Root multiplicities from quiver varieties

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with appendix coauthored by Colin Williams

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Outline

1. Background
   - What are Kac-Moody algebras and root multiplicities?
   - What are Crystals?
   - What are quiver varieties and how do they help?
2. Our method/Conjecture
3. Evidence
   - Exact Data
   - Heuristics

What are Kac-Moody algebras?

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$$\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$$

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$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & 0 & 0 \end{pmatrix}$$

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- $\mathfrak{sl}_3$:

1

1 1

2

1 1

1

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- $\widehat{\mathfrak{sl}}_2$:

	⋮	
⋮		⋮
	1	
1		1
	1	
1		1
	1	
1		1
	3	
1		1
	1	
1		1
	1	
1		1
	1	
⋮		⋮
	⋮	

What are Kac-Moody algebras?

- "Fibonacci": Cartan matrix $\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$

1		9		23		23		9		1
	2		9		16		9		2	
		3		9		9		3		
			4		6		4			
		1		4		4		1		
			1		3		1			
				2		2				
			1		1		1			
				1		1				
					1					
				1		1				
					2					
				1		1				
					1					
				1		1				
			1		1		1			
				2		2				
			1		3		1			
		1		4		4		1		
			4		6		4			
		3		9		9		3		
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- Formulae exist (Berman-Moody and Peterson), based on Weyl denominator identity. So, the point is “good," or maybe “combinatorial"
- We mostly consider the simplest hyperbolic case, and there there are combinatorial formulae (Kang-Melville, Carbone-Freyn-Lee, Kang-Lee-Lee), which use similar combinatorial objects to what we use...but there seem to be serious differences in the details and methods.

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- You can make a colored graph, where nodes are basis vectors, and arrows approximate actions of Chevalley generators.
- It has a subgraph for every highest weight integrable representation...but right now we don't really care about that.

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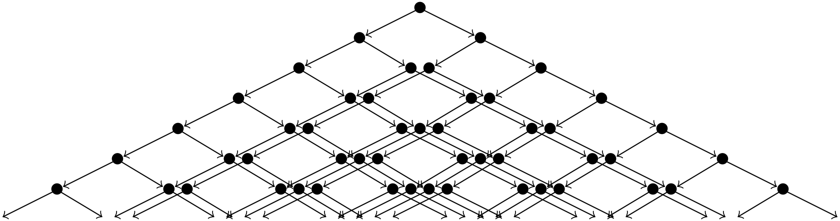
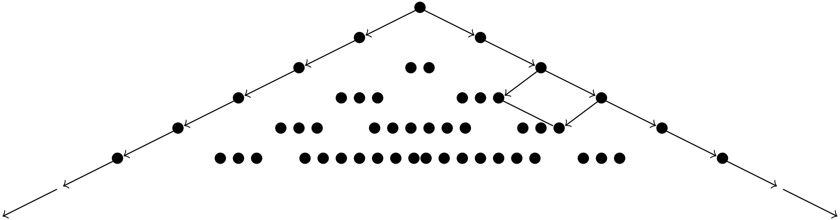


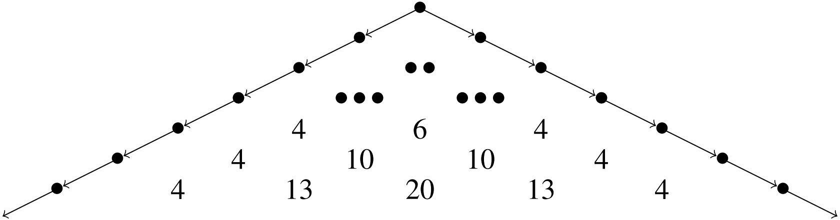
Examples of infinity crystals

- $\widehat{\mathfrak{sl}}_2$:



Examples of infinity crystals

- Hyperbolic with Cartan matrix $\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$



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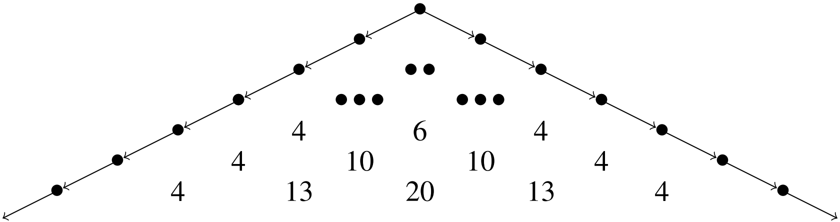

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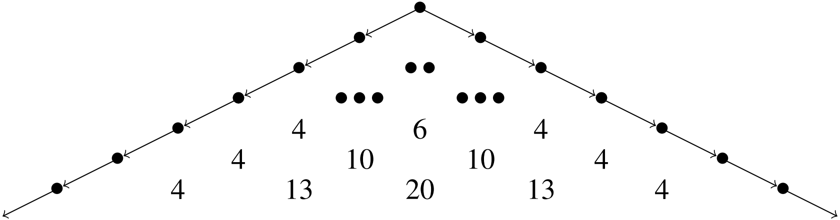

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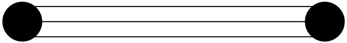
- These are given by Kostant partitions, so this is highly related.

