HALL ALGEBRA REALIZATIONS OF QUANTUM GROUPS

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1. Quivers

Fix a field K and let $Q = (Q_0, Q_1)$ be a loopless quiver. We consider finite-dimensional representations of Q over K. Each representation has a naturally associated dimension vector. Let $\Delta = (a_{ij})$ be the Cartan matrix associated to the underlying graph of Q and let $\mathfrak{g} = \mathfrak{g}(\Delta)$ be the associated Kac–Moody algebra.

Theorem 1.1 (Kac). *Assume K is algebraically closed. The dimension vectors of the indecomposable representations of Q are precisely the positive roots of $\mathfrak{g}$. Furthermore, there is a unique such indecomposable if and only if the positive root is real.*

In particular, Q has finitely many indecomposables if and only if the underlying graph of Q is a disjoint union of copies of type ADE graphs. In this case, the above theorem is true for any field. Hence we can identify the isomorphism class of a representation over a Dynkin quiver with a function from the positive roots of Δ to the nonnegative integers.

The path algebra of Q is hereditary, so $\mathrm{Ext}^i(M,N) = 0$ for all modules M, N and $i > 1$. Given two representations M, N, define the **Euler form**

$$\langle M, N\rangle = \dim_K \mathrm{Hom}(M,N) - \dim_K \mathrm{Ext}^1(M,N).$$

This descends to the Grothendieck group $K_0(Q)$ of Q, so the above number only depends on the dimensions of M and N. Furthermore, the symmetrized Euler form

$$(M,N) = \langle M,N\rangle + \langle N,M\rangle$$

recovers the Cartan pairing associated with the root system Δ.

For each vertex $i \in Q_0$, let S_i denote the simple module of Q of dimension 1 concentrated at i.

2. Hall algebras

Now assume that K is a finite field. Given representations $M, N_1, \dots, N_t$, let $\mathcal{F}^M_{N_1,\dots,N_t}$ be the set of filtrations

$$M = M_0 \supset M_1 \supset \cdots \supset M_t = 0$$

such that $M_{i-1}/M_i \cong N_i$ for $i = 1, \dots, t$ and let $F^M_{N_1,\dots,N_t}$ be the cardinality of $\mathcal{F}^M_{N_1,\dots,N_t}$.

Theorem 2.1 (Ringel [4]). *Let Q be a Dynkin quiver. Given representations $M, N_1, \dots, N_t$, there is a polynomial $P^M_{N_1,\dots,N_t}$ such that $P^M_{N_1,\dots,N_t}(\#K) = F^M_{N_1,\dots,N_t}$.*

The polynomials P are the **Hall polynomials**.

Let $H(Q)$ be the free $\mathbb{Z}[q]$-module spanned by isomorphism classes of representations of Q. For a dimension vector d, let $H(Q)_d$ be the submodule generated by representations of dimension d, so that we have a $\mathbb{N}^{Q_0}$-grading

$$H(Q) = \bigoplus_d H(Q)_d$$

Date: April 15, 2011.

We can put a product on $H(Q)$ via

$$[N_1] \diamond [N_2] = \sum_M P^M_{N_1,N_2}(q)[M].$$

Then $H(Q)$ becomes an associative, unital, graded $\mathbb{Z}[q]$-algebra.

Example 2.2. Consider the quiver $1 \to 2$. Let M be the isomorphism class of the indecomposable representation of dimension $(1,1)$. We have

$$\begin{aligned}
&[S_1] \diamond [S_1] \diamond [S_2] = (q+1)[2S_1] \diamond [S_2] = (q+1)([2S_1 \oplus S_2] + [M \oplus S_1])\\
&[S_1] \diamond [S_2] \diamond [S_1] = ([S_1 \oplus S_2] + [M]) \diamond [S_1] = (q+1)[2S_1 \oplus S_2] + [M \oplus S_i]\\
&[S_2] \diamond [S_1] \diamond [S_1] = [2S_1 \oplus S_2]
\end{aligned}$$

(we just use that $\#\mathbb{P}^1 = q+1$). Similarly, one has

$$\begin{aligned}
&[S_1] \diamond [S_2] \diamond [S_2] = (q+1)([M \oplus S_2] + [S_1 \oplus 2S_2])\\
&[S_2] \diamond [S_1] \diamond [S_2] = [M \oplus S_2] + (q+1)[S_1 \oplus 2S_2]\\
&[S_2] \diamond [S_2] \diamond [S_1] = (q+1)[S_1 \oplus 2S_2].
\end{aligned}$$

