LECTURE 4: RECOGNITION OF CRYSTALS

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The goal for today is to characterize crystals. we would perhaps like to give a set of axioms on the data B, e_i, f_i of a set B along with candidates e_i, f_i for the Kashiwara operators that would guarantee that we see the crystal of a representation V or $\mathfrak{g}$. However, this is difficult, and we take a sort of half-way ground: A axiomatize the notion of a combinatorial crystal, which includes things which are not crystal graphs of representations. We then study various ways to prove that such a combinatorial crystal does in fact arise as the crystal of a representation. That is what we mean by recognition theorems.

For today, $\mathfrak{g}$ is a symmetrizable Kac-Moody algebra.

1. Combinatorial crystals

Definition 1.1. An **integrable combinatorial** $\mathfrak{g}$-crystal is a set B with operators e_i, f_i with the properties:

(1) **integrable**: For all i and all $b \in B$, there is some $N > 0$ such that $e_i^N(b) = f_i^N(b) = 0$.

(2) **weighted**: Define

$$\varepsilon_i(b) = \max\{n \mid e_i^n b \neq 0\}, \quad \varphi_i(b) = \max\{n \mid f_i^n b \neq 0\}$$

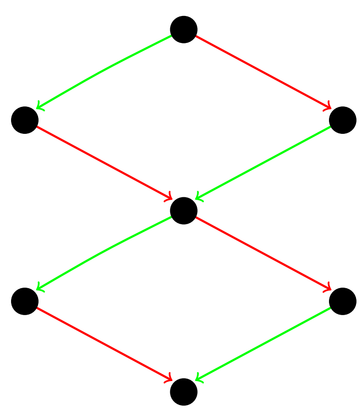
$$\varepsilon(b) = \sum_i \varepsilon_i(b)\omega_i, \quad \varphi(b) = \sum_i \varphi_i(b)\omega_i$$

$$\mathrm{wt}(b) = \varphi(b) - \varepsilon(b).$$

Then f_i has weight $-\alpha_i$ and e_i has weight α_i.

(3) **Partial permutations**: $f_i b = b'$ if and only if $e_i b' = b$. □

Example 1.2. This is not enough to characterize the crystals of integrable representations of $\mathfrak{g}$. For example



satisfies all the axioms, but is not the crystal of any representation. □

Remark 1.3. $B(\infty)$ is not covered by this definition. □

Definition 1.4. (see [K3, Section 7.2])A **combinatorial crystal** is a tuple $(B, f_i, e_i, \mathrm{wt}, \varepsilon_i, \varphi_i)$ where $\mathrm{wt}\colon B \to P$ (P is the weight lattice), $e_i, f_i : B \to B \sqcup \{0\}$, $\varepsilon_i, \varphi_i\colon B \to \mathbb{Z} \cup \{-\infty\}$ such that

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(1) $\varphi_i(b) - \varepsilon_i(b) = \langle \mathrm{wt}(b), \alpha_i \rangle$ whenever the left hand side is finite.
(2) e_i increases φ_i by 1 and decreases ε_i by 1.
(3) e_i has weight α_i.
(4) $f_i b = b'$ if and only if $e_i b' = b$.
(5) If $\varphi_i(b) = -\infty$, then $e_i b = f_i b = 0$. □

Remark 1.5. An integrable combinatorial crystal is in particular a combinatorial crystal, where one defines $\varphi_i(b) = \langle \varphi(b), \alpha_i^\vee \rangle$ and $\varepsilon_i(b) = \langle \epsilon(b), \alpha_i^\vee \rangle$.

Remark 1.6. The cases $\varphi_i(b) = -\infty$ are included to allow certain combinatorial constructions that come in handy. They don't ever really correspond to crystals of representations.

Example 1.7. $B(\infty)$ is a combinatorial crystal where we define $\mathrm{wt}(1) = 0$, $\varepsilon_i(b) = \max\{n \mid e_i^n b \neq 0\}$, and then allow this to determine $\varphi_i(b)$. □

2. Recognizing integrable crystals

The follow theorems will be our tools for showing that later constructions (using quiver varieties etc.) actually give the crystals of integrable representations of $\mathfrak{g}$.

Theorem 2.1 (Kashiwara)**.** *There is a unique family of integrable crystals $\{B(\lambda)\}$ indexed by dominant weights λ which is closed under $\otimes$ and taking connected components. These are the crystals for the highest weight representations $V(\lambda)$.*

Theorem 2.2 (Kashiwara)**.** *Assume $C(\lambda)$ is a graph with a unique source b_λ such that the length of the i root string leaving b_λ is $\langle \alpha_i^\vee, \lambda \rangle$, and such that, for each $i, j \in I$, the graph obtained by ignoring the other colors is an integrable rank 2 crystal for the corresponding rank 2 Kac–Moody algebra. Then $C(\lambda) = B(\lambda)$.*

Remark 2.3. For crystals of simply laced Kac-Moody algebras, Theorem 2.2 was enhanced by Stembridge [S] by giving a "local" characterization of integrable $\mathfrak{sl}_3$ crystals.

