CO-POISSON HOPF ALGEBRAS, DEFORMATION THEORY OF CO-POISSON HOPF ALGEBRAS AND QUANTUM GROUPS

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Let A be a Hopf algebra, and consider the monoidal category $\mathrm{Rep}(A)$. Is it possible to define an action of the braid group B_n on $\mathrm{Rep}(A)$ or, even better, an action of the symmetric group Σ_n on $\mathrm{Rep}(A)$? This of course depends on A, and in particular, depends on the existence of what is called an R-matrix. This is an element $R \in A \otimes A$ with some properties to be defined later.

We will be interested in the case $A = U_\hbar(\mathfrak{g})$, where A is a quantization of $U(\mathfrak{g})$. Suppose we start with the Hopf algebra $U_\hbar(\mathfrak{g}) = (U_\hbar, \mu_\hbar, \Delta_\hbar)$ and we define

$$\delta(x) = \frac{\Delta_\hbar^{\mathrm{op}}(a) - \Delta_\hbar(a)}{\hbar} \pmod{\hbar}$$

for $x \in U(\mathfrak{g})$ and $a = x \pmod{\hbar}$. This gives a map $\delta\colon U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$, which is a (co)-Poisson structure on $U(\mathfrak{g})$. By restricting to $\mathfrak{g}$, and getting a map $\delta\colon \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$, we get a Lie bialgebra structure on $\mathfrak{g}$.

What about going in the other direction? Starting with a map δ such that $\delta(a_1a_2) = \delta(a_1)\Delta(a_2) + \Delta(a_1)\delta(a_2)$, we can try to quantize $U(\mathfrak{g})$. It turns out that the answer is yes, provided δ is a Lie bialgebra structure. We will also see conditions on δ which will ensure that the resulting quantized Hopf algebra will have the nice properties discussed above.

All the results discussed today can be found in [CP], along with references to original sources and proofs.

1. Deformations of Poisson–Lie structures

To make everything clear let's start by defining what a quantization is. Lets recall what a quantization in the case of a Poisson algebra is.

Definition 1.1. Let $(A, \mu, \{,\})$ be a commutative Poisson algebra. A **quantization** of A is a deformation $A_\hbar = (A_\hbar, \mu_\hbar)$ such that

$$\{a, b\} = \frac{\mu_\hbar(a, b) - \mu_\hbar(b, a)}{\hbar} \pmod{\hbar}. \qquad \square$$

Before defining quantizations in the case of Hopf algebras, let's define the kind of structure that we want to quantize.

Definition 1.2. A **(co)-Poisson Hopf algebra** $A = (A, \mu, \Delta, \delta)$ is a Hopf algebra with a skey-symmetric map $\delta\colon A \to A \otimes A$ such that

(1) the composite

$$A \xrightarrow{\delta} A \otimes A \xrightarrow{\delta \otimes id} A \otimes A \otimes A \xrightarrow{c.p.} A \otimes A \otimes A$$

is zero, where c.p. means the sum over cyclic permutations.

(2) The co-Leibniz rule identity

$$(\Delta \otimes id)\delta = (id \otimes \delta)\Delta + \sigma_{23}(\delta \otimes id)\Delta$$

holds.

(3) $\delta(a_1a_2) = \delta(a_1)\Delta(a_2) + \Delta(a_1)\delta(a_2)$. $\square$

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As in the case of algebras, also here a quantization will be a deformation with an additional property.

Definition 1.3. A **deformation** of a Hopf algebra $A = (A, \mu, \Delta)$ over a field k is a topological Hopf algebra $A_\hbar = (A_\hbar, \mu_\hbar, \Delta_\hbar)$ over the ring $k[[\hbar]]$ such that

(1) $A_\hbar$ is isomorphic to $A[[\hbar]]$ as a $k[[\hbar]]$ module.
(2) $\mu_\hbar = \mu \pmod{\hbar}$ and $\Delta_\hbar = \Delta \pmod{\hbar}$.

□

Definition 1.4. A **quantization** of a (co)-Poisson co-commutative Hopf algebra $A = (A, \mu, \Delta, \delta)$ is a deformation $A_\hbar = (A_\hbar, \mu_\hbar, \Delta_\hbar)$ such that

$$\delta(x) = \frac{\Delta_\hbar^{\mathrm{op}}(a) - \Delta_\hbar(a)}{\hbar} \pmod{\hbar}.$$ □

where $x \in A$ and a is any element in $A_\hbar$ such that $x = a \pmod{\hbar}$

As we said before, we will now try to explain how to go from Lie bialgebras to (co)-Poisson Hopf algebras. For the sake of completeness, we will start from the notion of Poisson Lie group, and see how its Lie algebra inherits a bialgebra structure.

