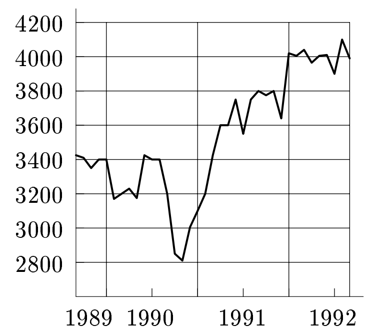
PART I

SINGLE VARIABLE EXAM QUESTIONS

Chapter 1 Exam Questions

Questions and Solutions for Section 1.1

1. The empirical function $W = f(t)$, given in the graph to the right, comes from the Wall Street Journal, September 4, 1992. From the graph, describe the domain of this function and the range of this function. In a sentence, apply the general definition of the word "function" to explain why you think that the given curve is in fact a function.



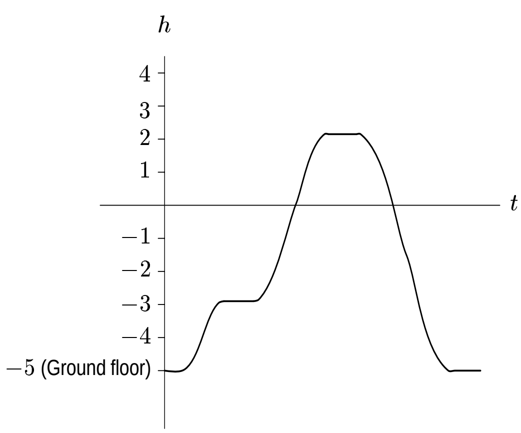
ANSWER:

Domain: September 1989 to August 1992. Range: 2810 (approx.) $\leq W \leq 4090$ (approx.). For every date in the domain, there is a unique value of W.

2. Consider a ten-story building with a single elevator. From the point of view of a person on the sixth floor, sketch a graph indicating the height of the elevator as a function of time as it travels. Remember to indicate when it stops. Try to take into account all *types* of cases that can happen, but do not worry about *every* possible situation. (There are many different possible graphs that could be drawn for this.)

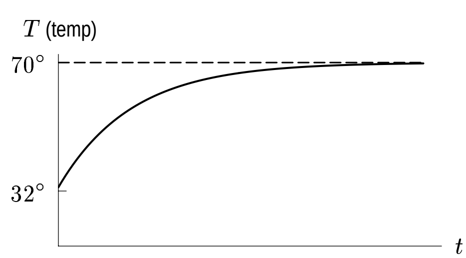
ANSWER:

A possible diagram: An elevator first goes from the ground floor to the third floor, then to the eighth floor, and finally back to the ground floor.



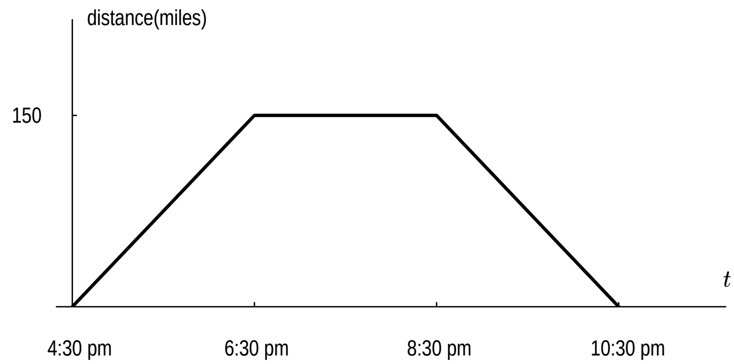
3. Draw a graph which accurately represents the temperature of the contents of a cup left overnight in a room. Assume the room is at 70° and the cup is originally filled with water slightly above the freezing point.

ANSWER:



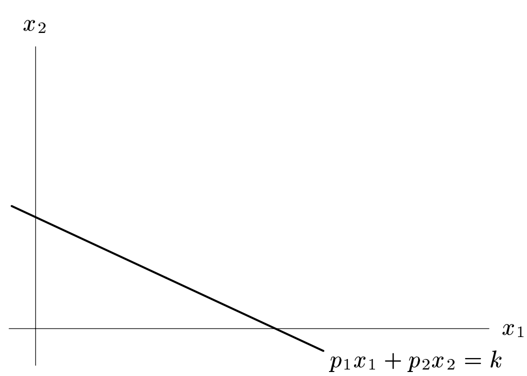
4. Suppose the Long Island Railroad train from Easthampton to Manhattan leaves at 4:30 pm and takes two hours to reach Manhattan, waits two hours at the station and then returns, arriving back in Easthampton at 10:30 pm. Draw a graph representing the distance of the train from the Farmingdale station in Easthampton as a function of time from 4:30 pm to 10:30 pm. The distance from Easthampton to Manhattan is 150 miles.

   ANSWER:



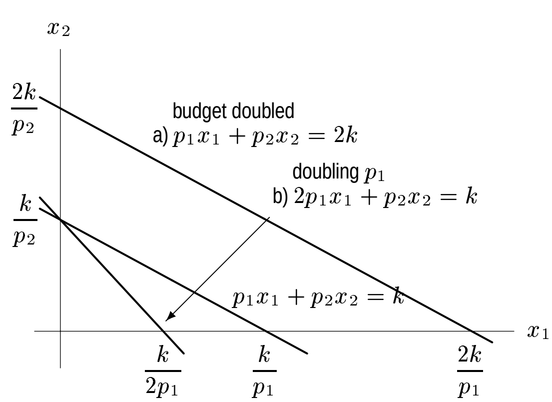
5. Suppose we buy quantities x_1 and x_2, respectively, of two goods. The following graph shows the budget constraint $p_1x_1 + p_2x_2 = k$, where p_1 and p_2 are the prices of the two goods and k is the available budget. On the graph, draw the lines that correspond to the following situations, and for each line, give the equation and the coordinates of both intercepts. Label each line clearly.

   (a) The budget is doubled, but prices remain the same.



   (b) The price of the first good is doubled, but everything else remains the same (the available budget is still k).

   ANSWER:



   (a) $x_2 = -\dfrac{p_1}{p_2}x_1 + \dfrac{k}{p_2}$. If the prices remain the same, the slope of the line remains the same. If the budget is doubled, push the line up, keeping the slope the same, but doubling the x_1 and x_2 intercepts.

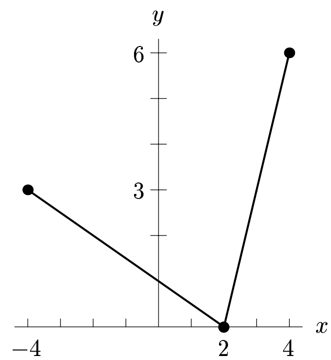
(b) $x_2 = -\frac{p_1}{p_2}x_1 + \frac{k}{p_2}$. When the price of the first good is doubled, we get $x_2 = -\frac{2p_1}{p_2}x_1 + \frac{k}{p_2}$. The slope of the line is double; y-intercept remains the same.

6. A function is linear for $x \leq 2$ and also linear for $x \geq 2$. This function has the following values: $f(-4) = 3$; $f(2) = 0$; $f(4) = 6$. Find formula(s) (or equation(s)) which describe this function.

ANSWER:

For $x \leq 2$, the slope is $\frac{-3}{6} = \frac{-1}{2}$ and the y-intercept is 1; thus $y = \frac{-1}{2}x + 1$. For $x \geq 2$, the slope is $\frac{6}{2} = 3$. To find the y-intercept we substitute: $6 = 3(4) + b$, $6 - 12 = b$, so that $b = -6$. Hence, $y = 3x - 6$.

This is an example of a *piece-wise function*: $f(x) = \begin{cases} -\frac{1}{2}x + 1 & \text{when } x \leq 2 \\ 3x - 6 & \text{when } x \geq 2 \end{cases}$



7. The empirical function $P = g(t)$ graphed below represents the population P of a city (in thousands of people) at time t. Describe the domain and range of this function.

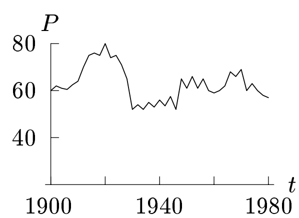


Figure 1.1.1

ANSWER:

Domain: 1900 to 1980. Range: $80,000 \leq P \leq 48,000$ (approximately).

8. A pond has a population of 500 frogs. Over a ten-year period of time the number of frogs drops quickly by 50%, then increases slowly for 5 years before dropping to almost zero. Sketch a graph to represent the number of frogs in the pond over the ten-year period of time.

ANSWER:

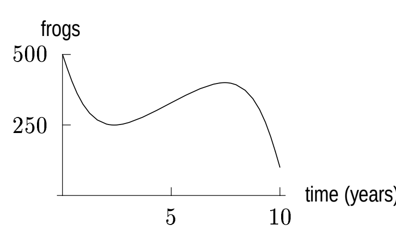


Figure 1.1.2

9. Below is the graph of a function $H = f(t)$ that represents the height of water in a reservoir. Write a short story to match the graph.

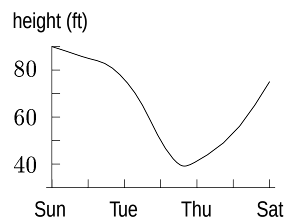


Figure 1.1.3

ANSWER:

The height of the water on Sunday is 90 feet. Over the course of the next two days the water level drops slowly to approximately 85 feet. The water level drops more quickly for the next day and a half and reaches its lowest level of 40 feet during Wednesday. The water level then rises steadily to a height of 75 feet on Saturday.

10. Suppose a container of water is placed in the freezer overnight. The next morning, it is put on the counter in a 70° room and then at the end of the day heated to the boiling point.
    (a) Sketch a graph representing the temperature of the water during the day.
    (b) Describe the domain and range of your graph from part (a).

    ANSWER:

    (a)

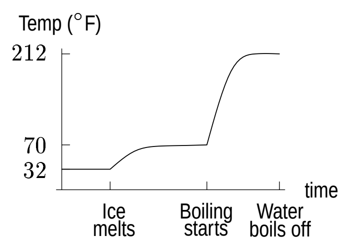


Figure 1.1.4

    (b) The domain is the time the container is put on the counter to the end of the day. The range is the temperature of frozen water ($32°$) to the boiling temperature ($212°$).

11. A school library opened in 1980. In January, 2000 they had 30,000 books. One year later, they had 30,480 books. Assuming they acquire the same number of books at the start of each month:
    (a) How many books did they have in January, 2003?
    (b) How many books did they have in July of 1980?
    (c) Find a linear formula for the number of books, N, in the library as a function of the number of years t the library has been open.
    (d) If you graph the function with domain 1980-2010, describe in words what the y-intercept of the graph means.

    ANSWER:

    (a) They acquire $30,480 - 30,000 = 480$ books per year. In January, 2003 the library will have 3×480 more books than they did in January, 2000 for a total of $31,440$ books.
    (b) From part (a), the number of books the library acquires each year is 480. January, 1980 was 20 years before January, 2000, therefore the number of books the library had in January, 1980 was $30,000 - (20 \times 480) = 20,400$. From part (a), the number of books acquired each month is $480/12 = 40$. By July, 1980 the library acquired $6 \times 40 = 240$ books therefore the total number of books in the library will be $20,400 + 240 = 20,640$.
    (c) We find the slope m and the intercept b in the linear equation $N = b + mt$. From part (a), we use m = 480. We substitute to find b:

$$30,000 = b + (480)(20)$$

$$b = 20,400$$

The linear formula is $N = 20,400 + 480t$.

(d) The y-intercept is the number of books the library had in 1980.

Questions and Solutions for Section 1.2

1. Table 1.2.1 defines three functions for $0 \le x \le 8$: $y_1 = f_1(x)$; $y_2 = f_2(x)$; and $y_3 = f_3(x)$. Identify which of the functions are linear, exponential, or neither. Write an equation for the functions which are exponential or linear.

Table 1.2.1

x	y_1	y_2	y_3
0	4.25	4.25	4.25
2	6.80	5.11	3.39
4	10.88	5.97	2.53
6	17.408	9.552	1.67
8	27.8528	15.2832	0.81

ANSWER:
For the y_1's: $\frac{6.8}{2.5} = 1.6$; $\frac{10.88}{6.8} = 1.6$; $\frac{17.408}{10.88} = 1.6$; $\frac{27.8528}{17.408} = 1.6$. Therefore, y_1 is an exponential function whose y-intercept is 4.25; $y_1 = f_1(x) = 4.25(1.6)^x$.
For the y_2's, there are no common ratios or common differences.
For the y_3's, the Δy's are: $3.39 - 4.25 = -0.86$; $2.53 - 3.39 = -0.86$; $1.67 - 2.53 = -0.86$; and $.81 - 1.67 = -0.86$. Thus y_1 is a linear function with slope $\frac{-0.86}{2} = -0.43$ and y-intercept 4.25; $y_3 = f_3(x) = -0.86x + 4.25$.

2. In Table 1.2.2, we are given the population of a small country over a ten year period.

Table 1.2.2 *Population by year*

Year	Population
1985	100,004
1987	108,104
1989	116,860
1991	126,326
1993	136,559
1995	147,620

During this same period, each year, the farmers of this country have produced more than enough food to support its population. Table 1.2.3 gives the number of people that this country's agriculture were able to support during this same period:

Table 1.2.3 *Food production by year*

Year	Number of people farmers can feed
1985	105,000
1987	115,650
1989	125,253
1991	134,847
1993	145,506
1995	155,100

We are interested in determining how long the farmers will be able to produce enough food to feed this population.

(a) Knowing that populations tend to grow exponentially and assuming that food production is linear, find equations that model these two sets of data.
(b) After studying these two sets of data, what can be said about the food supply for this population during this period?
(c) Using the equations that model population and food supply, how long will there be enough food for this population?

ANSWER:

(a) Since we are assuming the population will grow exponentially, we consider ratios of population for consecutive periods:

$$\frac{108,160}{100,004} \approx 1.081$$
$$\frac{116,986}{108,160} \approx 1.081$$

Do more if needed, but this tells us that an exponential model for this population can be given by

$$N(t) = 100,004(1.082)^{\frac{t}{2}}$$

with $t = 0$ for 1985 and $t = 2$ for 1987, etc.

To find the linear model for food produced, we find an equation of a straight line from this data. Let F represent food produced. Then,

$$\frac{F - 115,650}{t - 2} = \frac{115,650 - 105,000}{2 - 0}$$

or

$$F - 115,650 = 5325(t - 2)$$

or

$$F(t) = 5325(t - 2) + 115,650$$
$$F(t) = 5325t + 105,000.$$

Again, $t = 0$ represents 1985; $t = 2$, 1987, etc.

(b) Since the number of people the farmers can feed is greater than the population on any one of the given years, one can expect a happy, healthy, growing population.
(c) Note that both N and F are increasing functions. Graphing both N and F, one can see that they intersect at about $t = 20$, or during 2005. After this year, the food supply will be inadequate for the population.

3. One of the following tables of data is linear and one is exponential. Say which is which and give an equation that best fits each table. For the exponential table you do not have to use e if you do not want. An answer like $y = (3.73)(1.92)^{x/8}$ is fine.

(a)

x	0	0.50	1.00	1.50	2.00
y	3.12	2.62	2.20	1.85	1.55

(b)

x	0	0.50	1.00	1.50	2.00
y	2.71	3.94	5.17	6.40	7.63

ANSWER:

(a) This table is exponential and we find that the ratios of successive y-values are all 0.84 (when rounded to two decimals). An appropriate equation is therefore

$$y = 3.12(0.84)^{2x} = 3.12(0.7056)^x,$$

since $y(0) = 3.12$. (Check: When $x = 2$, $y(2) = 3.12(0.7056)^2 \approx 1.5534$.) This could equally well be written $y = 3.12e^{-0.3487x}$. Actually, there is a range of possible answers: $y = 3.12a^x$ for any a between 0.7046 and 0.7059 will give the values shown in the table, when rounded to two decimals.

(b) The second table is linear. Pick a line of the form $y = mx + b$. Since $y(0) = 2.71$, $b = 2.71$, and $y = mx + 2.71$. Using the first two points gives

$$m = \frac{3.94 - 2.71}{0.50 - 0} = 2.46,$$

so $y = 2.46x + 2.71$.

4. Identify the x-intervals on which the function graphed in Figure 1.2.5 is:

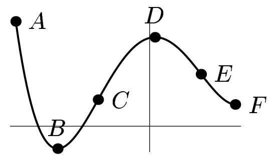


Figure 1.2.5

(a) increasing and concave downward.
(b) increasing and concave upward.
(c) decreasing and concave upward.
(d) decreasing and concave downward.

ANSWER:

(a) The function is increasing and concave downward on the x-interval between C and D.
(b) The function is increasing and concave upward on the x-interval between B and C.
(c) The function is decreasing and concave upward on the x-intervals between A and B and between E and F.
(d) The function is decreasing and concave downward on the x-intervals between D and E.

