THE GEOMETRIC SATAKE CORRESPONDENCE AND MV POLYTOPES

DINAKAR MUTHIAH (LIVE TEXED BY STEVEN SAM)

Mirkovic and Vilonen give a proof of the geometric Satake correspondence which provides a natural basis in each representation of a complex reductive group. After briefly reviewing the geometric Satake correspondence, I will discuss the geometry underlying the Mirkovic-Vilonen basis. In the course of this discussion, I will introduce the Anderson-Kamnitzer theory of MV polytopes and explain a surprising connection between MV polytopes and Lusztig's canonical basis.

1. Geometric Satake correspondence

Let G be a Chevalley group, and $G^\vee$ its Langlands dual (corresponding to the dual root datum). Let K be a non-Archimedean local field, and $\mathcal{O}$ be its ring of integers (for example, consider $K = \mathbb{Q}_p$ or $K = \mathbb{F}_q((t))$). Then classical Satake correspondence is an isomorphism

$$\mathbb{C}_c[G(\mathcal{O}) \setminus G(K)/G(\mathcal{O})] \xrightarrow{\cong} K_0 \operatorname{Rep} G^\vee \otimes \mathbb{C}$$

where the subscript c refers to compactly supported functions.

Now focus on the case $K = \mathbb{C}((t))$ and $\mathcal{O} = \mathbb{C}[[t]]$. We want to categorify this isomorphism. There is a tensor equivalence of categories

$$\begin{array}{ccc} \mathcal{P}erv(G(\mathcal{O}) \setminus G(K)/G(\mathcal{O})) & \xrightarrow{\cong} & \operatorname{Rep} G^\vee \\ \downarrow \mathbf{H}^* & & \downarrow F \\ \mathbf{Vect} & \xrightarrow{\cong} & \mathbf{Vect} \end{array}$$

where $\mathcal{P}erv$ denotes the category of perverse sheaves, $\mathbf{H}^*$ is hypercohomology, and F is the forgetful functor. This is due to Lusztig, Drinfeld, Ginzburg, Mirkovic–Vilonen. The Tannakian formalism states the left hand side is equivalent to a representation category, so the work goes into identifying the group.

2. MV bases and MV cycles

Perverse sheaves on $G(\mathcal{O}) \setminus G(K)/G(\mathcal{O})$ are the same as $G(\mathcal{O})$-equivariant perverse sheaves on the "affine grassmannian" $\mathcal{G}r(\mathbb{C}) := G(K)/G(\mathcal{O})$, so we need to study this object.

2.1. Geometry of the affine Grassmannian.

Fix a maximal torus $T \subset G$. Let W be the corresponding Weyl group. The coweight lattice is $X_*(T) = \operatorname{Hom}(\mathbb{G}_m, T)$. Given a map $\mathbb{G}_m \to T$, get a map $\operatorname{Spec} K \to G$ by composing the maps

$$\operatorname{Spec} K \to \mathbb{G}_m \to T \to G,$$

so this gives a map $X_*(T) \to G(K)$, which we can compose with $G(K)/G(\mathcal{O}) = \mathcal{G}r$. Call this map $\lambda \mapsto t^\lambda$.

It is an embedding, and the T-fixed points on $\mathcal{G}r$ are precisely the t^λ. Furthermore, each $G(\mathcal{O})$-orbit of $\mathcal{G}r$ contains the W-orbit of a unique dominant coweight t^λ. Hence we can index $G(\mathcal{O})$-orbits by dominant coweights, call them $\mathcal{G}r^\lambda$. Under the geometric Satake isomorphism, the intersection homology sheaf $\mathrm{IC}_{\mathcal{G}r^\lambda}$ is sent to $V(\lambda)$.

Date: April 29, 2011.

Now choose a pair of opposite Borel subgroups B and B^- such that $B \cap B^- = T$, and let N and N^- be the unipotent radicals of these Borel subgroups. Each $N(K)$ orbit contains a unique torus-fixed point. (This follows from the Iwasawa decomposition.) Call the $N(K)$-orbits S^ν where ν is the corresponding torus-fixed point. Similarly, we can index the $N^-(K)$-orbits by the coweight lattice and we call them T^ν.

For example, $S^0 \cong \mathcal{G}r_N = N(K)/N(\mathcal{O})$.