Definition 1.5. A **Poisson–Lie group** is a Lie group G with a Poisson structure on its ring of functions $C^\infty(G)$ such that

$$\mu^*(\{f_1 f_2\}_G) = \{\mu^* f_1, \mu^* f_2\}_{G\times G} \tag{1.6}$$

□

A Poisson structure on G comes from a section $\omega \colon G \to \bigwedge^2 T_G$ with the property that

$$\{f, g\} = \langle df \otimes dg, \omega\rangle$$

for all $f, g \in C^\infty(G)$. We can now define $\omega^R \colon G \to \mathfrak{g} \otimes \mathfrak{g}$ by $g \mapsto ((dR_g)_g \otimes (dR_g)_g)(\omega(g))$ and

$$\delta = (d\omega^R)_e \colon \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}.$$

Condition 1.6 implies that $\delta([X, Y]) = X.\delta(Y) - Y.\delta(X)$, so δ is a cocycle of $\mathfrak{g}$ with values in $\mathfrak{g} \otimes \mathfrak{g}$, where the action is given by $g \to (\mathrm{ad}(g) \otimes 1 + 1 \otimes \mathrm{ad}(g))$.

Definition 1.7. A structure of **bialgebra** on $\mathfrak{g}$ is a map $\delta \colon \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ such that

(1) δ is skew-symmetric,
(2) δ^* is a Lie bracket on $\mathfrak{g}^*$, and
(3) δ is a 1-cocycle with values in $\mathfrak{g} \otimes \mathfrak{g}$. □

So G being a Poisson–Lie group gives the structure of Lie bialgebra on $\mathfrak{g}$ via δ. This can be extended to a map $\delta \colon U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ via $\delta(a_1 a_2) = \delta(a_1)\Delta(a_2) + \Delta(a_1)\delta(a_2)$, which gives $U(\mathfrak{g})$ the structure of a co-Poisson Hopf algebra.

2. Nice deformations and the classical Yang-Baxter equation

Now let's start with a Lie algebra $\mathfrak{g}$. In the sea of cocycles we could look for coboundaries, i.e. cocycles δ defined by $\delta(x) = [x, r]$, for some $r \in \mathfrak{g} \otimes \mathfrak{g}$ (where the bracket is taken in the enveloping algebra $U(\mathfrak{g})$). Clearly this δ will be a cocycle, but it might not define a structure of Lie bialgebra on $\mathfrak{g}$. What does r have to satisfy to make $(\mathfrak{g}, \delta)$ a Lie bialgebra? It is easy to see that:

Defintiion 1.7 part (1) means that $r_{21} + r$ is $\mathfrak{g}$-invariant (where $r_{21} = b \otimes a$ if $r = a \otimes b$).

Defintiion 1.7 part (2) means that $[[r, r]] := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]$ is $\mathfrak{g}$-invariant. Here for $r = \sum a_i \otimes b_i$, then $[r_{12}, r_{13}] = \sum_{i,j} [a_i, a_j] \otimes b_i \otimes b_j$, or equivalently is the bracket of the elements $\sum a_i \otimes b_i \otimes 1$ and $\sum_j a_j \otimes 1 \otimes b_j$ in the associative algebra $U(\mathfrak{g}) \otimes U(\mathfrak{g}) \otimes U(\mathfrak{g})$.
In particular we could look for $r \in \mathfrak{g} \otimes \mathfrak{g}$ such that both of the above equations vanish:

(1) $r_{12} + r = 0$, and
(2) $[[r, r]] = 0$. This is called the **classical Yang–Baxter equation** (CYBE).

Lie bialgebras arising in this way are of particular importance for us since their enveloping algebra admits a quantization with good properties (i.e. they are triangular Hopf algebras).

Definition 2.1. A bialgebra $(\mathfrak{g}, r)$ is called **quasi-triangular** if it satisfies (CYBE). It is called **triangular** if is satisfies (CYBE) and $r_{12} + r = 0$. □

As we mentioned before, starting from triangular bialgebras, we can obtain (via quantization) Hopf algebras which are *triangular* as well. Since we haven't said what it means for a Hopf algebra to be *triangular* we will do it now.

Definition 2.2. An **almost cocommutative Hopf algebra** (A, μ, Δ) is a Hopf algebra such that $\Delta^{\mathrm{op}}(a) = R\Delta(a)R^{-1}$ for some invertible $R \in A \otimes A$. □

The fact that A^{op} is also a Hopf algebra forces conditions on R. These conditions (that we won't write here) are in particular satisfied if

(1) $(\Delta \otimes \mathrm{id})R = R_{13}R_{23}$ (if $R = a \otimes b$, then $R_{13} = a \otimes 1 \otimes b$)
(2) $(\mathrm{id} \otimes \Delta)R = R_{13}R_{12}$,

Definition 2.3. An almost cocommutative Hopf algebra (A, R) is called **quasi-triangular** if

(1) $(\Delta \otimes \mathrm{id})R = R_{13}R_{23}$ (if $R = a \otimes b$, then $R_{13} = a \otimes 1 \otimes b$)
(2) $(\mathrm{id} \otimes \Delta)R = R_{13}R_{12}$,

and (A, R) is called **triangular** if it is quasi-triangular and $R_{21} = R^{-1}$. □

Remark 2.4. (1) If (A, R) is almost commutative, then

$$\begin{array}{ccc} V \otimes W & \to & W \otimes V \\ v \otimes w & \mapsto & (1,2)R(v \otimes w) \end{array}$$

is an isomorphism (for $R = a \otimes b$, $(1, 2)(av \otimes bw) = bw \otimes av$).