5. A bar of soap starts out at 150 grams. In each of the following cases, write a formula for the quantity S grams of soap remaining after t days. The decrease is:

(a) 10 grams per day
(b) 10% per day
(c) half the rate as in part (b)

ANSWER:

(a) This is a linear function with slope -10 grams per day and intercept 150 grams. The function is $S = 150 - 10t$.
(b) Since the quantity is decreasing by a constant percent change, this is an exponential function with base 1 - 0.1 = 0.9. The function is $S = 150(0.9)^t$.
(c) Since the quantity is decreasing by a constant percent change, this is also an exponential function. In this case, it is an exponential function with base 1 - 0.1/2 = 0.95. The function is $S = 150(0.95)^t$.

6. A photocopy machine can reduce copies to 90% or 70% of their original size. By copying an already reduced copy, further reductions can be made.

(a) Write a formula for the size of the image, N, after the original image of size x has been reduced n times with the copy machine set on 90% reduction.
(b) Write a formula for the size of the image, Q, after the original image of size x has been reduced q times with the copy machine set on 70% reduction.
(c) Which will be larger: an image that has been reduced on the 90% setting 5 times or the same image after being reduced 7 times on the 70% setting?
(d) If an image is reduced on the 90% setting and a copy of the same original image is reduced the same number of times on the 70% machine, will one image ever be less than 50% the size of the other? If so, how many copies on each of the settings will it take?

ANSWER:

(a) $N = x(.9)^n$
(b) $Q = x(.7)^q$
(c) If an image of size x is reduced 5 times on the 90% reduction setting, its new size will be

$$x(.9)^5 = .59049x$$

If the image of size x is reduced 7 times on the 70% reduction setting, its new size will be

$$x(.7)^7 = .0823543x$$

So, the first image will be larger.
(d) We want to solve for n so that $(.5)(.9)^n \geq (.7)^n$.
After one copy is made, the images sizes are $.81x$ and $.49x$ for the 90% and 70% setting respectively. After two copies, the image sizes are $.729x$ and $.343x$. Therefore, $n = 2$.

7. Give a possible formula for the function in the following figure:

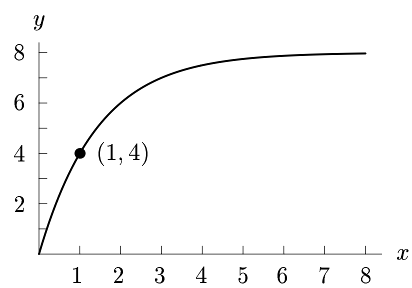


Figure 1.2.6

ANSWER:

The difference, D, between the horizontal asymptote and the graph appears to decrease exponentially, so we look for an equation of the form

$$D = D_0 a^x$$

Where $D_0 = 8 =$ difference when $x = 0$. Since $D = 8 - y$, we have

$$8 - y = 8a^x \text{ or } y = 8 - 8a^x = 8(1 - a^x)$$

The point $(1, 4)$ is on the graph, so $4 = 8(1 - a^1)$, giving $a = 1/2$.
Therefore $y = 8(1 - (1/2)^x) = 8(1 - 2^{-x})$.

8. Give a possible formula for the function in the following figure:

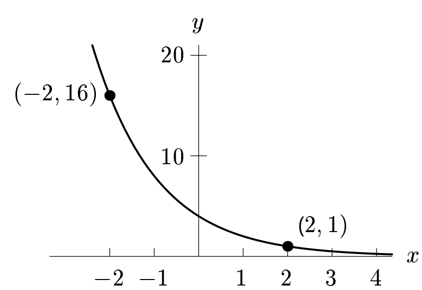


Figure 1.2.7

ANSWER:

We look for an equation of the form $y = y_0 a^x$ since the graph looks exponential. The points $(-2, 16)$ and $(2, 1)$ are on the graph, so

$$16 = y_0 a^{-2} \text{ and } 1 = y_0 a^2$$

Therefore $16/1 = y_0 a^{-2}/y_0 a^2 = 1/a^4$, giving $a = 1/2$, so $1 = y_0 a^2 = y_0(1/4)$, so $y_0 = 4$.

Hence, $y = 4(1/2)^x = 4(2^{-x})$.

9. Give a possible formula for the function in the following figure:

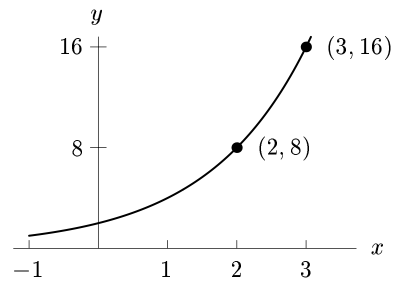


Figure 1.2.8

ANSWER:

We look for an equation of the form $y = y_0 a^x$ since the graph looks exponential. The points $(2, 8)$ and $(3, 16)$ are on the graph, so

$$8 = y_0 a^2 \text{ and } 16 = y_0 a^3$$

Therefore $8/16 = y_0 a^2 / y_0 a^3 = a^{-1}$, giving $a = 2$, so $16 = y_0 a^3 = y_0 (2)^3$, so $y_0 = 2$.

Hence, $y = 2(2^x)$.

10. These functions represent exponential growth or exponential decay.

$$P = 6(1.06)^t$$

$$Q = 4.2e^{0.04t}$$

$$S = 2e^{-0.2t}$$

$$R = 8(0.88)^t$$

Which functions represent growth and which represent decay?

ANSWER:

$P = 6(1.06)^t$ represents exponential growth because $1.06 > 1$. Since $e^{0.04t} = (e^{0.04})^t \approx (1.04)^t$, we have $Q = 4.2(1.04)^t$. This is exponential growth because $1.04 > 1$. $R = 8(0.88)^t$ represents exponential decay because $0.88 < 1$. Since $e^{-0.2t} = (e^{-0.2})^t \approx (0.82)^t$, we have $S = 2(0.82)^t$. This is exponential decay because $0.82 < 1$.

11. Joe and Sam each invested $\$20,000$ in the stock market. Joe's investment increased in value by 5% per year for 10 years. Sam's investment decreased in value by 10% for 5 years and then increased by 10% for the next 5 years.

(a) At the end of the 10 years, whose investment was worth more, Joe's or Sam's?

(b) If Sam's initial investment was $\$30,000$, but Joe's was still $\$20,000$, would that change whose investment would be worth more at the end of the 10 years?

ANSWER:

(a) The value of Joe's investment after ten years is $\$20,000(1.05)^{10} = \$32,577.89$. The value of Sam's investment after five years is $\$20,000(0.90)^5 = \$11,809.80$ and the value after ten years is $\$11,809.80(1.10)^5 = \$19,019.80$. This means that after ten years, Joe's investment will be worth more than Sam's.

(b) The value of Sam's investment after five years is $\$30,000(0.90)^5 = \$17,714.70$. After ten years, the value is $\$17,714.70(1.10)^5 = \$28,529.70$, therefore Joe's investment would still be worth more than Sam's at the end of ten years.

12. A bakery has 200 lbs of flour. If they use 5% of the available flour each day, how much do they have after 10 days? Write a formula for the amount of flour they have left after n days.

ANSWER:

Using $Q = Q_0(1-r)^t$, we have

$$Q = 200(1 - .05)^{10} = 200(0.95)^{10} = 119.75.$$

The amount of flour left after 10 days is 119.75 lbs. The amount of flour left after n days is $200(0.95)^n$.

13. A substance has a half-life of 56 years.
   (a) Write a formula for the quantity, Q, of the substance left after t years, if the initial quantity is Q_0.
   (b) What percent of the original amount of the substance will remain after 20 years?
   (c) How many years will it be before less than 10% of the substance remains?

   ANSWER:

   (a) The formula is $Q = Q_0(1/2)^{(t/56)}$.
   (b) The percentage left after 20 years is

$$\frac{Q_0(1/2)^{(20/56)}}{Q_0}$$

   The Q_0's cancel giving

$$(1/2)^{(20/56)} \approx 0.781,$$

   so 78.1% is left.
   (c) The percent left after t years is

$$(1/2)^{(t/56)}$$

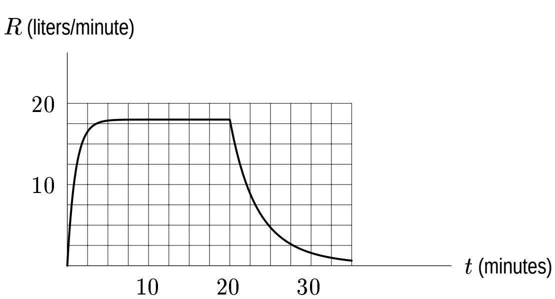
   To find the number of years it takes for there to be less than 10% remaining, we solve

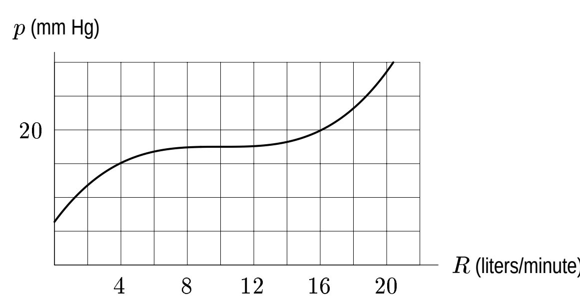
$$(1/2)^{(t/56)} = 0.1$$
$$t = 372,$$

   so after 372 years less than 10% of the substance will remain.

Questions and Solutions for Section 1.3

1. One of the graphs below shows the rate of flow, R, of blood from the heart in a man who bicycles for twenty minutes, starting at $t = 0$ minutes. The other graph shows the pressure, p, in the artery leading to a man's lungs as a function of the rate of flow of blood from the heart.





   (a) Estimate $p(R(10))$ and $p(R(22))$.
   (b) Explain what $p(R(10))$ represents in practical terms.

   ANSWER:

   (a) $p(R(10)) = p(18) = 23$ mm Hg
   $p(R(22)) = p(10) = 17.5$ mm Hg
   (b) $p(R(10))$ represents the pressure in the artery at $t = 10$.

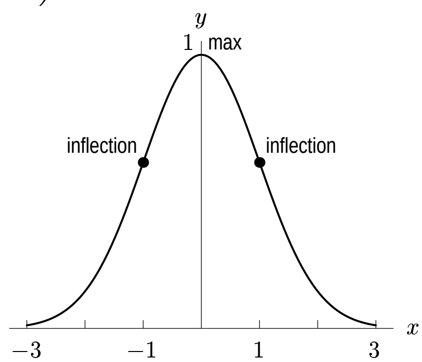
2. Given the function $y = f(x) = e^{-x^2/2}$:
   (a) Devise functions $g(x)$ and $h(x)$ so that $f(x) = g(h(x))$.
   (b) Graph the function $f(x)$; set the range of your calculator to $-3 \leq x \leq 3$. Copy your graph onto graph paper. Estimate the extrema (the maximum and/or minimum) point(s) and the inflection point(s) for this function.
   (c) The function $N(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}$ is one of the cornerstones of statistics. In a sentence, briefly describe what $\frac{1}{\sqrt{2\pi}}$ does to $f(x)$, i.e., briefly describe the curve $N(x)$ in terms of that of $f(x)$.

ANSWER:

(a) $h(x) = -\frac{x^2}{2}$, the "inside" function and $g(x) = e^x$.

Then $f(x) = g(h(x)) = g\left(-\frac{x^2}{2}\right) = e^{-\frac{x^2}{2}}$.

(b)



(c) The coefficient $\frac{1}{\sqrt{2\pi}}$ "dilates" the curve; since $\frac{1}{\sqrt{2\pi}} < 1$, the effect is to "squash the curve down" a bit.

This curve is the *Gaussian* or *Normal distribution*. The $\frac{1}{\sqrt{2\pi}}$ is chosen so that the area under the whole curve is 1. Also note that the x-axis is a horizontal asymptote.

3. Given the function $m(z) = z^2$, find and simplify $m(z+h) - m(z)$.

ANSWER:

$$\begin{aligned} m(z+h) - m(z) &= (z+h)^2 - z^2 \\ &= z^2 + 2zh + h^2 - z^2 \\ &= 2zh + h^2 \end{aligned}$$

4. Given the function $f(x) = \frac{2}{3x-7}$, do the following:

(a) Graph f, and from that graph, produce a graph of the inverse function of f.

(b) Find an algebraic expression for the inverse of f and check to see that the graph of the inverse function matches the graph found in part (a).

ANSWER:

(a) The graph of f is given in Figure 1.3.9.

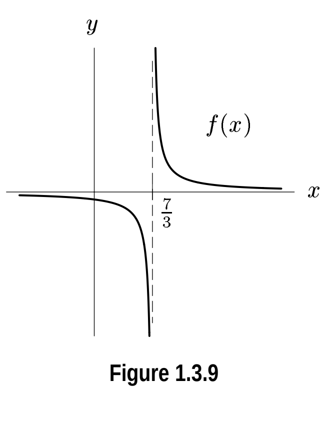


Figure 1.3.9

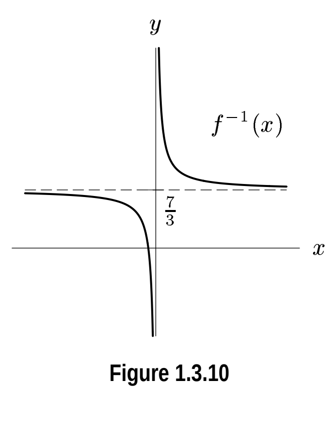


Figure 1.3.10

To find the graph of the inverse of f, we reflect the graph of f about the line $y = x$. This gives the graph shown in Figure 1.3.10.

(b) To find the inverse of f algebraically, set

$$y = \frac{2}{3x-7}$$

and solve for x in terms of y. This gives the following:

$$\begin{aligned} 3xy - 7y &= 2 \\ 3x - 7 &= \frac{2}{y} \\ 3x &= \frac{2}{y} + 7 \\ x &= \frac{2}{3y} + \frac{7}{3}. \end{aligned}$$

Rewriting this as a function with variable x gives

$$h(x) = \frac{2}{3x} + \frac{7}{3}.$$

This can be checked by showing that $f\left(h(x)\right) = x$ and $h\left(f(x)\right) = x$.

5. The graph of $y = f(x)$ is shown in Figure 1.3.11. Sketch graphs of each of the following. Label any intercepts or asymptotes that can be determined.
   (a) $y = 3f(x) - 4$
   (b) $y - 2 - 2f(x)$
   (c) $y = f(x) + 3$

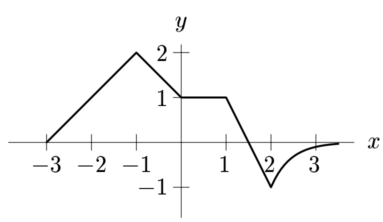


Figure 1.3.11

ANSWER:
Figure 1.3.12 shows the appropriate graphs. Note that asymptotes are shown as dashed lines and x- or y-intercepts are shown as filled circles.

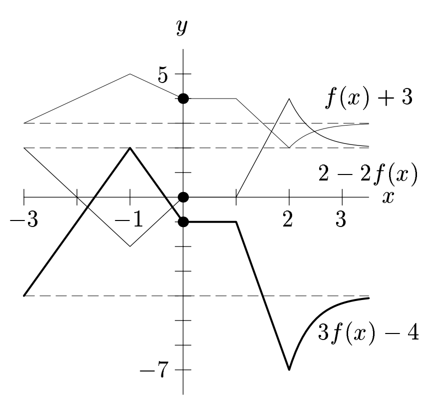


Figure 1.3.12

6. The graphs of $y = g(x)$ and $y = f(x)$ are given in Figure 1.3.13.

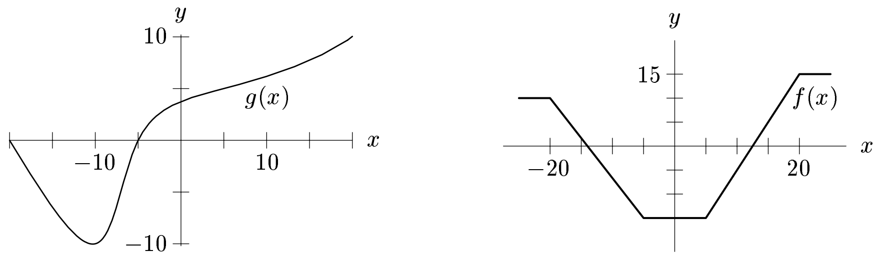


Figure 1.3.13

Estimate

(a) $f(g(5))$
(b) $g(f(5))$
(c) $f(g(-10))$
(d) $g(f(-5))$

ANSWER:

(a) $f(g(5)) \approx -15$
(b) $g(f(5)) \approx -7$
(c) $f(g(-10)) \approx -7$
(d) $g(f(-5)) \approx -7$

7. (a) Write an equation for the graph obtained by shifting the graph of $y = x^3$ vertically upward by 3 units, followed by vertically stretching the graph by a factor of 5.
   (b) Write the equation for a graph obtained by reflecting the graph for the function obtained in part (a) across the x-axis.

   ANSWER:

   (a) After a vertical shift upward by 3 units the equation is $y = x^3 + 3$. After vertically stretching the new graph by a factor of 5, the resulting equation is $y = 5(x^3 + 3) = 5x^3 + 15$.
   (b) $y = -(5x^3 + 15) = -5x^3 - 15$.

