The half Twist

(joint with Noah Snyder)

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Outline

- Quantum group knot invariants
- The half twist
- Half twist for $U_q(\mathfrak{g})$-representations
- Half twist Hopf algebras

Quantum group knot invariants

We need some topological categories

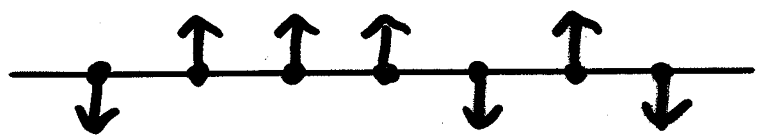
1) Tangle

Objects: $[n]$ =

Morphisms: Tangles

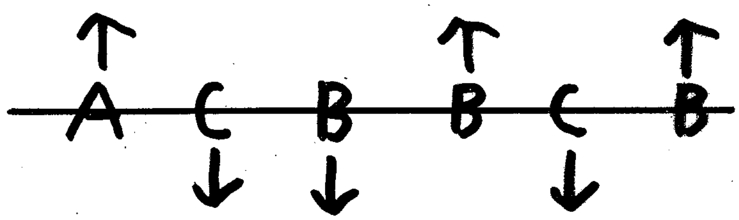
2) DTangle

Objects:



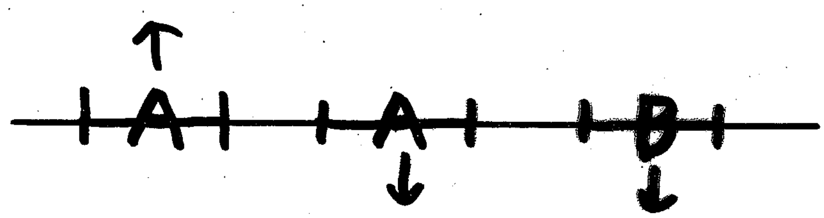
3) DTangle(X)

Objects:



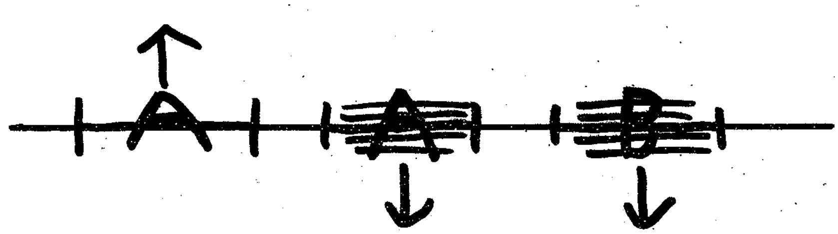
4) DRibbon(X)

Objects:

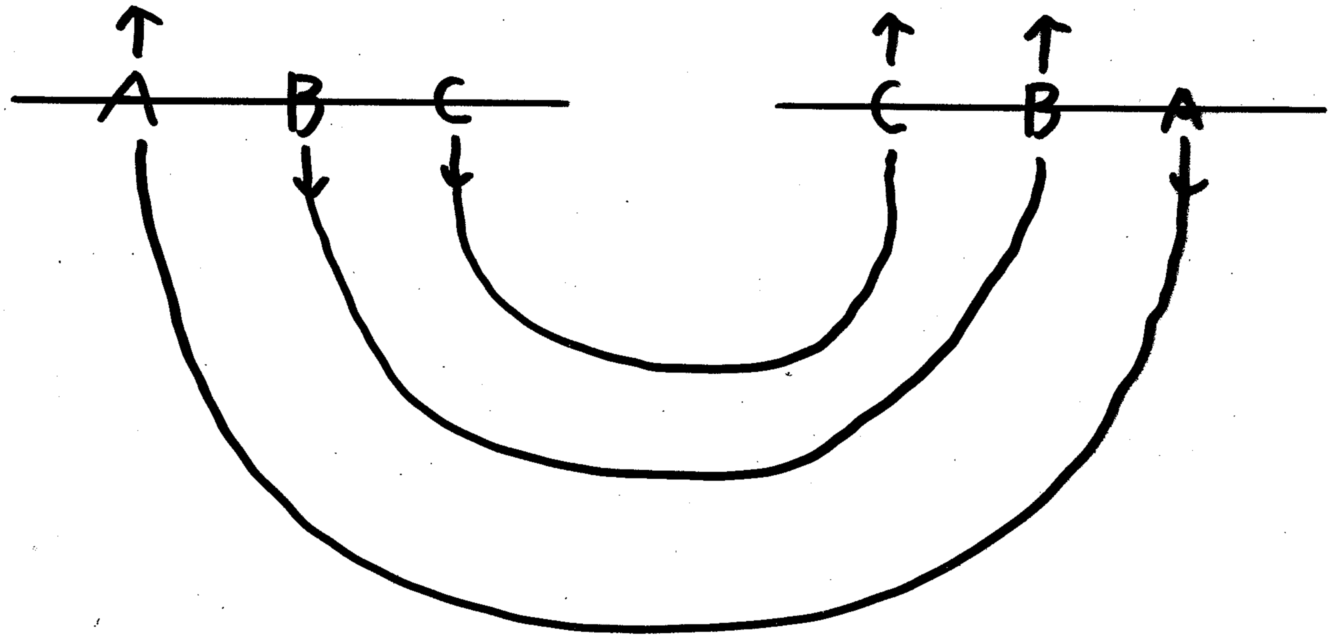


5) DHRibbon(X)

Objects:



Fact: DTangle(X) is a rigid monoidal category



Theorem (almost)

There is a tensor functor

$$\mathrm{DTangle}(X) \xrightarrow{\varphi} U_q(\mathfrak{g})\text{-rep}$$

$$\overset{\uparrow}{A} \longrightarrow A$$

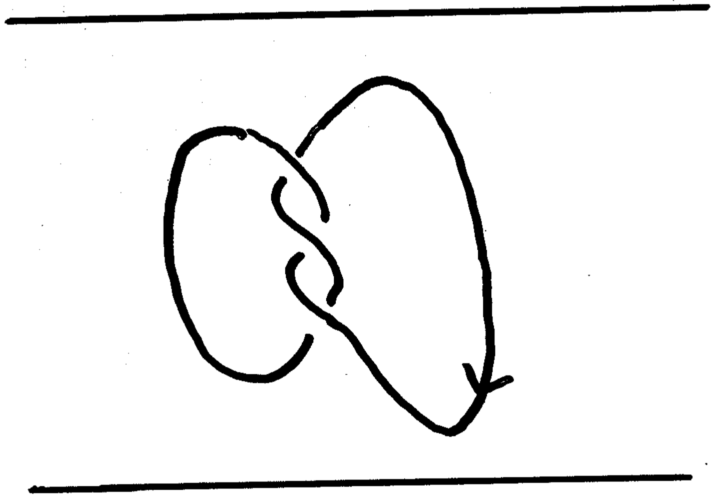
$$\underset{\downarrow}{A} \longrightarrow A^*$$