Theorem 2.1 (Mirkovic–Vilonen [MV])**.** *There is an isomorphism of functors*

$$\mathbf{H}^* \xrightarrow{\cong} \bigoplus_{\nu \in X_*(T)} \mathrm{H}_c^{2\rho(\nu)}(S^\nu, -) \colon (G(\mathcal{O})\text{−equivariant perverse sheaves}) \to \mathbf{Vect}.$$

Furthermore,

$$\begin{aligned} V(\lambda) = \mathbf{H}^*(\mathrm{IC}_{\overline{\mathcal{G}r^\lambda}}) &\cong \bigoplus_{\nu \in X_*(T)} \mathrm{H}_c^{2\rho(\nu)}(S^\nu \cap \mathcal{G}r^\lambda, \mathrm{IC}_{\overline{\mathcal{G}r^\lambda}}) \\ &= \bigoplus_{\nu \in X_*(T)} \mathrm{H}^{2\rho(\nu+\lambda)}(S^\nu \cap \mathcal{G}r^\lambda, \mathbb{C}). \end{aligned}$$

One consequence is that the irreducible components of $\mathcal{G}r^\lambda \cap S^\nu$ index a basis in $V(\lambda)_{-\nu}$. We may as well use the irreducible components of $\overline{\mathcal{G}r^\lambda \cap S^\nu}$, and these components are called the **Mirkovec–Vilonen (MV) cycles**.

3. Stable MV cycles and their moment polytopes

Recall that $V(\lambda)_{-\nu} \cong U(\mathfrak{n}_-)_{-\lambda-\nu}$ when λ is large compared to ν. Thus we expect the behavior of $\overline{\mathcal{G}r^\lambda \cap S^\nu}$ to stabilize when λ is large compared to ν. In fact, it stabilizes to $\overline{T^{-\lambda} \cap S^\nu}$. It turns out that $\overline{T^{-\lambda} \cap S^\nu} \cong \overline{T^{-\lambda-\nu} \cap S^0}$, so one consequence of this is that

$$\# \operatorname{Irr}(\overline{T^{-\gamma} \cap S^0}) = \dim U(\mathfrak{n}_-)_\gamma.$$

Elements of $\operatorname{Irr}(\overline{T^{-\gamma} \cap S^0})$ are called **stable MV cycles**. We want to describe this set explicitly, i.e., give a combinatorial parametrization of each irreducible component. The solution to this problem is due to Anderson and Kamnitzer, and it is to study the irreducible components via their moment polytopes.

There is an embedding

$$\mathcal{G}r \hookrightarrow \mathbb{P}(V(\omega_0))$$

where $V(\omega_0)$ is the basic representation of $\widehat{G(K)}$ (the unique non-trivial central extension of $G(K)$), and $\mathbb{P}(V(\omega_0))$ is its projectivization. This comes from the fact that $\mathcal{G}r = \widehat{G(K)}/\widehat{G(\mathcal{O})}$. There is a natural moment map

$$\mu \colon \mathbb{P}(V(\omega_0)) \to \operatorname{Lie}(T_\mathbb{R})^* \cong \operatorname{Lie}(T_\mathbb{R})$$

where the last isomorphism is via the Killing form. By a general result about moment map images of torus-invariant complete varieties we have the following: for $Z \in \operatorname{Irr}(\overline{S^0 \cap T^\gamma})$, we have $\mu(Z) = \operatorname{conv}\{\lambda \mid t^\lambda \in Z\} \subset \operatorname{Lie}(T_\mathbb{R})$ (the convex hull). The following result is due to Kamnitzer:

Theorem 3.1. [Kam]

(1) *The vertices of $\mu(Z)$ are indexed by W,*
(2) *The 1-skeleton of $\mu(Z)$ is the Cayley graph of W (with respect to the simple reflections),*
(3) *The edge between w and ws_i is parallel to $w(\alpha_i^\vee)$,*
(4) *The edges have "integral length", i.e., each edge has length equal to an integral multiple of the length of the coroot $w(\alpha_i^\vee)$ to which it is parallel.*

The paths from e to w_0 in the 1-skeleton of the MV polytope correspond to reduced expressions of w_0. Keeping track of edge lengths we get the following datum out of an MV polytope: for every reduced expression of w_0, we get a tuple of non-negative integers.

There is another object we encountered in this seminar that gives us the same type of combinatorial data, namely Lusztig's canonical basis. Let us fix an element b of the canonical basis. For every reduced expression of w_0, we can form a PBW basis, and for each PBW basis there is a unique element p that agrees with b when we specialize at $q = 0$. As elements of the PBW basis are naturally in bijection with tuples of non-negative integers, we get the following datum out of an element of the canonical basis: for every reduced expression of w_0, we get a tuple of non-negative integers.

Kamnitzer's theorem says that these two data are the same. Specifically, one can read off the Lusztig data as the lengths of the edges in a certain path in the MV polytope (the path depends on the expression for w_0). These paths determine the polytope, so one can also go the other way and construct the MV polytope directly from Lusztig data (although you need to consider Lusztig data with respect to all words for w_0).

References

[Kam] J. Kamnitzer, Mirković-Vilonen cycles and polytopes, *Ann. of Math. (2)* **171** (2010), 731–777.

[MV] I. Mirković and K. Vilonen, *Geometric Langlands duality and representations of algebraic groups over commutative rings*, *Ann. of Math. (2)* **166** (2007), 95–143.