8. If the graph of $y = f(x)$ is shrunk vertically by a factor of 1/2, then shifted vertically by 4 units, then stretched vertically by a factor of 2, is the resulting graph the same as the original graph?

   ANSWER:

   The final graph is not the same as the original graph. The equation for the graph after it is shrunk vertically by a factor of 1/2 is $y = .5f(x)$. After being shifted vertically by 4 units, the new equation is $y = .5f(x) + 4$. If the new graph is stretched vertically by a factor of 2, the resulting equation is $y = 2(.5f(x) + 4) = f(x) + 8$.

9. Complete Table 1.3.4 to show the values for functions f, g, and h given the following conditions:

   (a) $f(x)$ is even.
   (b) $g(x)$ is odd.
   (c) $h(x) = g(x)^2 - f(x)$

Table 1.3.4

x	$f(x)$	$g(x)$	$h(x)$
-3	7	27	
-2	2	8	
-1	-1	1	
0	-2	0	
1			
2			
3			

ANSWER:

Table 1.3.5

x	f(x)	g(x)	h(x)
-3	7	27	722
-2	2	8	62
-1	-1	1	2
0	-2	0	2
1	-1	-1	2
2	2	-8	62
3	7	-27	722

10. Decide if the function $y = f(x)$ is invertible:

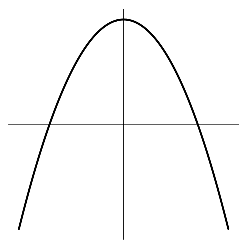


Figure 1.3.14

ANSWER:
The function is not invertible since there are many horizontal lines which hit the function twice.

11. Decide if the function $y = f(x)$ is invertible:

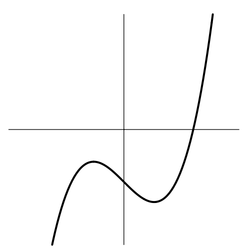


Figure 1.3.15

ANSWER:
The function is not invertible since there are horizontal lines which hit the function twice.

12. Decide if the function $y = f(x)$ is invertible:

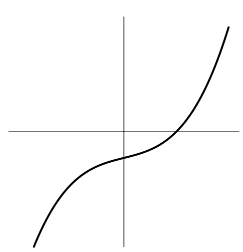


Figure 1.3.16

ANSWER:
The function is invertible since there are no horizontal lines which hit the function more than once .

13. Given the function $q(x) = x^3$, find and simplify
    (a) $q(2x + a) + q(x)$
    (b) $q(x^2) + q(x + a)$

    ANSWER:

    (a) $q(2x + a) + q(x) = (2x + a)^3 + x^3 = 9x^3 + 12ax^2 + 6a^2x + a^3$.
    (b) $q(x^2) + q(x + a) = (x^2)^3 + (x + a)^3 = x^6 + x^3 + 3ax^2 + 3xa^2 + a^3$.

14. Are the following functions even, odd, or neither?
    (a) $f(x) = x^4 + x^2 + x$
    (b) $f(x) = x^6 + x^3$
    (c) $f(x) = x^5 + x^3$

    ANSWER:

    (a) $f(-x) = (-x)^4 + (-x)^2 + (-x) = x^4 + x^2 - x$
    Since $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, this function is neither even nor odd.
    (b) $f(-x) = (-x)^6 + (-x)^3 = x^6 - x^3$
    Since $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, this function is neither even nor odd.
    (c) $f(-x) = (-x)^5 + (-x)^3 = -x^5 - x^3$
    Since $f(-x) \neq f(x)$ but $f(-x) = -f(x)$, this function is odd.

15. The cost of shipping r kilograms of material is given by the function $C = f(r) = 200 + 4r$.
    (a) Find a formula for the inverse function.
    (b) Explain in practical terms what the inverse function tells you.

    ANSWER:

    (a) The function f tells us C in terms of r. To get its inverse, we want r in terms of C, which we find by solving for r:

$$
\begin{aligned}
C &= 200 + 4r \\
C - 200 &= 4r \\
r &= (C - 200)/2 \\
&= f^{-1}(C).
\end{aligned}
$$

    (b) The inverse function tells us the number of kilograms that can be shipped for a given cost.

16. For $g(x) = 2x^2 - 2x$ and $h(x) = 3x - 1$, find and simplify
    (a) $g(x) + 2h(x)$
    (b) $g(h(x))$
    (c) $h(g(x))$

    ANSWER:

    (a) $g(x) + 2h(x) = 2x^2 - 2x + 2(3x - 1) = 2x^2 - 2x + 6x - 2 = 2x^2 + 4x - 2$
    (b) $(g(h(x)) = 2(3x - 1)^2 - 2(3x - 1) = 2(9x^2 - 6x + 1) - 6x + 2 = 18x^2 - 18x + 4$
    (c) $h(g(x)) = 3(2x^2 - 2x) - 1 = 6x^2 - 6x - 1$

Questions and Solutions for Section 1.4

1. In 1909, the Danish biochemist Sören Peter Lauritz Sörensen (1868-1939) introduced the pH function as a measure of the acidity of a chemical substance: pH= $f([H^+]) = -\log_{10}[H^+]$, where $[H^+]$ in the molecular concentration of hydrogen ions (moles per liter, M). Sörensen determined that, for $0 <$pH< 7, the substance is an acid; when pH$= 7$, the substance is neutral; and for pH> 7, the substance is a base or is said to be alkaline. The $[H^+]$ for blood is $3.16 \times 10^{-8}M$ and for milk is $4.0 \times 10^{-7}M$. Find the pH of blood and of milk, and categorize each as acid(s) and/or base(s). Reportedly, the worst known instance of acid rain occurred in Scotland in 1974, at which time the pH was determined to be 2.4 [Stewart, *et al.*, *College Algebra*, p. 331]. Find the hydrogen ion concentration for this acid rain.

    ANSWER:

    pH of blood $= -\log\left(3.16 \times 10^{-8}\right) = -\log 3.16 - \log 10^{-8} = -\log 3.16 + 8 \cdot \log 10$ so that the pH of blood $= -\log 3.16 + 8$.

When we evaluate this, we get pH of blood $\approx -.4997 + 8 = 7.5003$, and blood is slightly alkaline.
pH of milk $= -\log(4.0 \times 10^{-7}) = -\log 4 + 7 \approx -.6021 + 7 = 6.3979$, and we see that milk is mildly acidic.
For the acid rain, $2.4 = -\log[H^+]$, or $-2.4 = \log[H^+]$, so that $10^{-2.4} \approx 3.981 \times 10^{-3} = .003981M = [H^+]$.

2. Suppose that $N(t) = 100,000,000 \cdot 2^{t/30}$ gives the population of a certain country t years after a census was taken. A historian has a collection of documents that are not dated, but do refer to the population of this country at several times. In order to help the historian date these documents, find the inverse function for the function N.

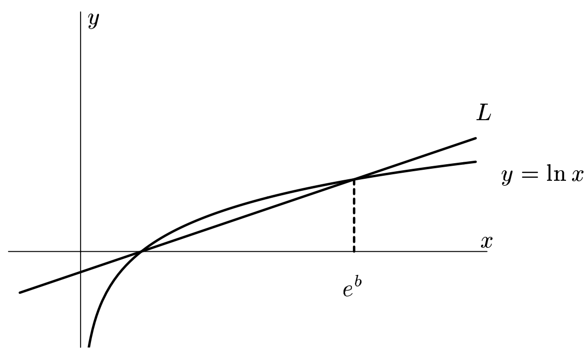
   ANSWER:
   To find the inverse of N, take the logarithm of both sides of the expression. This gives

$$\begin{aligned}\log N &= \log\left[100,000,000 \cdot 2^{\frac{t}{30}}\right] \\ &= \log[100,000,000] + \frac{t}{30}\log 2 \approx 8 + 0.01t\end{aligned}$$

   Solving this for t gives the desired function:

$$t = \frac{30}{\log 2}(\log N - 8) \approx 100 \log N - 797.$$

3. Find an equation for the line L shown below. Your answer will contain the positive constant b. Simplify your answer.



   ANSWER:
   The line L passes through the two points on the curve $y = \ln x$ specified by $y = 0$ and $x = e^b$, that is, $(1, 0)$ (since $1 = e^0$) and (e^b, b) respectively. The equation for this line is

$$\frac{y-0}{x-1} = \frac{b-0}{e^b-1},$$

   so

$$y = \frac{b}{e^b-1}(x-1).$$

4. Here are some data from a recent Scientific American article on Old World monkeys.

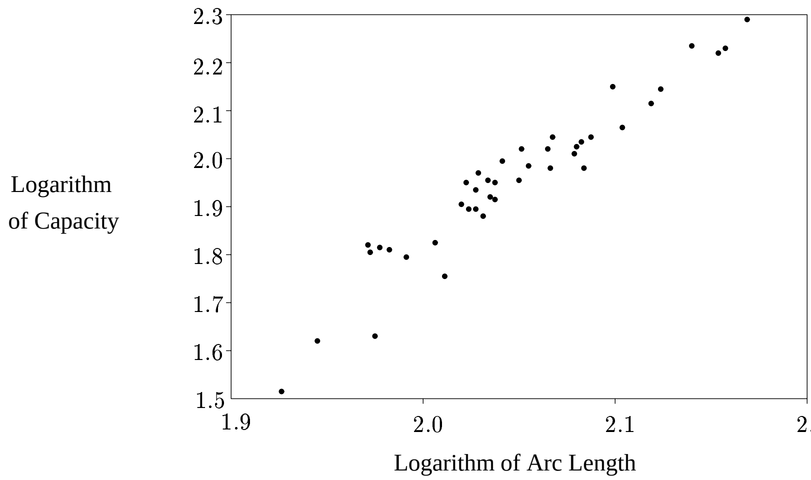


Figure 1.4.17: Cranial Capacity of contemporary Old World monkeys is related to arc length of skull as shown.

(a) From the data presented give an approximate formula for

$$C = \text{cranial capacity (in cm}^3\text{)}$$

as a function of

$$A = \text{arc length of skull (in cm)}.$$

[Hint: Fit a line through the data points. Logarithms are to base 10.]

(b) What type of function is $C = f(A)$ (logarithmic, exponential, trigonometric, power function,...)?

ANSWER:

(a) A line fit through the data points goes through $(2.18, 2.3)$ and $(1.9, 1.66)$, and so has slope $\dfrac{\Delta y}{\Delta x} = \dfrac{0.64}{0.28} = 2.3$. The equation is

$$y = 2.3x + b.$$

Solving for b yields

$$2.3 = (2.3)2.18 + b$$
$$b \approx -2.7.$$

So $y = 2.3x - 2.7$. But $x = \log(A)$ and $y = \log(C)$, so

$$\log C = 2.3 \log A + 2.7.$$

Exponentiating both sides yields

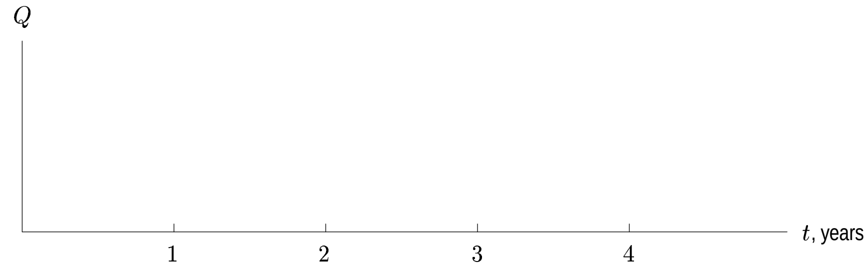
$$C = 10^{2.3 \log A + 2.7} \approx A^{2.3}\,(501)\,.$$

(b) This is a power function.

5. (a) Suppose there is an initial population of 100 rabbits on *Prosperity Island*. Assuming that the rabbits have more than enough of everything they need to live prosperously, we might expect the population to grow exponentially. If so, find a formula for $P(t)$, the number of rabbits on *Prosperity Island* at time t, given that after one year there are 120 rabbits on the island. (Assume t is in years.) When will there be 500 rabbits on *Prosperity Island*?

(b) Next, we turn our attention to *Cramped Quarters Island*, a tiny island which, although able to support a *limited* population of rabbits, doesn't have enough space or food supplies to support unlimited exponential growth. It is suggested that if $Q(t) =$ population of rabbits on *Cramped Quarters Island* at time t, then the quantity $(800 - Q(t))$ will be an exponentially decaying function of t. Given that there were 500 rabbits at time $t = 0$, and 600 rabbits one year later, what is the general formula for $Q(t)$, the population of rabbits on *Cramped Quarters Island* at time t?

(c) On the axes below, make a sketch of $y = Q(t)$. Choose an appropriate scale for the Q axis.

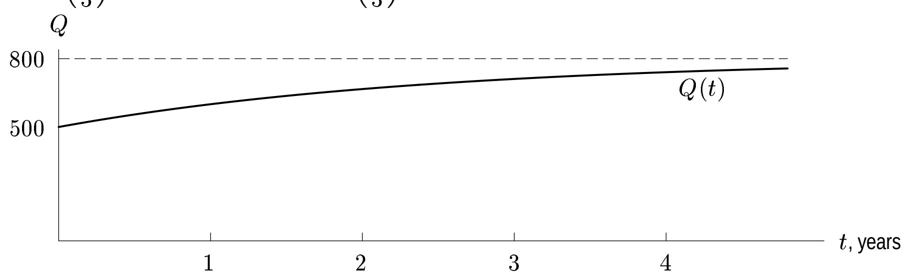


ANSWER:

(a) $P(t) = P_0 \cdot a^t$
$P(0) = P_0 = 100$
$P(1) = P_0 \cdot a^1 = 100 \cdot a = 120,$ so $a = 1.2$
so $P(t) = 100 \cdot (1.2)^t$
When will there be 500 rabbits on *Prosperity Island*?
Let t_0 be this number. Then $P(t_0) = 500 = 100{\cdot}(1.2)^{t_0}$. Therefore $5 = (1.2)^{t_0}$, so $\ln 5 = t_0 \cdot \ln 1.2$, so $t_0 = \dfrac{\ln 5}{\ln 1.2}$

(b) $800 - Q(t)$ is an exponential decay function.
Hence, $800 - Q(t) = f(t) = P_0 \cdot a^t$
$f(0) = 800 - Q(0) = 800 - 500 = 300 = P_0 a^t = P_0$
$f(t) = 800 - Q(1) = 800 - 600 = 200 = 300a^t$, so $a = \dfrac{2}{3}$
$f(t) = 300\left(\dfrac{2}{3}\right)^t$. So $Q(t) = 800 - 300\left(\dfrac{2}{3}\right)^t$.

(c)



6. An exponentially decaying substance was weighed every hour and the results are given below:

Time	Weight (in grams)
9 am	10.000
10 am	8.958
11 am	8.025
12 noon	7.189
1 pm	6.440

(a) Determine a formula of the form

$$Q = Q_0 e^{-kt}$$

which would give the weight of the substance, Q, at time t in hours since 9 am.

(b) What is the approximate half-life of the substance?

ANSWER:

(a) $Q = 10e^{-kt}$ since $Q_0 =$ initial value $= 10$.
When $t = 1$, $Q = 8.958$, so $8.958 = 10e^{-k(1)}$ and
$0.8958 = e^{-k}$ so $k = -\ln 0.8958 = 0.11$
Thus $Q = 10e^{-0.11t}$.

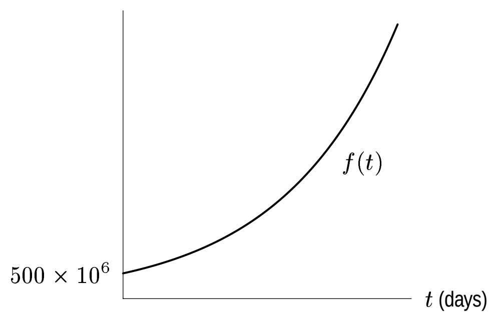
(b) Half life when $Q = \frac{1}{2}Q_0$: $\frac{1}{2}Q_0 = Q_0 e^{-0.11t}$ so $\ln\frac{1}{2} = -0.11t$, so $t = 6.3$ hours.

7. The number of bacteria in milk grows at a rate of 10% per day once the milk has been bottled. When the milk is put in the bottles, it has an average bacteria count of 500 million per bottle.

   (a) Write an equation for $f(t)$, the number of bacteria t days after the milk is bottled.
   (b) Graph the number of bacteria against time. Label the axes and intercepts.
   (c) Suppose milk cannot be safely consumed if the bacteria count is greater than 3 billion per bottle. How many days will the milk be safe to drink once it has been bottled?