caps / cups $\longrightarrow$ ev, coev, qtr, coqtr

$\longrightarrow$ $A \otimes B \longrightarrow B \otimes A$

standard braiding

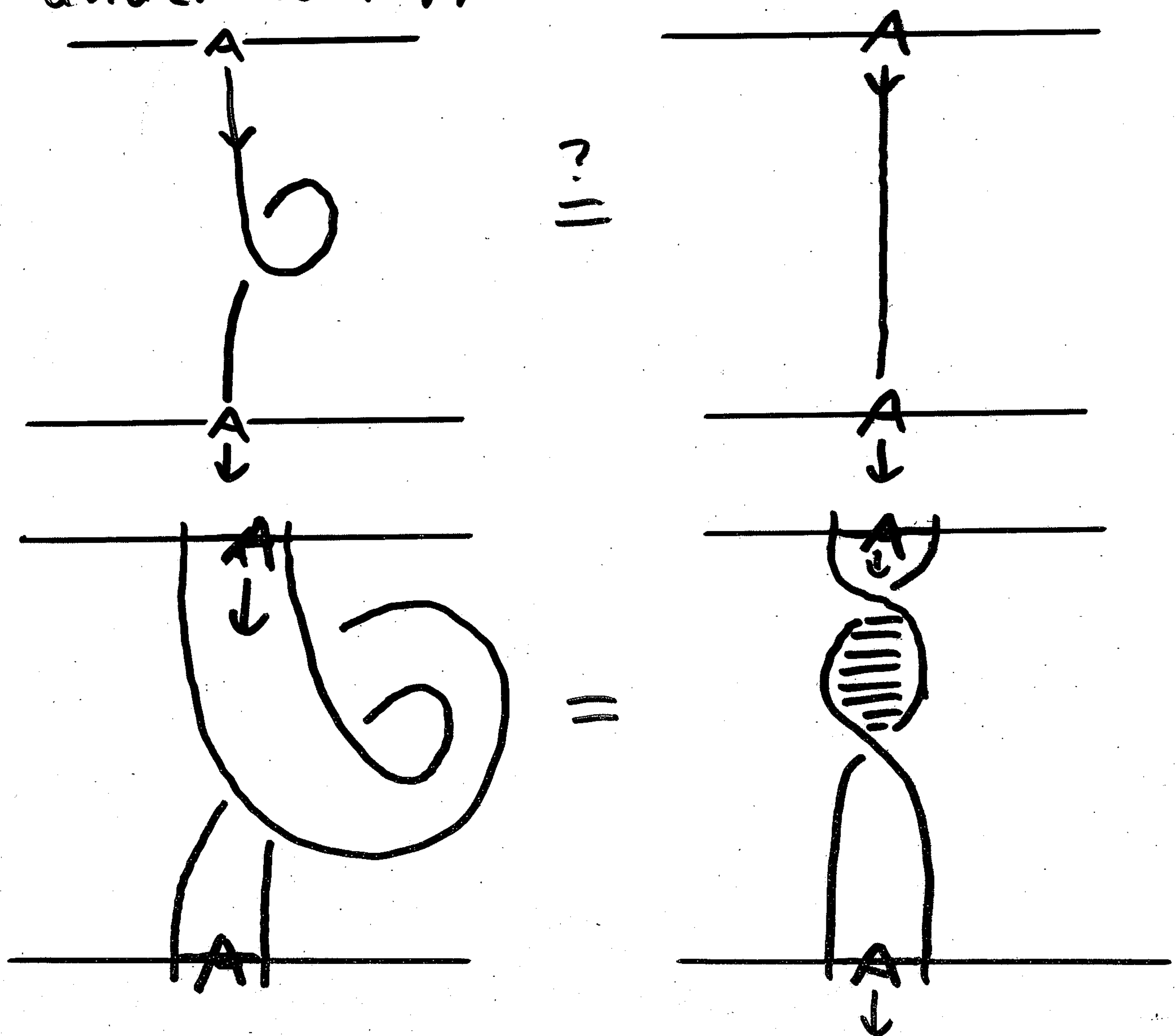
$= \mathrm{Flip} \circ R$



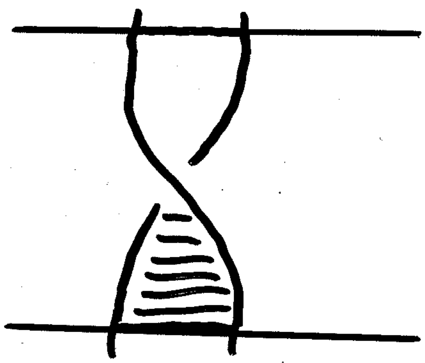
$\longrightarrow$

$$\begin{array}{c} \mathbb{C}(q) \\ \Big\downarrow \text{Scalar} \\ \mathbb{C}(q) \end{array}$$