(2) If (A, R) is quasi-triangular, then $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$. This is called **quantum Yang–Baxter equation**. Set $R^{\vee}_{12} = (1, 2)R$. In terms of $R^{\vee}$, this means $R^{\vee}_{23}R^{\vee}_{12}R^{\vee}_{23} = R^{\vee}_{12}R^{\vee}_{23}R^{\vee}_{12}$. This implies that the braid relations are satisfied and there is an action of the braid group on tensor products of representations of A.

(3) If (A, R) is triangular, then $(R^{\vee})^2 = (1, 2)R_{12}(1, 2)R_{12} = (1, 2)^2R_{21}R_{12} = I$, so actually get an action of the symmetric group on tensor products of representations of A. □

Summarizing, we have seen that starting with a bialgebra $\mathfrak{g}$, we get a co-Poisson structure on $U(\mathfrak{g})$. One might ask, Is it always possible to quantize this structure? And does the quantization $U_\hbar(\mathfrak{g})$ have nice properties like actions of B_n or Σ_n on its representation category? The answer in both cases is affirmative. that is:

Theorem 2.5. *Every finite dimensional lie bialgebra $\mathfrak{g}$ over a field k of characteristic zero admits a quantization.*

Theorem 2.6. *If $(\mathfrak{g}, r)$ is a triangular bialgebra, then there exists a quantization $U_\hbar(\mathfrak{g})$ of $U(\mathfrak{g})$ which is triangular (hence there is an action of Σ_n on Rep($U_\hbar(\mathfrak{g})$)).*

3. The case of $\mathfrak{sl}_2(\mathbb{C}$

In the case of $\mathfrak{sl}_2(\mathbb{C})$, up to equivalence (meaning up to isomorphism of the resulting Lie bialgebra), there are 3 possible choices of r.

(1) $r = 0$. This gives $\delta(E) = \delta(F) = \delta(H) = 0$, and we get the trivial quantization: if $x \in \mathfrak{g}$ and $a \in U_\hbar(\mathfrak{g})$ is such that $x = a \pmod{\hbar}$, then $\Delta_\hbar(x) = x \otimes 1 + 1 \otimes x$.

(2) $r = E \otimes F - F \otimes E$. Then

$$\begin{cases} \delta(E) = [E, E \otimes F] - [E, F \otimes E] = E \otimes H - H \otimes E \\ \delta(H) = 0 \\ \delta(F) = F \otimes H - H \otimes F \end{cases}$$

In this case, we can take

$$\begin{cases} \Delta_\hbar(H) = H \otimes 1 + 1 \otimes H \\ \Delta_\hbar(E) = E \otimes \exp(\hbar + 1) + 1 \otimes E \\ \Delta_\hbar(F) = F \otimes 1 + \exp(-\hbar H) \otimes F. \end{cases}$$

This is the usual quantization that gives the quantum group. Note that

$$\begin{aligned} \Delta_\hbar([E,F]) &= [\Delta_\hbar(E), \Delta_{hbar}(F)] \\ &= [E,F] \otimes \exp(\hbar H) + \exp(-\hbar H) \otimes [E,F] \neq \Delta_\hbar(H) \end{aligned}$$

From this we see that we need to modify the bracket in the Lie algebra as well and replace $[E,F] = H$ with $[E,F] = \frac{\exp(\hbar H) - \exp(-\hbar H)}{\exp(\hbar) - \exp -\hbar}$.

Although the given r is skew, it does not satisfy the classical Yang Baxter equation. However, it is equivalent to an r which is not skew, but does satisfy the classical Yang Baxter equation (see [CP, Chapters 2.1B and 2.1C]. Thus $\mathfrak{sl}_2(\mathbb{C})$ with this bialgebra structure is quasi-triangular but nit triangular.

(3) $r = H \otimes E - E \otimes H$. Here we have:

$$\begin{cases} \delta(H) = 2(H \otimes E - E \otimes H) \\ \delta(E) = 0 \\ \delta(F) = 2(E \otimes F - F \otimes E). \end{cases}$$

You can verify that the followings define a quantization of this $U(\mathfrak{sl}_2(\mathbb{C}))$.

$$\begin{cases} \Delta_\hbar(H) = H \otimes 1 + \exp(-2\hbar E) \otimes H \\ \Delta_\hbar(E) = E \otimes 1 + 1 \otimes E \\ \Delta_\hbar(F) = F \otimes 1 + \exp(-2\hbar E) \otimes E \end{cases}$$

provided that we modify the bracket in the following way:

$$\begin{cases} [E,F] = H \\ [H,E] = \frac{4}{\hbar}(1 - \exp(-2\hbar E)) \\ [H,F] = -2F - \hbar H^2. \end{cases}$$

In this case $\mathfrak{sl}_2(\mathbb{C})$ is triangular and this quantization is the triangular quantization that realizes the theorem.

References

[CP] V. Chari and A. Pressley, *A guide to quantum groups*, Cambridge university press, Cambridge (1994).