   ANSWER:

   (a) $f(t) = 500 \times 10^6 (1.1)^t$
   (b)



   (c) Find t making $f(t) = 3 \times 10^9$
   $3 \times 10^9 = 500 \times 10^6 (1.1)^t$

$$\frac{3000}{500} = (1.1)^y \text{ so } 6 = (1.1)^t$$

$$t = \frac{\ln 6}{\ln 1.1} \approx 18.8 \text{ days.}$$

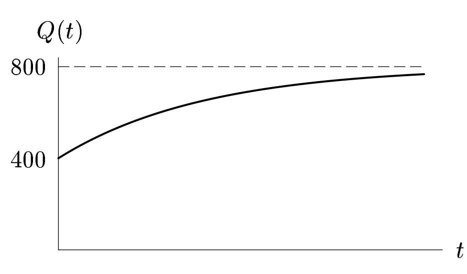
   Thus milk will be safe for 18 days; during the 19th day it will turn bad, according to this model.

8. *Cramped Quarters Island* is a tiny island which, although able to support a limited population of rabbits, does not have enough space or food supplies to support unlimited exponential growth. It is suggested that if $Q(t)$ = population of rabbits at time t, then the quantity $(800 - Q(t))$ will be an exponentially decaying function of t. If at time $t = 0$ there were 400 rabbits, and the population was increasing at an instantaneous rate of 100 rabbits per year, find a general formula for $Q(t)$.

   ANSWER:

   $Q(t)$ is the population of rabbits.

$800 - Q(t) = ae^{-kt}$
$Q(t) = 800 - ae^{-kt}$
$Q(0) = 400 = 800 - ae^{-k0}$
$a = 400$
$Q'(t) = (-a)(-k)e^{-kt} = ake^{-kt}$
$Q'(0) = 400k = 100 \Rightarrow k = \frac{1}{4} = .25$
$Q(t) = 800 - 400e^{-.25t}$



9. In 1992, the Population Crisis Committee wrote:

   > Large cities in developing countries are growing much faster than cities in the industrialized world ever have. London, which in 1810 became the first industrial city to top 1 million, now has a population of 11 million. By contrast, Mexico City's population stood at only a million just 50 years ago and now is 20 million.

Assume that the instantaneous percentage growth rates of London and Mexico City were constant over the last two centuries.

(a) How many times greater is Mexico City's percentage growth rate than London's? Show your calculations and reasoning.
(b) When were the two cities the same size? Show your calculations and reasoning.

ANSWER:

(a) Letting α and β be the two growth rates for London and Mexico City, respectively, we approximate the population growth in millions by two exponentials, $e^{\alpha t}$ and $e^{\beta t}$, both of which are set to have population 1 million when $t = 0$. Since 182 and 50 are, respectively, the times that have passed since each city had 1 million people, we get

$$11 \approx 1 \cdot e^{\alpha \cdot 182} \quad \text{and} \quad 20 \approx 1 \cdot e^{\beta \cdot 50}$$

and

$$\beta = \frac{\ln 20}{50} \approx 0.0599, \alpha = \frac{\ln 11}{182} \approx 0.0132, \quad \text{and} \quad \frac{\beta}{\alpha} = \frac{182 \ln 20}{50 \ln 11} \approx 4.5.$$

(Note that these are growth rates, not percentages, but the ratio is the same as if we did it in terms of percentages.)

(b) We measure from 50 years ago (1942), when the population in London was $e^{0.0132(132)}$ and the population is Mexico City was 1 million. The functions describing population in the two cities are then:

$$\text{Mexico City} : \text{Population} = e^{0.0599t}$$
$$\text{London} : \text{Population} = e^{0.0132(132)} e^{0.0132t}$$

Setting these equal and solving, we get:

$$\begin{aligned} e^{0.0599t} &= e^{0.0132(132)} e^{0.0132t} \\ 0.0599t &= 0.0132(132 + t) \\ t &= \frac{0.0132 \cdot 132}{0.0599 - 0.0132} \approx 37.2 \end{aligned}$$

So the populations were equal 37.2 years after 1942, that is, in 1979.

10. Given that $\ln 2 = 0.69$ and $\ln 5 = 1.61$ to two decimal places, find (without using calculator)

(a) $\ln 0.1$
(b) $\ln 100$

ANSWER:

(a) Using the properties of logarithms

$$\begin{aligned} \ln 0.1 &= \ln \frac{1}{10} \\ &= \ln 1 - \ln 10 \\ &= -\ln 10 \\ &= -\ln(2 \cdot 5) \\ &= -(\ln 2 + \ln 5) \\ &= -2.30. \end{aligned}$$

(b)

$$\begin{aligned} \ln 100 &= \ln \left(10^2\right) \\ &= 2 \ln 10 \\ &= 2(2.30) \\ &= 4.60. \end{aligned}$$

11. Simplify the expression as much as possible:

(a) $6e^{\ln(a^3)}$
(b) $3\ln(e^a) + 5\ln b^e$
(c) $\ln(ab) + \ln(1/e)$

ANSWER:

(a) Using the identity $e^{\ln x} = x$, we have $6e^{\ln(a^3)} = 6a^3$.
(b) Using the rules for $\ln$, we have $3a + 5e\ln(b)$.
(c) Using the rules for $\ln$, we have

$$\begin{aligned} \ln(ab) + \ln(1/e) &= \ln(a) + \ln(b) + \ln(1) - \ln(e) \\ &= \ln(a) + \ln(b) + 0 - 1 \\ &= \ln(a) + \ln(b) - 1. \end{aligned}$$

12. Solve for x using logs:
(a) $6^x = 12$
(b) $14 = 4^x$
(c) $2e^{4x} = 8e^{6x}$
(d) $3^{x+3} = e^{7x}$

ANSWER:

(a) Taking logs of both sides

$$\log 6^x = x\log 6 = \log 12$$

$$x = \frac{\log 12}{\log 6} \approx 1.39$$

(b) Taking logs of both sides

$$\log 14 = \log 4^x$$

$$\log 14 = x\log 4$$

$$x = \frac{\log 14}{\log 4} \approx 1.90$$

(c) Taking the natural logarithm of both sides

$$\ln(2e^{4x}) = \ln(8e^{6x})$$

$$\ln 2 + \ln(e^{4x}) = \ln 8 + \ln(e^{6x})$$

$$0.69 + 4x \approx 2.08 + 6x$$

$$2x \approx -1.39$$

$$x \approx -0.695$$

(d) Using the rules for $\ln$, we get

$$\ln(3^{x+3}) = \ln(e^{7x})$$

$$(x+3)\ln 3 = 7x$$

$$x\ln 3 + 3\ln 3 = 7x$$

$$x(\ln 3 - 7) = -3\ln 3$$

$$x = \frac{-3\ln 3}{\ln 3 - 7} \approx 0.558$$

13. What is the doubling time of prices which are increasing by
    (a) 7% a year
    (b) 14% a year

    ANSWER:

    (a) Since the factor by which the prices have increased after time t is given by $(1.07)^t$, the time after which the prices have doubled solves
    $$2 = (1.07)^t$$
    $$\log 2 = \log(1.07^t) = t\log(1.07)$$
    $$t = \frac{\log 2}{\log 1.07}$$
    $$t \approx 10.24 \text{ years.}$$
    (b) Since the factor by which the prices have increased after time t is given by $(1.14)^t$, the time after which the prices have doubled solves
    $$2 = (1.14)^t$$
    $$\log 2 = \log(1.14^t) = t\log(1.14)$$
    $$t = \frac{\log 2}{\log 1.14}$$
    $$t \approx 5.29 \text{ years.}$$

14. If the size of a bacteria colony doubles in 8 hours, how long will it take for the number of bacteria to be 5 times the original amount?

    ANSWER:

    Given the doubling time of 8 hours, we can solve for the bacteria's growth rate;
    $$2P_0 = P_0 e^{k8}$$
    $$k = \frac{\ln 2}{8}$$
    So the growth of the bacteria population is given by;
    $$P = P_0 e^{\ln(2)t/8}$$
    We want to find t such that
    $$5P_0 = P_0 e^{\ln(2)t/8}$$
    Therefore we cancel P_0 and apply $\ln$. We get
    $$t = \frac{8\ln(5)}{\ln(2)} = 18.575$$

Questions and Solutions for Section 1.5

1. Consider the function
   $$c(x) = \cos x + 0.5\cos 2x.$$
   (a) Is $c(x)$ a periodic function? If so, what is its smallest period?
   (b) Using your calculator, draw the graph of $c(x)$. Adjust the scales so you can see the patterns and symmetries clearly. Sketch your final version on the axes below, showing the scale you use, and describe what you see.
   (c) At what points x is $c(x)$ a maximum? Explain.
   (d) Let x_0 be the first positive number where $c(x_0) = 0$. Find an interval containing x_0 whose length is $< \frac{1}{10}$. Explain briefly how you did this.
   (e) Looking again at the symmetries in your graph in part (b), argue that the next positive number where $c(x_0) = 0$ is $2\pi - x_0$. Can you show this directly?
   (f) In terms of x_0, what are all the places where $c(x) = 0$ for $-2\pi \le x \le 4\pi$?

ANSWER:

(a) This is a periodic function with smallest period 2π.

(b) This function is similar to a cosine function, except that the peaks are slightly raised, and the troughs have small bumps in them.

(c) $c(x)$ has a local maximum at $n\pi$, where n is an integer. $c(x)$ takes on its greatest value, 1.5, at every 2π.

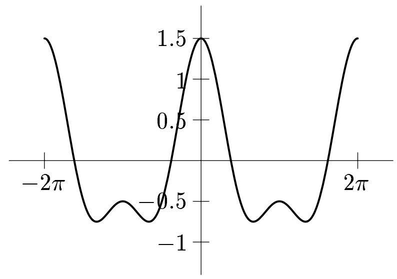


Figure 1.5.18

(d) In order to confine x_0 within an interval whose length is less than $\frac{1}{10}$, try picking values of x and looking at the sign of $c(x)$. Start with $x = 1$, since it looks like it's near the root.

$$\begin{aligned} c(1) &= \cos(1) + 0.5\cos(2) \\ &\approx 0.5403 - 0.2081 > 0 \\ c(2) &= \cos(2) + 0.5\cos(4) \\ &\approx -0.4161 - 0.3268 < 0 \end{aligned}$$

So x_0 is between 1 and 2.

$$\begin{aligned} c(1.5) &= \cos(1.5) + 0.5\cos(3) \\ &\approx 0.0707 - 0.495 < 0 \end{aligned}$$

So x_0 is between 1 and 1.5.

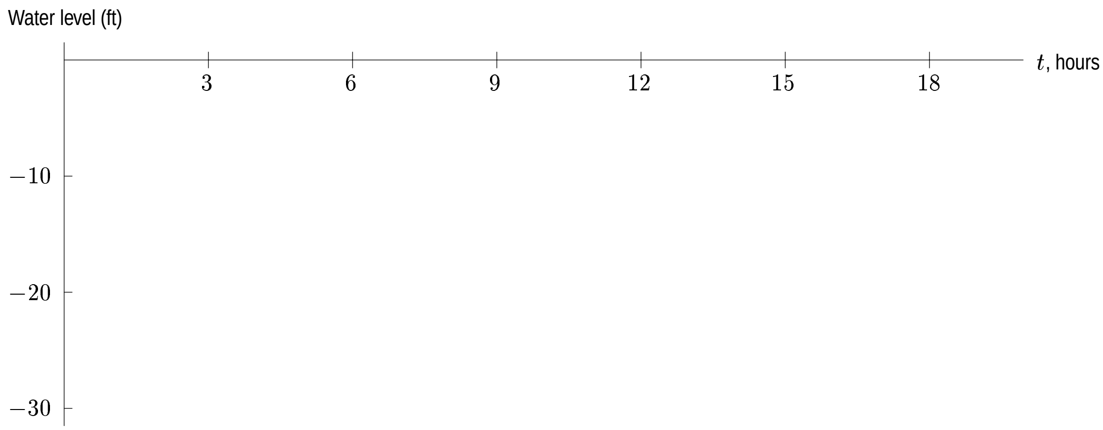
$$\begin{aligned} c(1.2) &= \cos(1.2) + 0.5\cos(2.4) \\ &\approx 0.3624 - 0.3686 < 0 \\ c(1.1) &= \cos(1.1) + 0.5\cos(2.2) \\ &\approx 0.4536 - 0.2942 > 0 \end{aligned}$$

So x_0 is between 1.1 and 1.2.

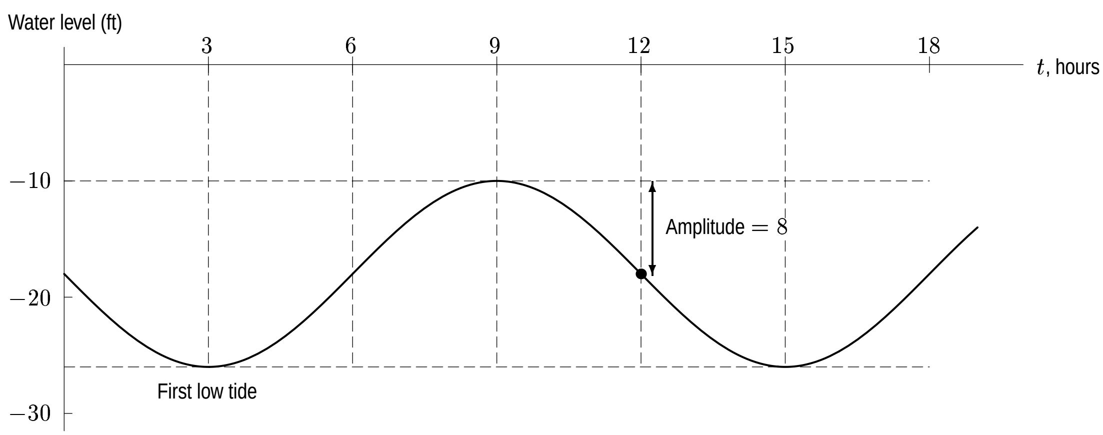
(e) Since $c(x)$ is even, there will be a root at $-x_0$. Since this function has period 2π, there will be a root at $-x_0 + 2\pi$. From the graph, it is clear that this is the next root of $c(x)$.

(f) $x_0, -x_0, 2\pi - x_0, x_0 - 2\pi, x_0 + 2\pi, 4\pi - x_0$

2. At high tide, the water level is 10 feet below a certain pier. At low tide the water level is 26 feet below the pier. Assuming sinusoidal behavior, sketch a graph of $y = f(t) =$ the water level, relative to the pier, at time t (in hours) if at $t = 0$ the water level is -18 feet and falling, until it reaches the first low tide at $t = 3$. Based on your sketch and the information provided above, give a formula for $f(t)$.



ANSWER:

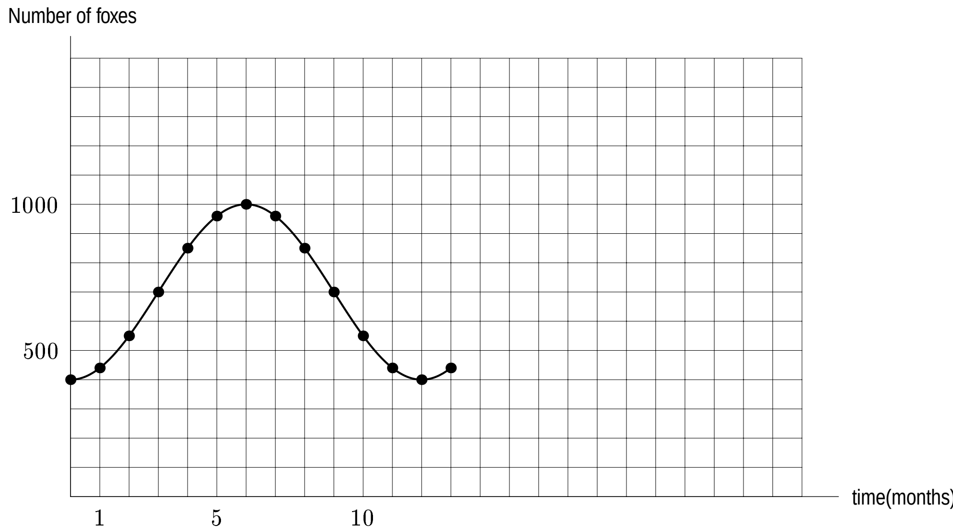


$$f(t) = A\sin(B(t+C)) + D$$

$$\text{period} = 12 \text{ hrs} \qquad \frac{2\pi}{B} = 12 \qquad B = \frac{\pi}{6}$$

$$\begin{aligned} f(t) &= 8\sin\left(\frac{\pi}{6}(t+6)\right) - 18 \\ &= 8\sin\left(\frac{\pi}{6}t + \pi\right) - 18 \\ &= -8\sin\frac{\pi}{6}t - 18 \end{aligned}$$

3. In nature, the population of two animals, one of which preys on the other (such as foxes and rabbits) are observed to oscillate with time, and are found to be well approximated by trigonometric functions. The population of foxes is given by the graph below.



(a) Find the amplitude.
(b) Find the period.
(c) Give a formula for the function.
(d) Give an estimate for three times when the population is 500.