This implies the relations

$$\begin{aligned}
&[S_1] \diamond [S_1] \diamond [S_2] - (q+1)[S_1] \diamond [S_2] \diamond [S_1] + q[S_2] \diamond [S_1] \diamond [S_1] = 0\\
&q[S_2] \diamond [S_2] \diamond [S_1] - (q+1)[S_2] \diamond [S_1] \diamond [S_1] + [S_1] \diamond [S_2] \diamond [S_2] = 0.
\end{aligned}$$

□

We see from this example that the relations between $[S_1]$ and $[S_2]$ depend on the direction of the arrow. If we want to relate this to quantum groups, we would want to remove this dependency. To do this, we modify the product $\diamond$.

Let $A = \mathbb{Z}[v, v^{-1}]$ and set $q = v^2$. Let $H_*(Q) = H(Q) \otimes_{\mathbb{Z}[q]} A$. We define a multiplication on $H_*(Q)$ via

$$[N_1] * [N_2] = v^{\langle \dim N_1, \dim N_2 \rangle}[N_1] \diamond [N_2].$$

This $H_*(Q)$ is an associative, unital, graded A-algebra. A direct computation gives some relations.

$$\begin{aligned}
[S_i] * [S_j] - [S_j] * [S_i] &= 0 \quad \text{if } a_{ij} = 0,\\
[S_i] * [S_i] * [S_j] - (v + v^{-1})[S_i] * [S_j] * [S_i] + [S_j] * [S_i] * [S_i] &= 0 \quad \text{if } a_{ij} = -1.
\end{aligned}$$

Let $\mathfrak{n}_+$ be the upper triangular part of $\mathfrak{g}$ generated by E_i and let $U_q(\mathfrak{n}_+)$ be the subalgebra of the quantum group of $\mathfrak{g}$ generated by the divided powers $E_i^{(n)}$. We see that we can define an A-algebra homomorphism

$$\begin{aligned}
\eta\colon U_q(\mathfrak{n}_+) &\to H_*(Q)\\
E_i &\mapsto [S_i]
\end{aligned}$$

Theorem 2.3 (Ringel [5])**.** *The map η is an isomorphism.*

2.1. **Canonical bases.** Given a module M, define $\langle M \rangle = q^{\dim_K \mathrm{End}(M) - \dim_K M}[M]$ where $\dim_K M$ means the sum of the entries of the dimension vector of M. We define a bar involution on $H_*(Q)$ by $\bar{v} = v^{-1}$ and $[\bar{S_i}] = [S_i]$. This is just the image under η of the bar involution on $U_q(\mathfrak{n}_+)$.

Let α be a function from the positive roots of Δ to $\mathbb{N}$. There is a uniquely associated module $M(\alpha)$ given by taken a direct sum of the indecomposable modules associated to each positive root with multiplicities specified by α.

Choose a total ordering on the positive roots Φ^+ in such a way that $\mathrm{Hom}(M(r), M(r')) \neq 0$ implies that $r \leq r'$. Then write them in order $r_1 < r_2 < \cdots$. For functions $\alpha, \beta\colon \Phi^+ \to \mathbb{N}$, say that

$\beta < \alpha$ if there exists j such that $\beta(r_i) = \alpha(r_i)$ for all $i < j$ and $\beta(r_j) > \alpha(r_j)$. For each α, there is a unique element $C(\alpha) \in H_*(Q)$ such that $\overline{C(\alpha)} = C(\alpha)$ and

$$C(\alpha) = \langle M(\alpha)\rangle + \sum_{\beta<\alpha} u_{\alpha,\beta}\langle M(\beta)\rangle$$

for $u_{\alpha,\beta} \in v^{-1}\mathbb{Z}[v^{-1}]$. Applying η^{-1} to the collection of $C(\alpha)$ gives the canonical basis of $U_q(\mathfrak{n}_+)$.