How do quiver varieties help?



- Preprojective algebra is path algebra mod a generic quadratic relation.
- Elements of $B(\infty)$ correspond to irreducible components of the variety of nilpotent representations of this algebra.
- These irreducible components can be identified by the form of the Harder-Narasimhan filtration of their points (work with Kamnitzer Baumann).
- Note: only two irreps, Which we call **0** and **1**. We will identify representations (or families of representations) by a socle filtration.

Example

- Here are the HN filtrations of the irreducible components of the variety of irreducible representations on $\mathbb{C}^2 + \mathbb{C}^3$:

$$
\begin{array}{c} 1 \oplus 1 \oplus 1 \\ \hline 0 \oplus 0 \end{array}
\qquad
\begin{array}{c} 1 \oplus 1 \\ \hline 0 \\ 1 \\ \hline 0 \end{array}
\qquad
\begin{array}{c} 1 \\ \hline 0 \\ 11 \\ \hline 0 \end{array}
\qquad
\begin{array}{c} 0 \\ 111 \\ \hline 0 \end{array}
\qquad
\begin{array}{c} 11 \\ \hline 00 \\ 1 \end{array}
$$

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$$

- Correctly predicts that $B(\infty)$ has 10 elements in this degree.
- There are exactly two with a trivial filtration, which corresponds to the root multiplicity of $2\alpha_0 + 3\alpha_1$ being 2.

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  - The corresponding component is stable.
- This idea was partly suggested to me by Alex Feingold.

Translating conditions to combinatorics

- The conditions on the previous page aren't very tractable.
- We will translate them into
  - two combinatorial conditions
  - An error term.
- In fact, we can start adding more combinatorial conditions and get better estimates, but I have no great hope of describing them all.

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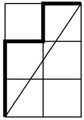

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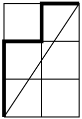

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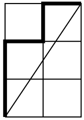

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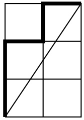

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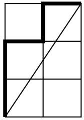

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- For string data $(a_1, a_2, \ldots, a_{2k})$, for all $0 \le x < y < k$,

$$\frac{a_1 + \cdots + a_{2x-1} + (a_{2x+2} + \cdots + a_{2y}) - a_{2x+3} - \cdots - a_{2y+1}}{a_2 + \cdots + a_{2y}}$$

is at most the slope of the Dyck path. Rules out e.g. $1^3 0^2 1^5 0^5$.

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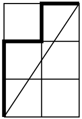

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- Many more conditions...but they all seem to be weak:

Conjectures

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For this rank 2 algebra, the Number of rational Dyck paths satisfying the ratio condition is a good estimate of the root multiplicity of $m\alpha_0 + n\alpha_1$ *provided* $gcd(m,n) = 1$ *and* $m\alpha_0 + n\alpha_1$ *is far inside the imaginary cone.*

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- I hope/believe this means the number of rational Dyck paths satisfying the ratio condition for e.g. $(n+1)\alpha_0 + n\alpha_1$ is $\mathcal{O}$ of the correct answer. Or at least the error grows extremely slowly.
- Something similar should hold going out along any line.
- Something similar should be true in other types.

Data

Calculated in SAGE with my student Colin Williams

Root	Estimate using only ratio	Estimate with next condition	Actual multiplicity
$15\alpha_0 + 14\alpha_1$	278335	271860	271860
$16\alpha_0 + 15\alpha_1$	837218	815215	815214
$15\alpha_0 + 16\alpha_1$	1234431	817505	815214

Our estimates are generally more accurate for roots $m\alpha_0 + n\alpha_1$ with $m > n$. Here is the one word we over-counted for $16\alpha_0 + 15\alpha_1$:

$$1^{10}0^3 1^5 0^{13}.$$

It should be ruled out because the quotient $1^5 0^{13}$ generates $10^2 1^5 0^{13}$, which has the submodule 10^2.

Monte-Carlo data

- We also estimated large root multiplicities by sampling Dyck paths, and estimating the proportion that satisfy each condition.

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- Here are some results. Each took about 24 hours on a 2018 laptop.

Root	Paths sampled	First estimate	*Better* estimate
$51\alpha_0 + 50\alpha_1$	10^9	2.2283×10^{23}	2.0419×10^{23}
$50\alpha_0 + 51\alpha_1$	10^9	3.4013×10^{23}	2.0476×10^{23}

Heuristics

- For large k, the expected number of returns a random rational Dyck path makes to distance r from the diagonal stays around $4r+4$. Does not grow!
- Stability fails when consecutive edge lengths a_k, a_{k+1} generate a problematic submodule, but this only has “local" effect.
- You need to both be close to the boundary and close to the ratio at once....unlikely.
- I can’t prove it is unlikely enough though.

Thanks for listening!!!!!!!!