Need to check invariance under isotopy



Annoyance

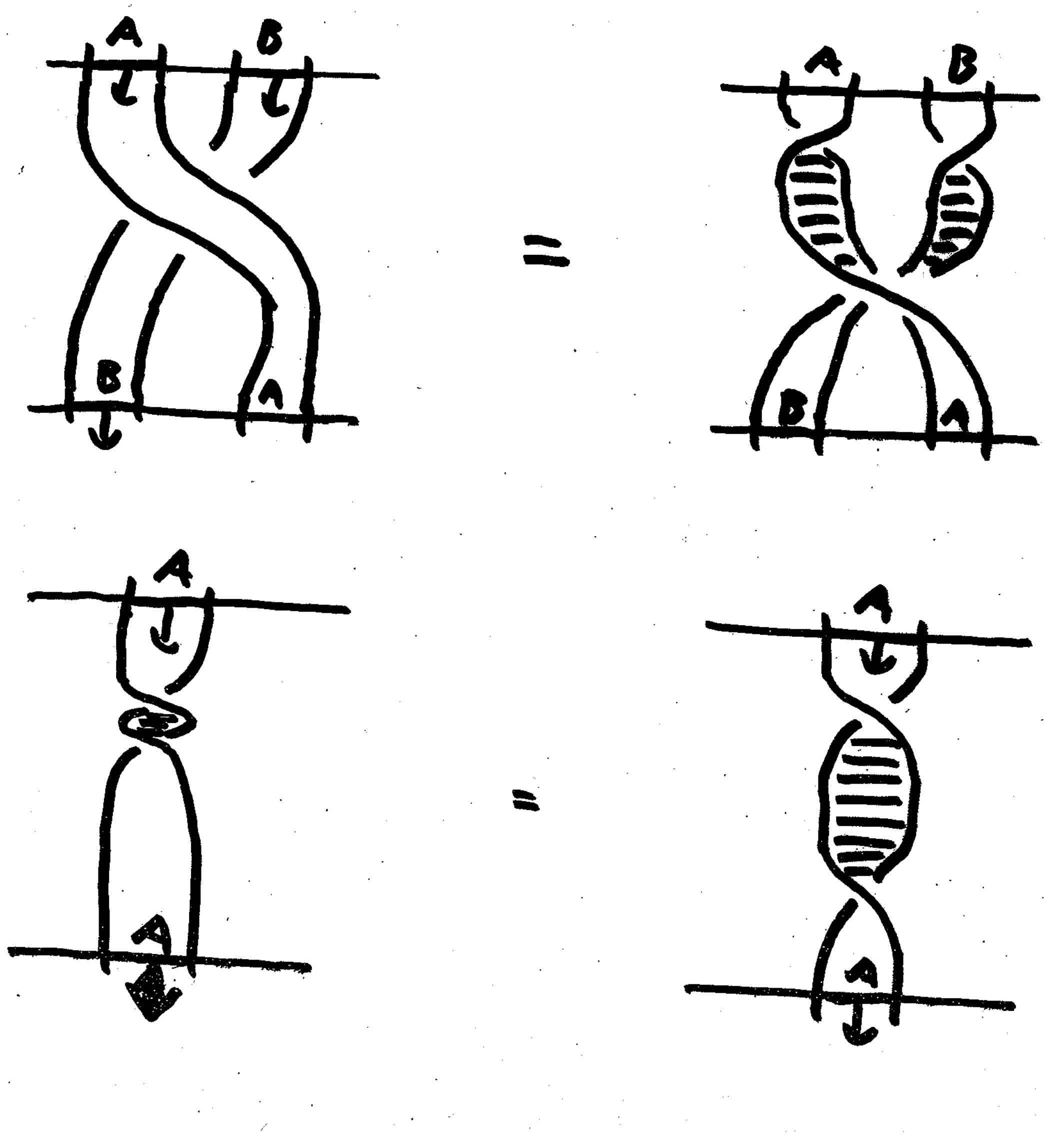


is not allowed

The half twist

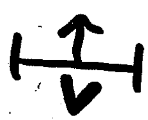
Morphisms in DHRibbon(X) are generated by