2.2. **Complements.** Green [1] introduced a coproduct structure on $H_*(Q)_{\mathbb{Q}}$ which makes $H_*(Q)_{\mathbb{Q}}$ into a sort of "twisted bialgebra" (here it becomes relevant that Ext^2 is identically 0). Given a module $[M]$, we set $a_M = \#\mathrm{Aut}(M)$ and

$$\Delta([M]) = \sum_{N_1,N_2} v^{\langle \dim N_1, \dim N_2\rangle} \frac{a_{N_1}a_{N_2}}{a_M} P^M_{N_1,N_2}(q)[N_1]\otimes[N_2]$$
$$\epsilon([M]) = \delta_{M,0}.$$

$H_*(Q)_{\mathbb{Q}}$ is not a bialgebra, since the map Δ is not a morphism of algebras when $H_*(Q)_{\mathbb{Q}}\otimes H_*(Q)_{\mathbb{Q}}$ is given the usual tensor product algebra structure. However, if the multiplication on $H_*(Q)_{\mathbb{Q}}\otimes H_*(Q)_{\mathbb{Q}}$ is twisted as follows:

$$([M_1]\otimes[N_1])\cdot([M_2]\otimes[N_2]) = v^{\langle \dim N_1,\dim M_2\rangle}([M_1]*[M_2])\otimes([N_1]*[N_2]),$$

then Δ does become an algebra morphism.

One can get an honest bialgebra by enlarging $H_*(Q)_{\mathbb{Q}}$ to $\widetilde{H}(Q) = H_*(Q)_{\mathbb{Q}}\otimes_{\mathbb{Q}[q]}\mathbb{Q}[K_0(Q)]$ where $K_0(Q)$ is the usual Grothendieck group of Q, and $\mathbb{Q}[K_0(Q)]$ denotes its group algebra (which is isomorphic to the ring of Laurent polynomials in $\#Q_0$ variables). Given $\alpha\in K_0(Q)$, we let $k_\alpha \in \mathbb{Q}[K_0(Q)]$ denote the corresponding element. Then we define a multiplication on $\widetilde{H}(Q)$ via

$$k_\alpha[M] = v^{\langle\alpha,\dim M\rangle}[M]k_\alpha.$$

We define a new comultiplication on this enlarged algebra by

$$\Delta([M]k_\alpha) = \sum_{N_1,N_2} v^{\langle \dim N_1,\dim N_2\rangle}\frac{a_{N_1}a_{N_2}}{a_M}P^M_{N_1,N_2}(q)[N_1]k_{\dim N_2+\alpha}\otimes[N_2]k_\alpha.$$

With these definitions, $\widetilde{H}(Q)$ is a bialgebra. Xiao [7] introduced an antipode S so that $\widetilde{H}(Q)$ becomes a Hopf algebra. Given a module M and an integer r, let $S_{M,r}$ denote the set of strict filtrations $L_\bullet$ of the form

$$0 \neq L_r \subsetneq L_{r-1}\subsetneq\cdots\subsetneq L_2\subsetneq L_1 = M.$$

We define

$$S([M]k_\alpha) = k^{-1}_{\dim M+\alpha}\Big(\sum_{r\geq 1}\sum_{L_\bullet\in S_{M,r}}\frac{(-1)^r}{a_M}\Big(\prod_{i=1}^r v^{\langle\dim L_i/L_{i+1},\dim L_{i+1}\rangle}a_{L_i/L_{i+1}}\Big)[L_1/L_2]*[L_2/L_3]*\cdots*[L_r]\Big)$$

Then η can be extended to an isomorphism of Hopf algebras $\widetilde{H}(Q)\to U_q(\mathfrak{h}\oplus\mathfrak{n}^+)$.

Much of the above extends to the case of an arbitrary loopless quiver Q, but some changes are necessary. First, in general, an indecomposable representation need not be indecomposable after base change of the field. The existence of Hall polynomials is known for affine quivers [2] but not in general. Second, one considers only nilpotent representations of Q (otherwise, for example, the Grothendieck group is not generated by the representations S_i). However, one can still define a map like η for any specialized value of q. Instead of being an isomorphism, η in general is an embedding of Hopf algebras [6, Theorem 3.16].

Finally, let us mention the work of Peng and Xiao [3] which allows one to reconstruct the universal enveloping algebra of $\mathfrak{g}$ (in finite type) using a triangulated category associated with the quiver. More precisely, letting T be the shift functor, this is the orbit category of the derived category of representations of Q where we have set $T^2 = 1$. However, one does not get *quantized* enveloping algebra from this construction.

References

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