ANSWER:

(a) $\dfrac{1000-400}{2} = 300$
(b) Period=12 months
(c) Average value = 700 and graph looks like an upside-down cosine, so $F = 700 - 300\cos(kt)$ since $k(12) = 2\pi$, $k = \dfrac{\pi}{6}$ so $F = 700 - 300\cos\left(\dfrac{\pi t}{6}\right)$
(d) First time is between $t = 1$ and $t = 2$, so let's say about $t \approx 1.5$. From graph, next values are $t \approx 10.5$ and then $t \approx 13.5$ months

4. One of the functions below is a quadratic, one is a cubic, and one is a periodic function. Which is which? Why? [Note: You don't have to find formulas for these functions.]

x	$f(x)$
0.2	-0.42
0.4	-0.65
0.6	0.96
0.8	-0.15
1.2	0.84

x	$g(x)$
1.3	0.41
1.7	0.81
2.5	0.65
3.0	-0.10
3.5	-1.35

x	$h(x)$
0.5	-1.13
1.2	0.13
1.8	0.03
2.0	0.00
2.2	0.05

ANSWER:

$f(x)$ changes direction three times, so it cannot be cubic or quadratic, and thus it is trigonometric. $h(x)$ changes direction twice, so it cannot be a quadratic, and thus it must be cubic. This leaves only $g(x)$ (which changes direction only once), so it must be a quadratic.

5. Find possible formulas for the following sinusoidal function as a
   (a) transformation of $f(t) = \sin t$
   (b) transformation of $f(t) = \cos t$

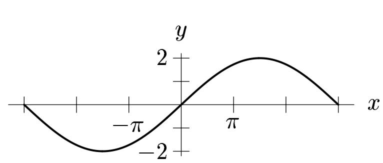


Figure 1.5.19

ANSWER:

(a) This function looks like a sine function with amplitude 2, so $f(t) = 2\sin(Bt)$. Since the function executes one full oscillation between $t = 0$ and $t = 6\pi$, when t changes by 6π, the quantity Bt changes by 2π. This means $B(6\pi) = 2\pi$, so $B = 1/3$. Therefore, $f(t) = 2\sin(t/3)$ has the graph shown.

(b) The function also looks like a cosine function with amplitude 2 that has a period of 6π that been translated horizontally by 4.5π. Therefore, $f(t) = 2\cos(t/3 - 4.5\pi)$.

6. Temperatures in Town A oscillate daily between 30°F at 4am and 60°F at 4pm. Write the following formulas:

(a) Temperature in Town A, in terms of time where time is measured in hours from 4am.

(b) Temperature in Town A, in terms of time where time is measured in hours from midnight.

(c) Temperature in Town B, where the temperatures are consistently 10°F colder than in Town A and measured from 4am.

ANSWER:

(a) We use a cosine of the form

$$H = A\cos(Bt) + C$$

and choose B so that the period is 24 hours, so $2\pi/B = 24$, giving $B = \pi/12$.

The temperature oscillates around an average value of 45°F, so $C = 45$. The amplitude of the oscillation is 15°F. To arrange that the temperature be at its lowest when $t = 0$, we take A negative,

so

$$H = -15\cos\left(\frac{\pi}{12}t\right) + 45.$$

(b) The formula is the answer from (a), shifted to the right by four hours,

$$H = -15\cos\left(\frac{\pi}{12}(t-4)\right) + 45.$$

(c) The shape of the graph will be the same as (a), only translated down 10. So the formula is

$$H = -15\cos\left(\frac{\pi}{12}t\right) + 35.$$

7. Temperatures in a room oscillate between the low of -10°F (at 5am) and the high of 40°F (reached at 5pm).

(a) Find a possible formula for the temperature in the room in terms of time from 5am.

(b) Sketch a graph of temperature in terms of time.

ANSWER:

(a) We use a cosine of the form

$$H = A\cos(Bt) + C$$

and choose B so that the period is 24 hours, so $2\pi/B = 24$, giving $B = \pi/12$.

The temperature oscillates around an average value of 15°F, so $C = 15$. The amplitude of the oscillation is 25°F. To arrange that the temperature be at its lowest when $t = 0$, we take A negative,

so

$$H = -25\cos\left(\frac{\pi}{12}t\right) + 15.$$

(b)

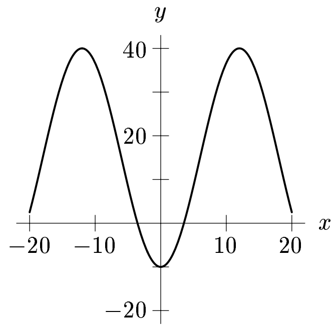


Figure 1.5.20

8. Consider the function $g(t) = 10 + \cos 2t$.
   (a) What is its amplitude?
   (b) What is its period?
   (c) Sketch its graph.
   (d) Write a short scenario that could be described by the behavior of $g(t)$.

   ANSWER:

   (a) The amplitude is 1.
   (b) The period is $2\pi/2 = \pi$.
   (c)

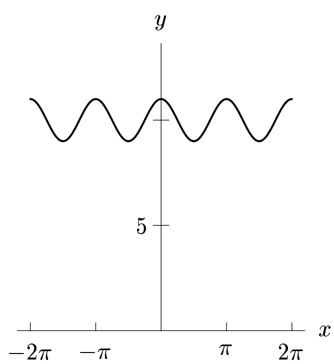


Figure 1.5.21

   (d) Answers will vary.

9. Consider the functions $f(x) = 5 + \sin 3x$ and $g(x) = 3 \sin x$. Describe how these functions are the same and how they are different. Include a description of amplitude, period and general shape of the graph.

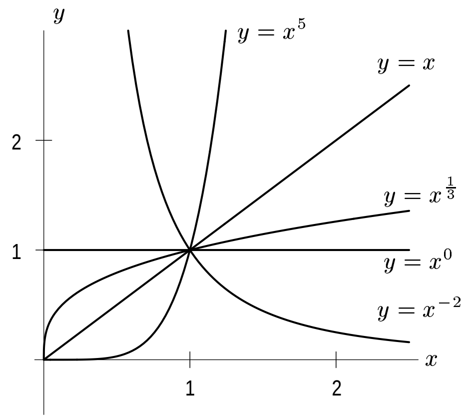
   ANSWER:

   The amplitude of $g(x)$ is 3 times the amplitude of $f(x)$. The period of $g(x)$ is 2π while the period of $f(x)$ is $2\pi/3$, thus $f(x)$ is oscillating more quickly than $g(x)$. The y-intercept of $g(x)$ is zero whereas $f(x)$ has y-intercept 5.

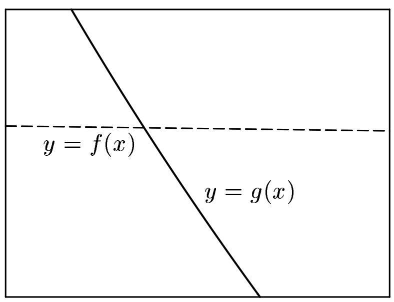
Questions and Solutions for Section 1.6

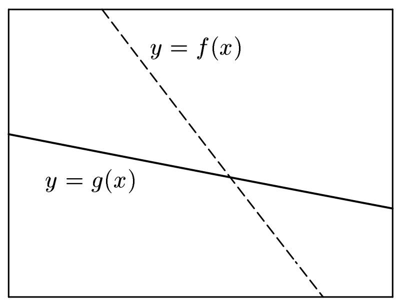
1. Give rough sketches, for $x > 0$, of the graphs of $y = x^5$, $y = x$, $y = x^{1/3}$, $y = x^0$, and $y = x^{-2}$.

   ANSWER:



2. Consider the functions $f(x) = 10\left(\frac{2}{3}\right)^x$ and $g(x) = \frac{1}{\sqrt{x}}$. Graph both of these functions on your graphing calculator. By zooming in on appropriate regions, you should be able to make the two graphs appear as shown in the diagrams below.

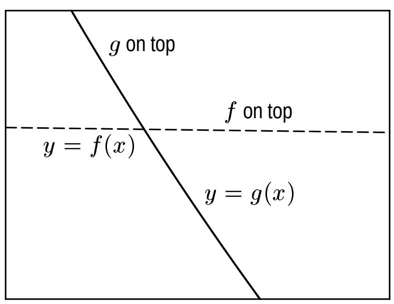


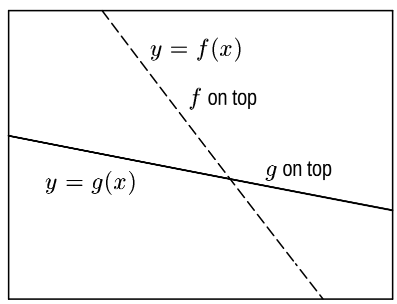


(a) Find the x-coordinate of the point of intersection of the two graphs indicated in the **left**-hand diagram. Give an answer accurate to within one decimal place..

(b) Find the x coordinate of the point of intersection of the two graphs indicated in the **right**-hand diagram. Give an answer accurate to within one decimal place.

ANSWER:





(a) The first diagram: This intersection occurs "near" $x = 0$. Try
x min$= 0$, x max$= 0.1$
y min$= 0$, y max$= 11$
Then zoom in. Get $x \approx 0.010$.

(b) The second diagram: This intersection occurs for y-values near zero. Try
x min$= 0$, x max$= 10$
y min$= 0$, y max$= 1$.
Then zoom. Get $x \approx 8.3$

3. (a) Use your calculator to find all the solutions to the equation

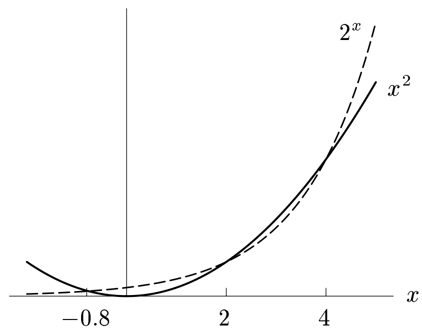
$$2^x = x^2.$$

Give your answers to one decimal place. Sketch the graphs drawn by your calculator as part of the explanation for your answer.

(b) For what values of x is $2^x > x^2$?

ANSWER:

(a)

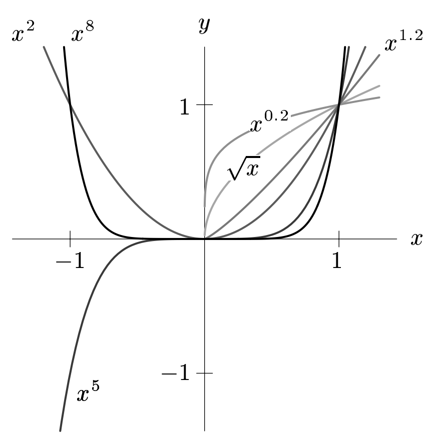


Solutions : $x = -0.8$ (by zooming), $x = 2, 4$.

(b) $2^x > x^2$ for $-0.8 < x < 2$ or $x > 4$.

4. On the given set of axes, graph and clearly label: (A) $y = \sqrt{x}$; (B) $y = x^{.2}$; (C) $y = x^{1.2}$; (D) $y = x^2$; (E) $y = x^5$; (F) $y = x^8$.

ANSWER:



5. A spherical cell takes in nutrients through its cell wall at a rate proportional to the area of the cell wall. The rate at which the cell uses nutrients is proportional to its volume.

(a) Write an expression for the rate at which nutrients enter the cell as a function of its radius, r.

(b) Write an expression for the rate at which the cell uses nutrients as a function of its radius,r.

(c) Sketch a possible graph showing the rate at which nutrients enter the cell against the radius r (put r along the horizontal axis). On the same axes, sketch a possible graph for the rate at which the cell uses nutrients.

(d) Show algebraically why there must be a radius r_0 (other than $r_0 = 0$) at which the rate at which nutrients are used equals the rate at which nutrients enter the cell. Mark r_0 on your graph.

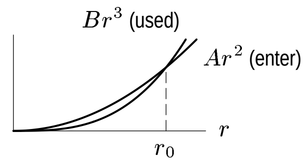
(e) What happens to the cell when $r > r_0$? When $r < r_0$? What does this tell you about the radius of the cell in the long run?

ANSWER:

(a) Rate at which nutrients enter cell $= k4\pi r^2 = Ar^2 \quad (A > 0) \quad k, c$ are constants of proportionality.

(b) Rate at which nutrients are used $= c\frac{4}{3}\pi r^3 = Br^3 \quad (B > 0)$.

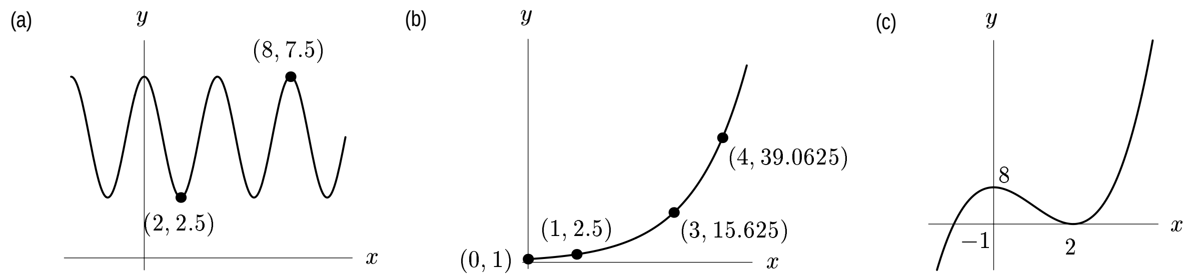
(c)



(d) $Ar^2 = Br^3$ for $r = 0$ and $r = \dfrac{A}{B}$ so $r_0 = \dfrac{A}{B}$

(e) When $r > r_0$, rate used $>$ rate enter so cell shrinks.
When $r < r_0$, rate used $<$ rate enter so cell grows.
In long run, cell's radius $\to r_0$.

6. Find a possible formula for each of the following functions. Check that your formula fits the data points.



ANSWER:

(a) This function looks like a cosine function with period 4 displaced upward by 5 with amplitude 2.5. A possible formula is

$$5 + 2.5\cos\left(\frac{2\pi x}{4}\right).$$

(b) Notice that $f(1)/f(0) = f(4)/f(3) = 2.5$. This suggests that f is exponential. In fact, $f(x) = 2.5^x$ fits this data.

(c) This is a cubic with double zero at 2 and another at -1. So let $f(x) = C(x-2)^2(x+1)$. If $f(0) = 8$, then $C(-2)^2(1) = 8$, so we have $C = 2$.

7. (a) Using the standard viewing rectangle $(-10 \le x \le 10, -10 \le y \le 10)$, I graph a cubic polynomial and see two more or less vertical lines.

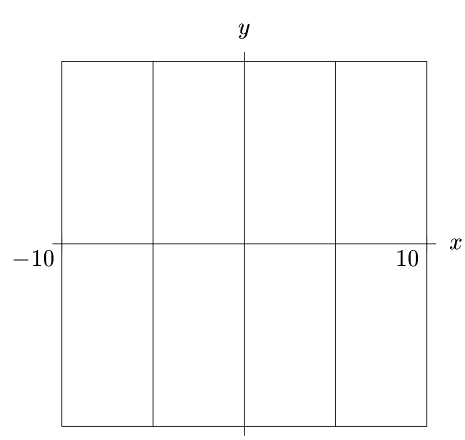


Figure 1.6.22

Need there be another root? Explain. Sketch some of the possibilities for the complete graph. Explain.

(b) Once again, using the standard viewing rectangle, I graph $y = x^2 - e^{0.1x}$ and I see what appears to be a parabola.

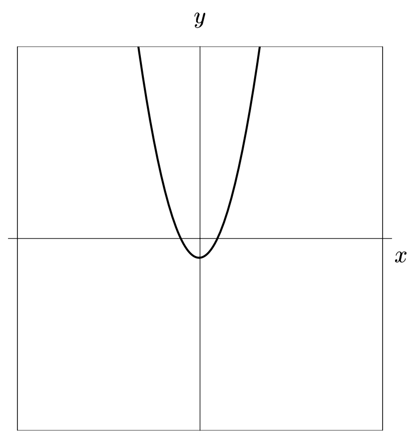


Figure 1.6.23

Is this all there is to the graph? Sketch what you think the complete picture should be. Explain.

ANSWER:

(a) There must be another real root. A cubic polynomial can have one real root, a real root and a double root, or three real roots. Since two roots are shown in the picture, of which neither is a double, there must be another. These are some possibilities:

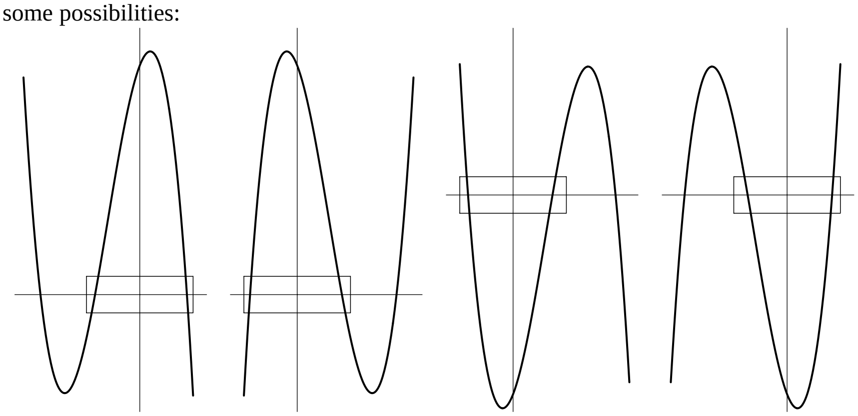


Figure 1.6.24 **Figure 1.6.25** **Figure 1.6.26** **Figure 1.6.27**

(b) As x approaches negative infinity, $e^{0.1x}$ approaches 0, the graph becomes closer and closer to pure parabolic as we move out leftwards from $x = 0$.