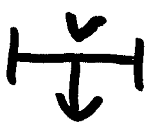
- Caps and cups
- The half twist



Definition: A half ribbon category is a rigid monoidal category $\mathcal{C}$ with a tensor functor $\mathrm{DHRibbon}(X) \to \mathcal{C}$ for any finite subset X of $\mathrm{Ob}(\mathcal{C})$

Theorem: $U_q(\mathfrak{g})$-rep is a half ribbon category

 $\longrightarrow V$

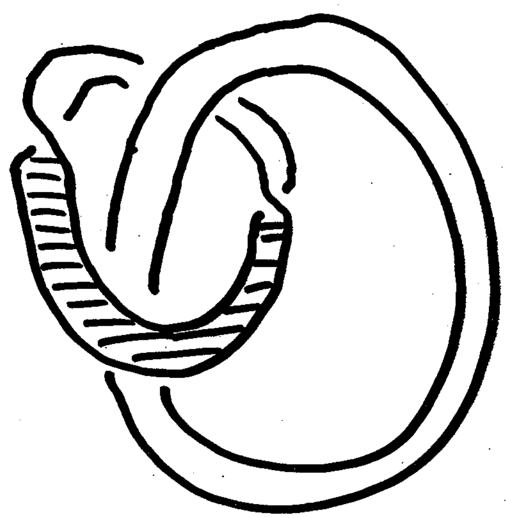
 $\longrightarrow V^*$

 $\longrightarrow V^A$

 $\longrightarrow (V^*)^A = {}^*(V^A)$

$$\begin{cases} A(E_i) = -F_i \\ A(F_i) = -E_i \\ A(K_i) = K_i^{-1} \end{cases}$$

$$\chi_{V_\lambda} : v_\lambda \longrightarrow q^{\langle\lambda,\lambda\rangle/2 + \langle\lambda,\rho\rangle} v_\lambda^{low}$$



<u>Proof</u>:

By a theorem of Kirilov - Reshetikhin and Levendorski - Soibelman

$$R = (X^{-1} \otimes X^{-1}) \Delta X$$

where $X = \chi$ as a vector space map

Half twist Hopf algebras

- Need a Hopf algebra $\mathcal{H}$, with a chosen element $t \in \mathcal{H}$
- Define $C_t : \mathcal{H} \longrightarrow \mathcal{H}$

  $u \to t u t^{-1}$
- C_t must be a coalgebra anti-aut

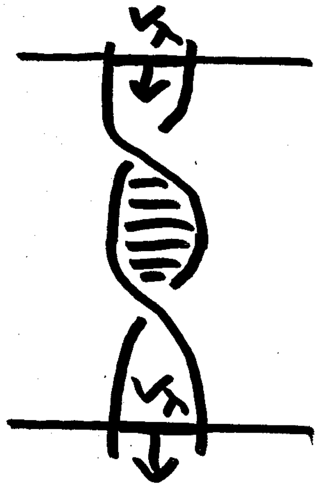
  $\Rightarrow$ $V \otimes W \xrightarrow{\simeq} W \otimes V$

  $v \otimes w \to \mathrm{Flip} \circ t^{-1} \otimes t^{-1} \circ \Delta t$
- t must satisfy
  - t^2 is central
  - $\varepsilon(t) = 1$
  - $S(t)t^{-1} = S(t^{-1})t$ is grouplike

$\Rightarrow$ Ribbon element & qtrace

- $R = (t^{-1} \otimes t^{-1})\Delta t$ must be a quasi-triangular structure

A consequence in $U_q(sl_2)$



$=$

Normal: $q^{\langle\lambda,\lambda\rangle}$.

Here : $(-1)^{\langle 2\lambda,\rho\rangle} q^{\langle\lambda,\lambda\rangle}$

Try to connect to Temperley - Leib algebra

Normal: Disoriented version (Morrison - Walker)

Here: Unoriented