\widetilde{W}/\overline{W}$, where $\widetilde{W}$ is the extended affine Weyl group and $\overline{W}$ is the classical Weyl group.

Extending Sanderson's work [36], Ion [15] showed that for all untwisted affine root systems except $B_n^{(1)}$, $C_n^{(1)}$, $F_4^{(1)}$, $G_2^{(1)}$ the specialization of the nonsymmetric Macdonald polynomials $E_\lambda(x;q,t)$ at $t=0$ coincide with Demazure characters of level one affine integrable modules (see [36] for type A):

$$E_{\tilde{\lambda}}(x;q,0) = q^c \operatorname{ch}(V_{-\lambda}(\Lambda_0)). \tag{7.3}$$

Here c is a specific exponent, and the affine Demazure character is specialized by setting $x^{\Lambda_0} = 1$ and $q := x^{-\delta}$. Also, $\tilde{\lambda} = w_0\lambda$.

Example 7.2 *The nonsymmetric Macdonald polynomial of type $A_2^{(1)}$ indexed by $(0,0,2)$ is given by*

$$E_{(0,0,2)}(x;q,0) = x_1^2 + (q+1)x_1x_2 + x_2^2 + (q+1)x_1x_3 + (q+1)x_2x_3 + x_3^2.$$

The weight $\widetilde{\lambda} = (0,0,2)$ corresponds to $\lambda = 2\omega_1$. The translation $t_{-2\omega_{\theta(1)}} = t_{-2\omega_2}$ is given by $t_{-2\omega_2} = \tau s_2 s_1 s_0 s_2$. Hence the Demazure crystal is given by

$$\begin{array}{cccccccc} 2\otimes 1 \xrightarrow{2} 3\otimes 1 & \xrightarrow{0} 1\otimes 1 \xrightarrow{1} 1\otimes 2 & & \xrightarrow{1} 2\otimes 2 \xrightarrow{2} 2\otimes 3 \xrightarrow{2} 3\otimes 3 \\ & \overset{1}{\searrow} \quad 3\otimes 2 & & \overset{2}{\searrow} \quad 1\otimes 3 \quad \overset{1}{\nearrow} \end{array}$$

From this it is easy to verify that $E_{(0,0,2)}(x,q,0) = q^2 \mathrm{ch}(V_{-2\omega_2})(\Lambda_0)$.

7.3 Whittaker functions

Gerasimov, Lebedev, Oblezin [9, Theorem 3.2] showed that q-deformed $\mathfrak{gl}_n$-Whittaker functions are Macdonald polynomials specialized at $t=0$, which by our previous discussion also gives a link to Demazure characters. The q-deformed $\mathfrak{gl}_n$-Whittaker functions are simultaneous eigenfunctions of a q-deformed Toda chain. It would be interesting to generalize this to other types.

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