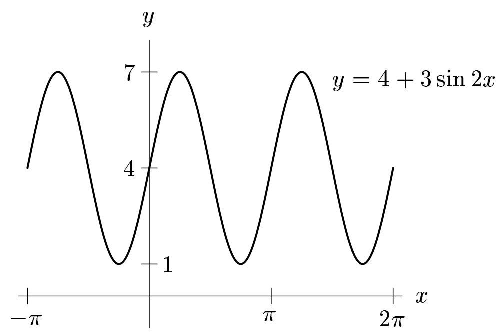
As x approaches infinity, $e^{0.1x}$ approaches infinity. In fact, $e^{0.1x}$ increases so fast with x that it will eclipse the x^2 term. There will be thus a downward turn somewhere to the right of $x = 0$, and the curve will cross the x-axis and head for negative infinity.

8. Give rough sketches of the graphs of the following functions. In each case, give a scale along the x-axis and y-axis.

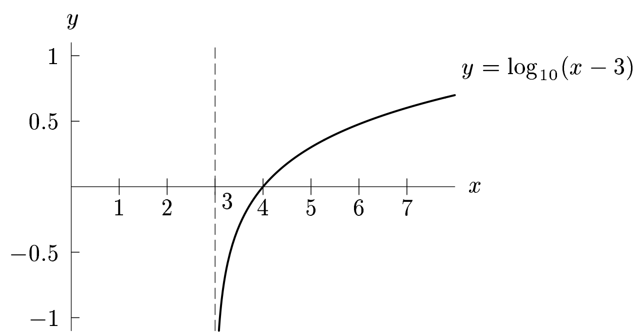
(a) $y = 4 + 3\sin 2x$
(b) $y = \log_{10}(x - 3)$
(c) $y = -5(x + 2)x^2(x - 1)$

ANSWER:

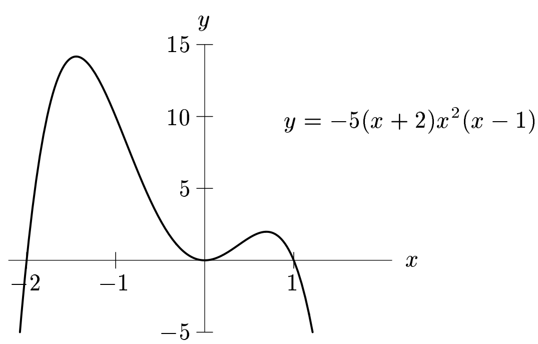
(a)



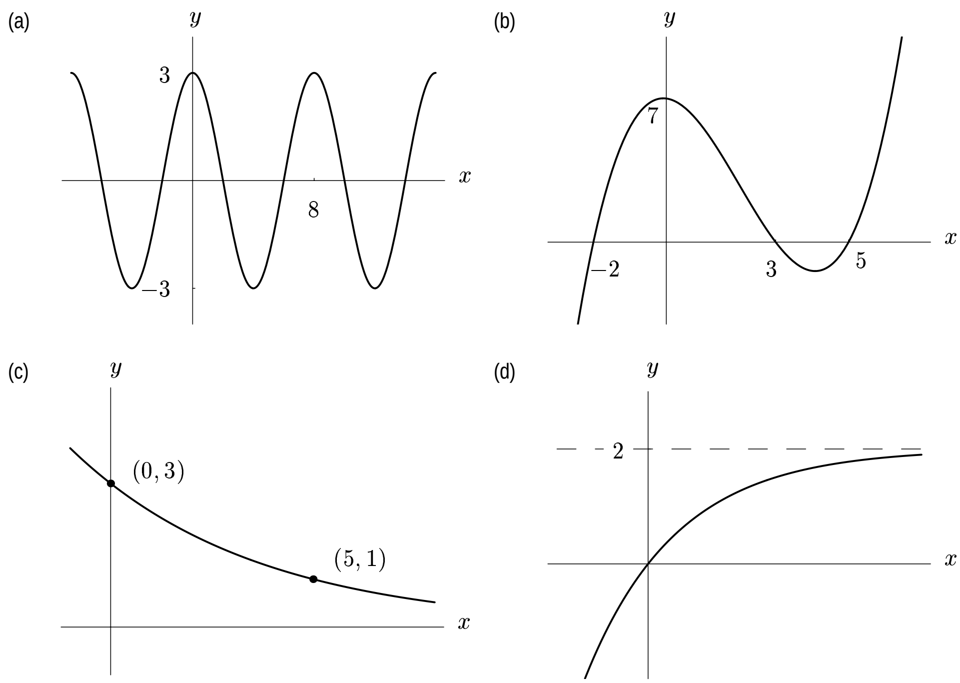
(b)



(c)



9. Give a possible function for each curve.



ANSWER:

(a) This graph is periodic with amplitude 3 and period 8 and has a maximum at $x = 0$. So a reasonable solution is $y = 3\cos\left(\frac{\pi}{4}x\right)$.

(b) This curve appears to be a cubic polynomial with roots at $x = -2, 3$ and 5. Thus $y = k(x+2)(x-3)(x-5)$ is a first guess. Since $y(0) = 7$,

$$\begin{aligned} 7 &= k(2)(-3)(-5) \\ k &= \frac{7}{30} \end{aligned}$$

So, $y = \frac{7}{30}(x+2)(x-3)(x-5)$ is a possible answer.

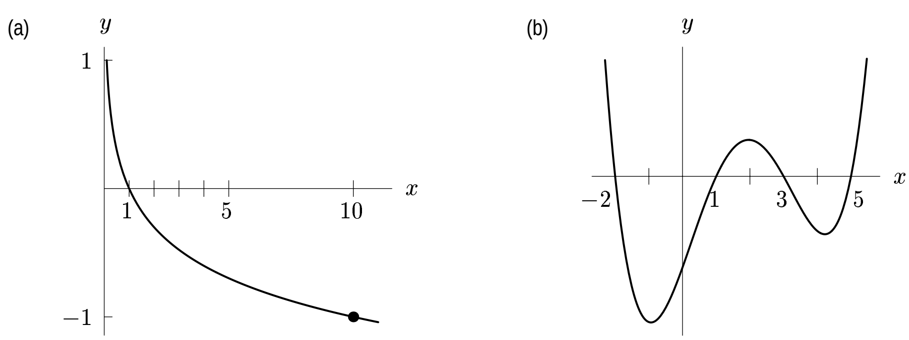
(c) This appears to be an exponential decay curve of the form $y = Ak^{-x}$. Since $y(0) = 3$, $y = 3k^{-x}$. Since $y(5) = 1$, we have

$$\begin{aligned} 1 &= 3k^{-5} \\ k &= \left(\frac{1}{3}\right)^{-\frac{1}{5}} \end{aligned}$$

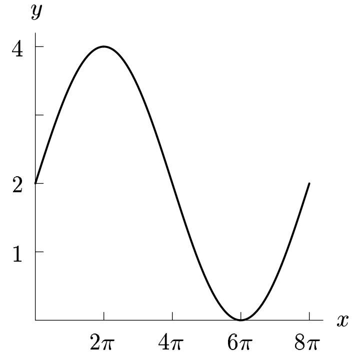
So, $y = 3\left(\frac{1}{3}\right)^{\frac{1}{5}x} = 3^{(1-\frac{x}{5})}$ is a possible answer.

(d) This graph appears to be of the form $y = a(1 - e^{-kx})$. As $x \to \infty$, $y \to a$, and the graph approaches 2, so $y = 2(1 - e^{-kx})$. Any positive k will work, since no scale is indicated for the x-axis.

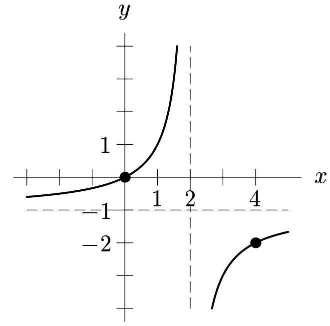
10. For each of the graphs below, find an equation which defines the function. In (D), the numerator is a linear function.



(c)



(d)



ANSWER:

(a) $y = -\log_{10} x$ which is the pH curve from problem 7!

(b) This is a 4-th degree polynomial whose roots are -2, 1, 3, 5. Therefore, $y = K(x+2)(x-1)(x-3)(x-5) = K(x^4 - 7x^3 + 5x^2 + 4x - 30)$, where $K >$ is a constant greater than 0. We don't have enough information to find out anything more.

(c) This is a sine curve with amplitude 2, period 8π and vertical shift 2. Thus $y - 2 = 2\sin\left(\frac{x}{4}\right)$.

(d) Since $x = 2$ is the vertical asymptote, $x - 2$ can be the denominator. We are told the numerator is linear, so $y = \frac{ax+b}{x-2}$.

From the fact that the y-intercept is 0, we have $y = \frac{b}{-2} = 0$, and thus $b = 0$; now $y = \frac{ax}{x-2}$, where "a" is constant. We can find "a" by substitution: $-2 = \frac{4a}{4-2} = \frac{4a}{2} = 2a$, so that $a = -1$. Finally, we have $y = \frac{-x}{x-2}$.

Questions and Solutions for Section 1.7

1. For the following function, find an interval where the function is continuous, and if possible, an interval where the function is not continuous.

$$\frac{1}{\sin x}$$

ANSWER:
Possible answer: continuous on $\pi/4 \le x \le \pi/2$, not continuous on $\pi/2 \le x \le 3\pi/2$.

2. For the following function, find an interval where the function is continuous, and if possible, an interval where the function is not continuous.

$$\frac{1}{\sqrt{x^2-9}}$$

ANSWER:
Possible answer: continuous on $4 \le x \le 9$, not continuous on $0 \le x \le 4$.

3. For the following function, find an interval where the function is continuous, and if possible, an interval where the function is not continuous.

$$\frac{x}{\cos x}$$

ANSWER:
Possible answer: continuous on $-\pi/2 \le x \le \pi/2$, not continuous on $\pi/2 \le x \le \pi$.

4. Find an equation $y = f(x)$ for a function that is *not* continuous at $x = 3$ but is continuous at $x = 0$.
ANSWER:
Many possible answers, for example: $y = \frac{1}{(x-3)^2}$

5. Find an equation for a function that is not continuous at $x = 0$ or at $x = 2$.
   ANSWER:
   Many possible answers, for example: $y = \dfrac{1}{x(x-2)}$
6. $f(x)$ and $g(x)$ are continuous functions. Discuss the continuity of:
   (a) $f(x) + g(x)$
   (b) $3f(x)$
   (c) $f(3x)$
   (d) $g(x) - f(x)$
   (e) $g(x)/f(x)$
   (f) $g(x)/3$
   ANSWER:
   (a) Since $f(x)$ and $g(x)$ are continuous functions, the function you get by adding them is also continuous.
   (b) Since $f(x)$ is continuous, multiplying it by a constant also give a continuous function.
   (c) Since $f(x)$ is continuous, $f(3x)$ is also continuous.
   (d) Since both functions are continuous, the function give by their difference is also continuous.
   (e) The quotient of two continuous functions is not necessarily continuous, therefore we can not determine the continuity of $g(x)/f(x)$.
   (f) Since $g(x)$ is continuous, a constant multiple of $g(x)$ (in this case, 1/3) is also continuous.
7. Sketch the graphs of two different functions that are continuous on $-5 \leq x \leq 5$ and that have the values given in Table 1.7.6. The first function is to have exactly one zero in the interval $[-5, 0]$, the second function is to have at least 3 zeros in the interval $[-1, 5]$.

Table 1.7.6

x	-5	-3	-1	1	3	5
f(x)	4	2	3	0	-1	-1

ANSWER:

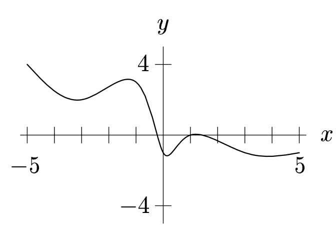


Figure 1.7.28

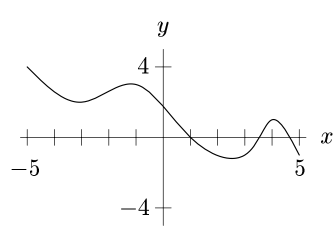


Figure 1.7.29

8. Sketch a graph on the interval $[-5, 5]$ with exactly 2 zeros and at least 2 places where the function is not continuous.
   ANSWER:
   Many possible answers, for example:

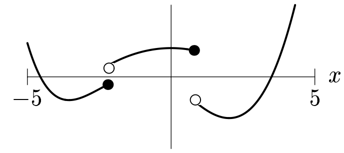


Figure 1.7.30

Review Questions and Solutions for Chapter 1

For Problems 1– 2, decide whether each statement is true or false, and provide a short explanation or a counterexample.

1. The function described by the following table of values is exponential:

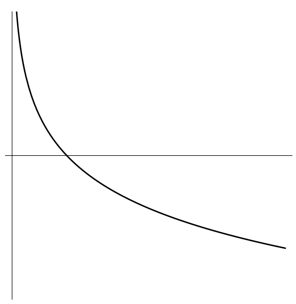
x	5.2	5.3	5.4	5.5	5.6
$f(x)$	27.8	29.2	30.6	32.0	33.4

ANSWER:
FALSE. The function is linear; for every increase of 0.1 in x, there is an increase of 1.4 in $f(x)$.

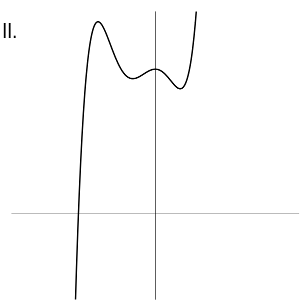
2. A quantity Q growing exponentially according to the formula $Q(t) = Q_0 5^t$ has a doubling time of $\frac{\ln 2}{\ln 5}$.
ANSWER:
TRUE. To calculate the doubling time, T, we use $2Q_0 = Q_0 5^T$ which gives $T = \frac{\ln 2}{\ln 5}$.

3. Match the following graphs with the formulas.

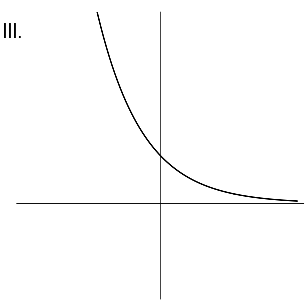
I.



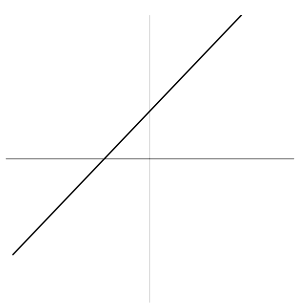
II.



III.



IV.



(a) $\ln(e^x) + 1$
(b) $-2\ln x$
(c) e^{-x}
(d) $x^5 + 2x^4 - x^3 - 2x^2 + 5$
(e) $\dfrac{1}{x+1}$

ANSWER:

I. This curve has the appearance of an upside-down $\ln$ curve and crosses the x-axis at a positive x-value. Thus (b) is the corresponding equation.

II. This curve has four "wiggles" in it and thus looks like it corresponds to a degree 5 polynomial. Hence, (d) is the correct equation.

III. This curve is always positive, decreasing, and concave up. we conclude that (c) is the corresponding equation.

IV. This is a linear function. Equation (a), $\ln(e^x) + 1$ is actually a linear function in disguise, since $\ln(e^x) = x$. Thus, (a) is the correct equation.

V. This graph approaches 0 as $x \to \pm\infty$ and has a vertical asymptote at $x = -1$. Thus, (e) is the correct answer.

4. Give expressions for $f(x), g(x), h(x)$ which agree with the following table of values.

x	$f(x)$	$g(x)$	$h(x)$
0	-7	0	–
1	-4	2	5
2	-1	8	2.50
3	2	18	1.66 . . .
4	5	32	1.25
5	8	50	1

ANSWER:

All the values of $f(x)$ jump by 3 for a change in x of 1, so $f(x)$ is linear with slope 3. Since $f(0) = -7$, we have $f(x) = 3x - 7$.

The differences between successive values of $g(x)$ are as follows: 2, 6, 10, 14, 18., so $g(x)$ is not linear. Notice that the values in the g column, 0, 2, 8, 18 ... are exactly twice the values of the well known function, x^2. So $g(x) = 2x^2$.

$h(x)$ is a decreasing, concave up function that is infinite when $x = 0$ and equal to 1 when $x = 5$. $h(x) = 5/x$ fits the requirements.

5. You are offered two jobs starting on July 1^{st} of 1994. Firm A offers you $\$40{,}000$ a year to start and you can expect an annual raise of 4% every July 1^{st}. At firm B you would start at $\$30{,}000$ but can expect an annual 6% increase every July 1^{st}. On July 1^{st} of which year would the job at firm B first pay more than the job at firm A?