3. Recognizing $B(\infty)$

The previous section is only concerned with integrable crystals, so cannot recognize $B(\infty)$. For that, we need to introduce more structure. Recall that we have an inner product $(,)$ on U^-, $\mathcal{L}(\infty)$, and $\mathcal{L}(\infty)/q\mathcal{L}(\infty)$. Working over $\mathbb{Z}$, $B(\infty) \cup -B(\infty) = \{b \mid (b, b) = 1\}$.

Definition 3.1. $*$ is the anti-algebra involution on U^- which fixes each F_i. . □

Proposition 3.2 ([K2, Proposition 5.2.1])**.** *For all $u, v \in U^-$, we have $(u, v) = (*u, *v)$. In particular, $*$ preserves the set $\{(b, -b) \mid b \in B(\infty)\}$ (but not pointwise). By ignoring signs, we get an involution, also denoted $*$, on $B(\infty)$. Also, define $e_i^* = * \circ e_i \circ *$, and define $\varphi_i^*, \varepsilon_i^*$ in the obvious way.*

Remark 3.3. It was shown by Lusztig that, at least in types ADE, the signs in the above are all $+$. This is conjectured to hold in other types as well.

Definition 3.4 (see [K3, Section 7.5])**.** Let $B^{(i)}$ be the following crystal

$$\cdots b^{(i)}(1) \xrightarrow{i} b^{(i)}(0) \xrightarrow{i} b^{(i)}(-1) \xrightarrow{i} b^{(i)}(-2) \cdots$$

where $wt(b^{(i)}(k)) = k\alpha_i$, $\varphi_i(b^{(i)}(k)) = k$, $\varepsilon_i(b^{(i)}(k)$, and for $j \neq i$, $\varphi_i(b^{(i)}(k) = \varepsilon(b^{(i)}(k)) = -\infty$. Here the arrows show the action of f_i. □

Theorem 3.5. [K1, Theorem 2.2.1] *For each i, there is a morphism of crystals*

$$\Phi_i \colon B(\infty) \to B(\infty) \otimes B^{(i)}$$
$$u_0 \mapsto u_0 \otimes b^{(i)}(0).$$

Furthermore, Ψ satisfies
(1) $\Phi_i(b) = (e_i^*)^{\varepsilon_i^*(b)}(b) \otimes b^{(i)}(-\varepsilon_i^*(b))$.
(2) $\operatorname{image} \Phi_i = \{b \otimes b^i(k) \mid k \leq 0,\ \varepsilon_i^*(b) = 0\}$. □

Furthermore $B(\infty)$ can be characterized as the unique crystal for which the above holds. This is made precise as follows

Theorem 3.6 ([KS, Proposition 3.2.3])**.** *Let $\{B, e_i, f_i, \mathrm{wt}, \varepsilon_i, \varphi_i\}$ be a combinatorial crystal with an element b_+ such that $\mathrm{wt}(b) = 0$, $\varepsilon_i(b) = 0$ for all $i \in I$, and $\{f_{i_N} \cdots f_{i_1} b_+ : i_1, \ldots, i_N \in I\} = B$. Assume also that*
(1) *$\varepsilon_i(b) \in \mathbb{Z}$ for every i.*
(2) *For every i, there is a strict embedding (that is, an embedding of crystals) $\Phi_i : B \to B \otimes B^{(i)}$.*
(3) *$\Phi_i(B) \subset B \otimes \{b_i(-k) : k \geq 0\}$.*
(4) *For each $b \neq b_+ \in B$, there some $i \in I$ such that $\Phi_i(b) = b' \otimes b^{(i)}(-k)$ with $k > 0$.*
Then B is isomorphic to $B(\infty)$.

The proof that the properties from Theorem 3.6 uniquely characterize the resulting crystal is quite simple. Essentially, one can see that these properties imply that B is isomorphic to the crystal generated by

$$\cdots b^{(i_2)}(0) \otimes b^{i_1}(0) \in \cdots B^{(i_2)} \otimes B^{(i_1)},$$

for appropriate $\ldots i_2, i_1$. It then follows from Theorem 3.5 that this unique crystal is in fact $B(\infty)$.

The following is another wording of this result which makes the role of $*$ more obvious.

Corollary 3.7. *Let B be a highest weight combinatorial crystal with highest weight element b_+ such that, for all $b \in B$ and all i, $\epsilon_i(b) \geq 0$, and $\epsilon_i(b_+) = 0$. Fix an involution $*$ on B, and define $e_i^* = *e_i*$ and $\epsilon_i^*(b) = \epsilon_i(*b)$. Define $\Phi_i : B \to B \otimes B^{(i)}$ by*

$$\Phi_i(b) = (e_i^*)^{\varepsilon_i^*(b)}(b) \otimes b^{(i)}(-\varepsilon_i^*(b)).$$

If Φ_i is an embedding of crystals for all i, then $B = B(\infty)$ and $*$ is Kashiwara's involution.

References

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