ANSWER:

After n July 1^{st}'s, Firm A pays $40000(1.04)^n$, and Firm B pays $30000(1.06)^n$. So they offer equal salaries when

$$40000(1.04)^n = 30000(1.06)^n,$$

or

$$\frac{(1.06)^n}{(1.04)^n} = \frac{4}{3} \approx 1.333,$$

or

$$\left(\frac{1.06}{1.04}\right)^n = 1.333.$$

But $1.06/1.04 \approx 1.0192$, so $(1.0192)^n = 1.3333$. Taking logs of both sides yields

$$\begin{aligned} n \ln 1.0192 &= \ln 1.3333 \\ n &= \frac{\ln 1.3333}{\ln 1.0192} \approx 15.1. \end{aligned}$$

So when $n = 16$, in the year 2010, Firm B offers more than Firm A.

6. You have \$500 invested in a bank account earning 8.2% compounded annually.

(a) Write an equation for the money M in your account after t years.

(b) How long will it take to triple your money?

(c) Suppose the interest were compounded monthly instead, that is you earned $\frac{8.2}{12}$% interest each month. What interest would you then earn for 1 year?

ANSWER:

(a) $M = 500(1.082)^t$.

(b) To triple your money, set $M = 1500$, so

$$\begin{aligned} \frac{1500}{500} &= (1.082)^t \\ 3 &= e^{(\ln 1.082)t} \\ \ln 3 &= t \ln 1.082 \\ t &= \frac{\ln 3}{\ln 1.082} \\ &\approx 13.9 \text{ years} \end{aligned}$$

(c) If interest is compounded monthly, then we get $M = 500\left(1 + \frac{0.082}{12}\right)^{12t}$, where t is still measured in years. So $M \approx 500(1.0068333)^{12t}$. After $t = 1$ year,

$$\begin{aligned} M &\approx 500(1.0068333)^{12} \\ &\approx 500(1.08516) \\ &\approx 542.58 \end{aligned}$$

so the interest earned is $\$42.58$.

Chapter 2 Exam Questions

Questions and Solutions for Section 2.1

1. For any number r, let $m(r)$ be the slope of the graph of the function $y = (2.1)^x$ at the point $x = r$.

   (a) Complete the table to the right:

r	0	1	2	3	4
$m(r)$			3.27	6.87	14.43

   (b) Explain in a few complete sentences what you did to fill in this table, and why you did it. (If you include pictures, make sure they are carefully labeled.)

   (c) What you have done in part (a) gives you some points on the graph of the function $m(r)$. Graph the points and guess the general shape of the graph of the function $m(r)$ by "fitting a curve" through this data. Give the equation of the curve.

   ANSWER:

   (a)

r	0	1	2	3	4
$m(r)$	0.74	1.56	3.27	6.87	14.43

   (b) The slope of the tangent line to $y = (2.1)^x$ at $x = 1$ is approximately the same as that of the secant line through $x = 1$ and $x = 1.001$:

$$m(1) \approx \frac{(2.1)^{1.001} - (2.1)^1}{0.001} = 1.56.$$

   Similarly,

$$m(0) \approx \frac{(2.1)^{0.001} - (2.1)^0}{0.001} = 0.74.$$

   (c) For comparison, let's tabulate $y = (2.1)^x$ along with m:

r	0	1	2	3	4
$m(r)$	0.74	1.56	3.27	6.87	14.43
$y(r)$	1	2.1	4.41	9.26	19.45
$\frac{m}{y}$	0.74	0.74	0.74	0.74	0.74

   We see that the m values are directly proportional to the y values, with a ratio of approximately 0.74. (Which happens to be $\ln 2.1$.) So $m(r) = \ln 2.1(2.1)^r$.

2. (a) If $x(V) = V^{1/3}$ is the length of the side of a cube in terms of its volume, V, then calculate the average rate of change of x with respect to V over the intervals $0 < V < 1$ and $1 < V < 2$.

   (b) What might we conclude about this rate as the volume V increases? Is it increasing? Decreasing?

   ANSWER:

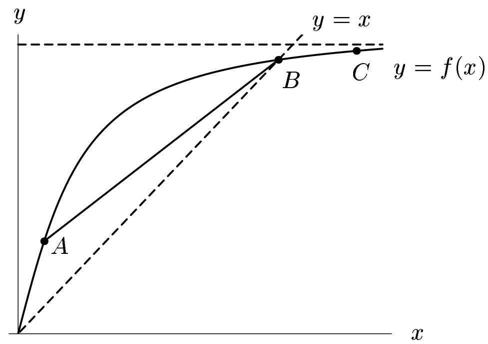
   (a) Average rate of change of $x = \dfrac{1^{1/3} - 0^{1/3}}{1 - 0} = 1$ for $0 < V < 1$.

   average rate of change of $x = \dfrac{2^{1/3} - 1^{1/3}}{2 - 1} \approx 0.26$ for $1 < V < 2$.

   (b) We conclude that as V increases, the rate of change of x decreases.

3. If the graph of $y = f(x)$ is shown below, arrange in ascending order (i.e., smallest first, largest last):

   $f'(A)$ $f'(B)$ $f'(C)$ slope AB the number 1 the number 0



ANSWER:

By eye, we can see that $f'(C) < f'(B) < f'(A)$. We can also see that $f'(B) <$ slope $AB < f'(A)$, so we have $f'(C) < f'(B) <$ slope $AB < f'(A)$. Finally, we note that all the slopes on this graph are positive, and that $f'(A)$ is the only slope that is greater than the slope of $y = x$, namely 1. So we have $0 < f'(C) < f'(B) <$ slope $AB < 1 < f'(A)$.

4. The height of an object in feet above the ground is given in Table 2.1.7.

Table 2.1.7

t (sec)	0	1	2	3	4	5	6
y (feet)	10	45	70	85	90	85	70

(a) Compute the average velocity over the interval $0 \leq t \leq 3$.
(b) Compute the average velocity over the interval $2 \leq t \leq 4$.
(c) If the height of the object is doubled, how do the answers to (a) and (b) change?

ANSWER:

(a) During the interval of $(3-0) = 3$ seconds, the object moves $(85-10) = 75$ feet. The average velocity is $75/3 = 25$ ft/sec.
(b) During the interval of $(4-2) = 2$ seconds, the object moves $(90-70) = 20$ feet. The average velocity is $20/2 = 10$ ft/sec.
(c) If the height is doubled, the average velocity also doubles. The answer to (a) would be 50 ft/sec., and the answer to (b) would be 20 ft/sec.

5. (a) Sketch a graph of height in terms of time for the object in the previous problem over the interval $0 \leq t \leq 6$.
(b) Indicate the average velocity for $0 \leq t \leq 3$ on the graph from (a).

ANSWER:

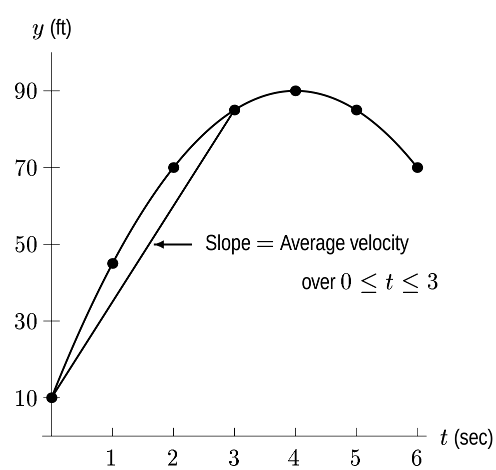


Figure 2.1.31

6. The graph of $p(t)$ in Figure 2.1.32 gives the position of a particle at time t. List the following quantities in order, smallest to largest:

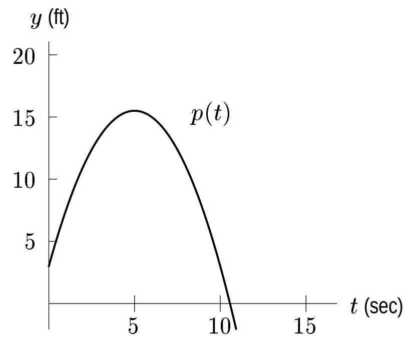


Figure 2.1.32

A, average velocity on $1 \le t \le 3$.

B, average velocity on $8 \le t \le 10$.

C, instantaneous velocity at $t = 1$.

D, instantaneous velocity at $t = 3$.

E, instantaneous velocity at $t = 10$.

ANSWER:

Since $p(t)$ is concave down on $1 \le t \le 3$, the average velocity between the two times should be less than the instantaneous velocity at $t = 1$, but greater than the instantaneous velocity at $t = 3$, so **D** $<$ **A** $<$ **C**. For analogous reasons, **E** $<$ **B**. Finally, note that $p(t)$ is decreasing over the interval $8 \le t \le 10$, but increasing at $t = 0$, so **D** > 0. Therefore, **E** $<$ **B** $<$ **D** $<$ **A** $<$ **C**.

7. The following graph describes the position of a car at time t. Write a short story of the trip that corresponds to the graph. Be sure to discuss the average velocity.

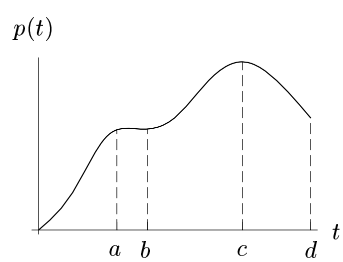


Figure 2.1.33

ANSWER:

Answers will vary. Average velocity on $0 \le t \le a$ is positive, on $a \le t \le b$ is approximately zero, positive on $b \le t \le c$ and negative on $c \le t \le d$. Average velocity on $b \le t \le c$ is smaller than on $0 \le t \le a$.

8. Estimate the limit by substituting smaller and smaller values of h. Give answer to two decimal places. For trigonometric functions, use radians.

(a) $\lim_{h \to 0} \frac{(2+h)^2 - 4}{h}$

(b) $\lim_{h \to 0} \frac{\sin(h^2)}{h}$

ANSWER:

(a) Using $h = 0.1,\ 0.01,\ 0.001$, we see

$$\frac{(2+0.1)^2-4}{0.1} = 4.1$$

$$\frac{(2+0.01)^2-4}{0.01} = 4.01$$

$$\frac{(2+0.001)^2-4}{0.001} = 4.001$$

These calculations suggest $\lim_{h\to 0} \frac{(2+h)^2-4}{h} \approx 4$.

(b) Using radians,

Table 2.1.8

h	$(\sin(h^2))/h$
0.01	.00999
0.001	.00099
0.0001	.0001

These calculations suggest $\lim_{h\to 0} \frac{1-\sin(h)}{h} = 0$.

9. Estimate the limit by substituting smaller and smaller values of h. Give answer to three decimal places.

(a) $\lim_{h\to 0} \frac{3^h-1}{h}$

(b) $\lim_{h\to 0} \frac{e^{1+h}-e}{2h}$

ANSWER:

(a) Using $h = 0.1,\ 0.01,\ 0.001$, we see

$$\frac{3^{0.1}-1}{0.1} = 1.161$$

$$\frac{3^{0.01}-1}{0.01} = 1.1046$$

$$\frac{3^{0.001}-1}{0.001} = 1.0992$$

These calculations suggest $\lim_{h\to 0} \frac{3^h-1}{h} \approx 1.099$.

(b) Using $h = 0.1,\ 0.01,\ 0.001$, we see

Table 2.1.9

h	$(e^{1+h}-e)/2h$
0.1	1.429
0.01	1.366
0.001	1.359

These calculations suggest $\lim_{h\to 0} \frac{e^{1+h}-e}{2h} \approx 1.359$

Questions and Solutions for Section 2.2

1. Sketch a graph $y = f(x)$ with the following limits:

$$\lim_{x\to -3} f(x) = -1$$
$$\lim_{x\to 0} f(x) = 5$$
$$\lim_{x\to 2} f(x) = 1$$

ANSWER:
Answers will vary. One example:

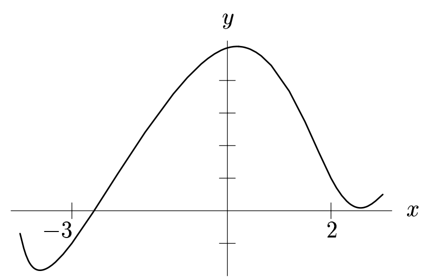


Figure 2.2.34

2. Use Figure 2.2.35 to give approximate values for the following limits (if they exist).

(a) $\lim_{x\to -4} f(x)$
(b) $\lim_{x\to -1} f(x)$
(c) $\lim_{x\to 2} f(x)$
(d) $\lim_{x\to 6} f(x)$

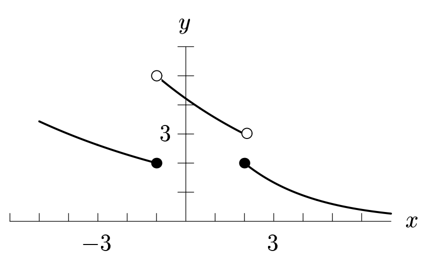


Figure 2.2.35

ANSWER:

(a) As x approaches -2 from either side, the values of $f(x)$ get closer and closer to 3, so the limit appears to be about 3.
(b) As x approaches -1 from the left, $f(x)$ gets closer to 2. As x approaches -1 from the right, $f(x)$ gets closer to 5. Thus, the limit does not exist.
(c) As in (b), the limit does not exist.
(d) As x approaches 6 from both sides, $f(x)$ appears to be approximately 0.5, thus the limit is about 0.5.

3. Use algebra to evaluate the limits.

(a) $\lim_{h\to 0} \dfrac{(3+h)^2 - 9}{h}$
(b) $\lim_{h\to 0} \dfrac{\sqrt{9-h} - 3}{h}$

ANSWER:

(a) $\lim_{h\to 0}\frac{(3+h)^2-9}{h}=\lim_{h\to 0}\frac{9+6h+h^2-9}{h}=\lim_{h\to 0}\frac{6h+h^2}{h}=6$

(b) $\sqrt{9-h}-3=\frac{(\sqrt{9-h}-3)(\sqrt{9-h}+3)}{\sqrt{9-h}+3}=\frac{9-h-9}{\sqrt{9-h}+3}=\frac{-h}{\sqrt{9-h}+3}$

Therefore, $\lim_{h\to 0}\frac{\sqrt{9-h}-3}{h}=\lim_{h\to 0}\frac{-1}{\sqrt{9-h}+3}=-\frac{1}{6}$.

4. Does $\lim_{x\to -4}\frac{|x+4|}{x+4}$ exist? Explain why or why not.

ANSWER:

The right-hand limit and the left-hand limit are different. For $x>-4$, we have $|x+4|=x+4$, so as x approaches -4 from the right,

$$\lim_{x\to -4^+}\frac{|x+4|}{x+4}=\lim_{x\to -4^+}\frac{x+4}{x+4}=\lim_{x\to -4^+}1=1$$

Similarly, if $x<-4$, then $|x+4|=-x-4$, so

$$\lim_{x\to -4^-}\frac{|x+4|}{x+4}=\lim_{x\to -4^-}\frac{-x-4}{x+4}=\lim_{x\to -4^-}(-1)=-1$$

Since the limits from both sides are not the same, $\lim_{x\to -4}\frac{|x+4|}{x+4}$ does not exist.

5. Use algebra to find a maximum distance between x and 3 which ensures that x^2+1 is within 0.1 of 10. Use a similar argument to show that $\lim_{x\to 3}x^2+1=10$.

ANSWER:

We write $x=3+h$. We want to find the value of h making x^2+1 within 0.1 of 10. We know

$$x^2+1=(3+h)^2+1=9+6h+h^2+1=10+6h+h^2,$$

so x^2+1 differs from 10 by $6h+h^2$. Since we want x^2+1 to be within 0.1 of 10, we need

$$|x^2+1-10|=|6h+h^2|=|h||6+h|<0.1$$

Assuming $0<|h|<1$, we know $|6+h|<7$, so that

$$|x^2-9|<7|h|$$

Hence we want $7|h|<0.1$

Thus, if we choose h such that $0<|h|<0.1/7=0.014$, then x^2+1 is less than 0.1 from 10.

An analogous argument using any small ϵ instead of 0.1 shows that if we take $\delta=\epsilon/7$, then

$$|(x^2+1)-10|<\epsilon \text{ for all } |\mathrm{x}-3|<\epsilon/7.$$

Therefore, $\lim_{x\to 3}x^2+1=10$.

6. By referring to the properties of limits, give justifications for each step in the following limit calculation:

(a)

$$\lim_{x\to 2}\frac{2x^2+6x}{3x}=\frac{\lim_{x\to 2}(2x^2+6x)}{\lim_{x\to 2}(3x)}$$

(b)

$$=\frac{2\cdot\lim_{x\to 2}(x^2+3x)}{3\cdot\lim_{x\to 2}(x)}$$

(c)

$$=\frac{2\cdot\lim_{x\to 2}(x)\cdot\lim_{x\to 2}(x+3)}{3\cdot\lim_{x\to 2}(x)}$$

(d)

$$=\frac{2\cdot 2\cdot(2+3)}{3\cdot 2}=\frac{10}{3}$$

ANSWER:

(a) $\lim_{x\to c}\dfrac{f(x)}{g(x)}=\dfrac{\lim_{x\to c}f(x)}{\lim_{x\to c}g(x)}$, provided $\lim_{x\to c}g(x)\neq 0$ and in this case $\lim_{x\to 2}3x=6\neq 0$ so the property applies.

(b) If b is a constant, then $\lim_{x\to c}(b(f(x))=b\left(\lim_{x\to c}f(x)\right)$.

(c) $\lim_{x\to c}(f(x)g(x))=\left(\lim_{x\to c}f(x)\right)\left(\lim_{x\to c}g(x)\right)$.

(d) For any constant k, $\lim_{x\to c}k=k$ and $\lim_{x\to c}x=c$.

7. Use algebra to evaluate the left- and right-hand limits of $f(x)=\dfrac{|x-5|}{3x-15}$ at $x=5$.

ANSWER:

$$f(x)=\frac{|x-5|}{3x-15}=\begin{cases}\frac{x-5}{3x-15}, & x>5\\ \frac{x-5}{3x-15}, & x<5\end{cases}=\begin{cases}\frac{1}{3}, & x>5\\ -\frac{1}{3}, & x<5\end{cases}$$

$$\lim_{x\to 5^+}f(x)=\frac{1}{3}\text{ and }\lim_{x\to 5^-}\text{f(x)}=-\frac{1}{3}.$$

8. Use Figure 2.2.36 to give appropriate values for the following limits (if they exist).

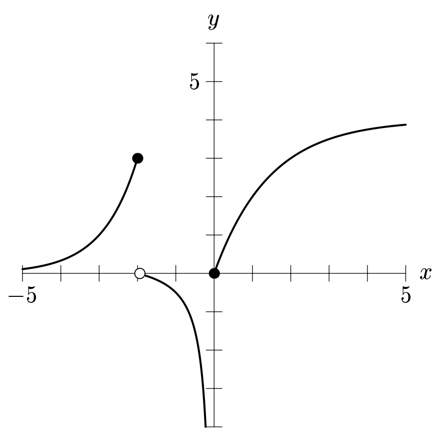


Figure 2.2.36

(a) $\lim_{x\to -2^+}f(x)$
(b) $\lim_{x\to -\infty}f(x)$
(c) $\lim_{x\to 0}f(x)$

ANSWER:

(a) $\lim_{x\to -2^+}f(x)=0$
(b) $\lim_{x\to -\infty}f(x)=0$
(c) $\lim_{x\to 0}f(x)$ does not exist.

9. If $f(x)=x^2-4$ and $g(x)=3x-5$, evaluate the following limits if they exist. If they do not exist, explain why.

(a) $\lim_{x\to 0}f(x)+g(x)$
(b) $\lim_{x\to 2}f(x)/g(x)$
(c) $\lim_{x\to 2}g(x)/f(x)$
(d) $\lim_{x\to 3}(f(x)+2g(x))$

ANSWER:

(a) $\lim_{x\to 0}f(x)+g(x)=\lim_{x\to 0}(x^2-4+3x-5)=\lim_{x\to 0}(x^2+3x-9)=-9$.

(b) $\lim_{x\to 2} f(x)/g(x) = \lim_{x\to 2} \frac{x^2-4}{3x-5} = \frac{2^2-4}{3(2)-5} = 0.$

(c) $\lim_{x\to 2} g(x)/f(x) = \lim_{x\to 2} \frac{3x-5}{x^2-4}$. The limit does not exist because $f(2) = 0$.

(d) $\lim_{x\to 3} (f(x)+2g(x)) = \lim_{x\to 3} (x^2-4+2(3x-5)) = \lim_{x\to 3} (x^2-4+6x-10) = \lim_{x\to 3} (x^2+6x-14) = (3^2+6(3)-14) = 13.$

Questions and Solutions for Section 2.3

1. Given the following data about a function, f,

x	3	3.5	4	4.5	5	5.5	6
$f(x)$	10	8	7	4	2	0	-1

(a) Estimate $f'(4.25)$ and $f'(4.75)$.
(b) Estimate the rate of change of f' at $x = 4.5$.
(c) Find, approximately, an equation of the tangent line at $x = 4.5$.
(d) Use the tangent line to estimate $f(4.75)$.
(e) Estimate the derivative of f^{-1} at 2.

ANSWER:

(a) $f'(4.25) \approx \frac{f(4.5)-f(4)}{4.5-4} = \frac{4-7}{0.5} = -6$
$f'(4.75) \approx \frac{f(5)-f(4.5)}{5-4.5} = \frac{2-4}{0.5} = -4$
(b) $f''(4.5) \approx \frac{f'(4.75)-f'(4.25)}{0.5} = \frac{-4+6}{0.5} = 4$
(c) $f'(4.5) \approx \frac{f(5)-f(4.5)}{0.5} = -4$, thus $y-4 = -4(x-4.5)$ is the equation of the tangent line.
(d) $f(4.75) \approx f(4.5) + .25 \cdot f'(4.5) \approx 3$
(e) $(f^{-1}(2))' \approx \frac{f^{-1}(4)-f^{-1}(2)}{4-2} = \frac{4.5-5}{2} = -\frac{1}{4}$.

2. (a) Explain how the average rate of change of a function f can be used to find the instantaneous rate of change of f at a point x_0.
(b) Give a geometric interpretation of the instantaneous rate of change.

ANSWER:

(a) By taking points $x_1, x_2, \ldots$ closer and closer to x_0 and calculating the average rate of change of f over the interval $[x_0, x_n]$, we get a sequence which approaches the instantaneous rate of change of f at x_0.
(b) The instantaneous rate of change at a given point is the slope of a tangent to the curve at that point.

3. (a) Estimate $f'(0)$ when $f(x) = 2^{-x}$.
(b) Will your estimate be larger or smaller than $f'(0)$? Explain.

ANSWER:

(a) To estimate $f'(0)$, find the average slope over intervals that get smaller and smaller but still contain $x = 0$:

Interval Size	Average Slope
0.1	$\frac{f(0.1)-f(0)}{0.1} \approx -0.670$
0.01	$\frac{f(0.01)-f(0)}{0.01} \approx -0.691$
0.001	$\frac{f(0.001)-f(0)}{0.001} \approx -0.693$

$f'(0)$ appears to be about -0.693.

(b) The average slopes in the chart above seem to approach a limiting value (which turns out to be $\ln\frac{1}{2} \approx -0.69315$) from above; this indicates that our estimate of $f'(0)$ is probably an overestimate.

4. Given the following data about a function f,

x	3.0	3.2	3.4	3.6	3.8
$f(x)$	8.2	9.5	10.5	11.0	13.2

(a) Estimate $f'(3.2)$ and $f'(3.5)$.
(b) Give the average rate of change of f between $x = 3.0$ and $x = 3.8$.
(c) Give the equation of the tangent line at $x = 3.2$.

ANSWER:

(a) Estimate the slope at 3.2 by finding the average slope over the interval $[3.2, 3.4]$:

$$\text{Slope } = \frac{f(3.4) - f(3.2)}{3.4 - 3.2} = \frac{10.5 - 9.5}{0.2} = 5$$

To estimate the slope at 3.5, we have to look at the average slope over $[3.4, 3.6]$, which is $\dfrac{11.0 - 10.5}{0.2} = 2.5$.

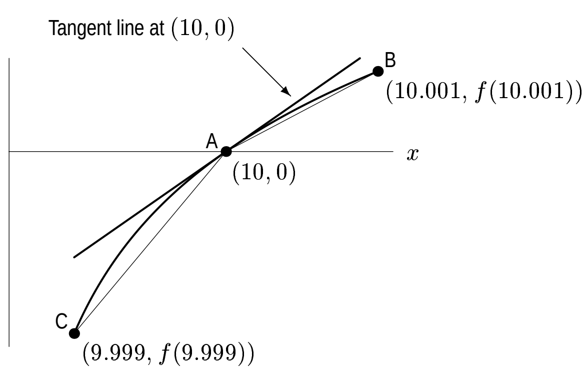
(b) The average rate of change is $\frac{13.2-8.2}{3.8-3.0} = 6.25$.

(c) At $x = 3.2$, $f(x) = 9.5$ and the slope ≈ 5 by part (a). So

$$\begin{aligned} y - 9.5 &= 5(x - 3.2) \\ y &= 9.5 + 5x - 16 \\ &= 5x - 6.5 \end{aligned}$$

5. Let $f(x) = \log(\log(x))$. (This is the "common log" which uses "base 10.") Our goal is to approximate the derivative of this function at the point $x = 10$. Give an UPPER BOUND (call it "U") and a LOWER BOUND (call it "L") for $f'(10)$ that agree up to three decimal places. Explain how you know that U is an upper bound for $f'(10)$ and that L is a lower bound for $f'(10)$. Include a sketch to explain your reasoning. You may assume the graph of $y = f(x)$ is concave down. Note: An "upper bound" for $f'(10)$ is simply a number which is larger than $f'(10)$.

ANSWER:



Note: The figure is not drawn to scale.

Since the graph of f is concave down, the slope of the line joining points C and A will be larger than the slope of the tangent line at point A and similarly, the slope of the line joining the points A and B will be smaller than the slope of the tangent line at A. Since the slope of the tangent line at A is $f'(10)$, we get:

$$\underbrace{\frac{\log(\log(10.001)) - \overbrace{\log(\log(10))}^{0}}{0.001}}_{L} < f'(10) < \frac{\overbrace{\log(\log(10))}^{0} - \log(\log(9.999))}{0.001} = U$$

$L \approx \underline{0.01885}98173 \qquad U \approx .\underline{0.01886}25224$

After rounding, they agree to 0.01886, which is better than the required accuracy.

6. For $f(x) = \log x$, estimate $f'(1)$ by finding the average slope over intervals which get smaller and smaller but still contain the value $x = 1$.

ANSWER:

Taking average slopes over smaller and smaller intervals to the right of $x = 1$, we obtain the following table:

Interval Length	Average Slope
0.1	$\dfrac{f(1.1) - f(1)}{0.1} \approx 0.414$
0.01	$\dfrac{f(1.01) - f(1)}{0.01} \approx 0.432$
0.001	$\dfrac{f(1.001) - f(1)}{0.001} \approx 0.434$
0.0001	$\dfrac{f(1.0001) - f(1)}{0.0001} \approx 0.434$

Thus, $f'(1)$ appears to be about 0.434.

7. There is a function used by statisticians, called the error function, which is written

$$y = \text{erf}(x).$$

Suppose you have a statistical calculator, which has a button for this function. Playing with your calculator, you discover the following:

x	$\text{erf}(x)$
1	0.29793972
0.1	0.03976165
0.01	0.00398929
0	0

(a) Using this information alone, give an estimate for $\text{erf}'(0)$, the derivative of erf at $x = 0$. Only give as many decimal places as you feel reasonably sure of, and explain why you gave that many decimal places.

(b) Suppose that you go back to your calculator and find that

$$\text{erf}(0.001) = 0.000398942.$$

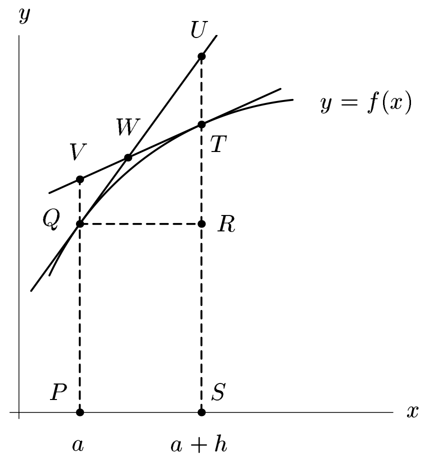
With this extra information, would you refine the answer you gave in (a)? Explain.

ANSWER:

(a) Since $\text{erf}'(0) = \lim_{h\to 0} \dfrac{\text{erf}(h) - \text{erf}(0)}{h - 0} = \lim_{h\to 0} \dfrac{\text{erf}(h)}{h}$, we approximate $\text{erf}'(0)$ by $\dfrac{\text{erf}(h)}{h}$ where h is small. As $\dfrac{\text{erf}(0.1)}{0.1}$ and $\dfrac{\text{erf}(0.01)}{0.01}$ agree in the first two decimal places, it seems safe to estimate $\text{erf}'(0) = 0.39$.

(b) The new value for $\text{erf}(0.001)$ gives us agreement out to four decimal places between $\dfrac{\text{erf}(0.01)}{0.01}$ and $\dfrac{\text{erf}(0.001)}{0.001}$, so we can refine our answer to 0.3989.

8. Each of the quantities below can be represented in the picture. For each quantity, state whether it is represented by a length, a slope or an area. Then using the letters on the picture, make clear exactly which length, slope or area represents it. [Note: The letters P, Q, R, etc., represent points.]



(a) $f(a+h) - f(a)$
(b) $f'(a+h)$
(c) $f'(a)h$
(d) $f(a)h$

ANSWER:

(a) $f(a+h) - f(a)$ is represented by the length TR.
(b) $f'(a+h)$ is the slope of the line TV.
(c) $f'(a)h$ is the length RU.
(d) $f(a)h$ is the area of the rectangle $PQRS$.

9. Given the following table of values for a Bessel function, $J_0(x)$, what is your best estimate for the derivative at $x = 0.5$?

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$J_0(x)$	1.0	.9975	.9900	.9776	.9604	.9385	.9120	.8812	.8463	.8075	.7652

ANSWER:
We can approximate $J_0'(0.5)$ by the difference quotient with $h = 0.1$ to the right of 0.5:.

$$J_0'(0.5) \approx \frac{J_0(0.6) - J_0(0.5)}{0.1} = \frac{0.9120 - 0.9385}{0.1} = -0.265.$$

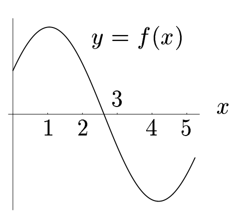
However, to obtain a better approximation, we approximate $J_0'(0.5)$ by the average of the difference quotients with $h = 0.1$ to the left and right of 0.5.

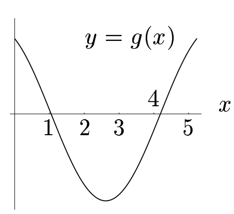
$$J_0'(0.5) \approx \frac{1}{2}\left(\frac{J_0(0.5) - J_0(0.4)}{0.1} + \frac{J_0(0.6) - J_0(0.5)}{0.1}\right)$$

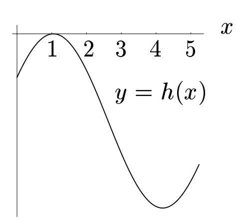
$$= \frac{1}{2}\left(\frac{0.9385 - 0.9604}{0.1} + \frac{0.9120 - 0.9385}{0.1}\right) = -0.242.$$

Questions and Solutions for Section 2.4

1. Which of the functions below could be the derivative of which of the others? (Hint: try all combinations.)

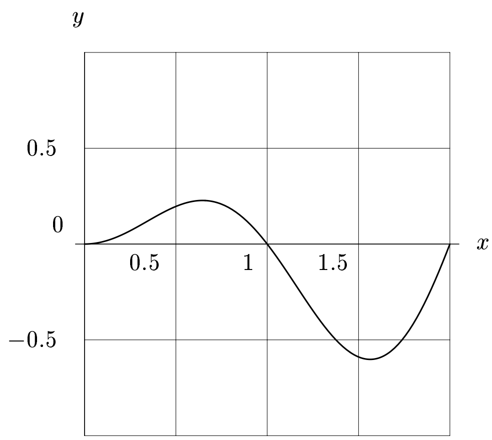




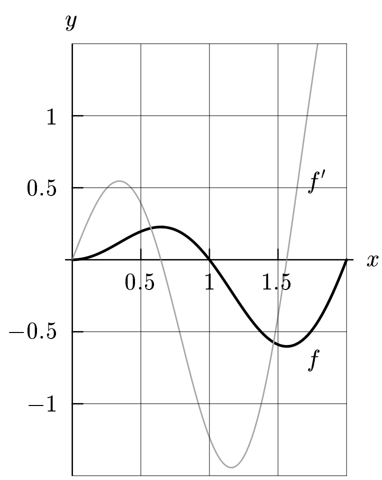


ANSWER:
$g(x)$ could be the derivative of $h(x)$ or $f(x)$

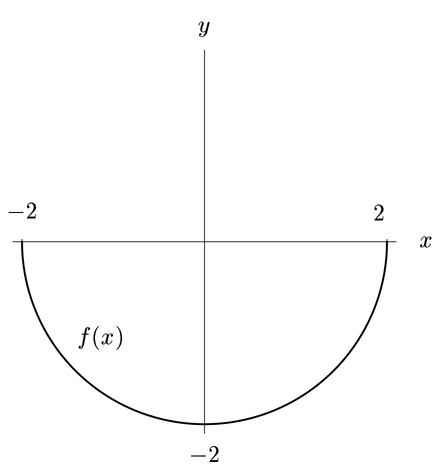
2. Below is the graph of a function f. Sketch the graph of its derivative f' on the same axes.



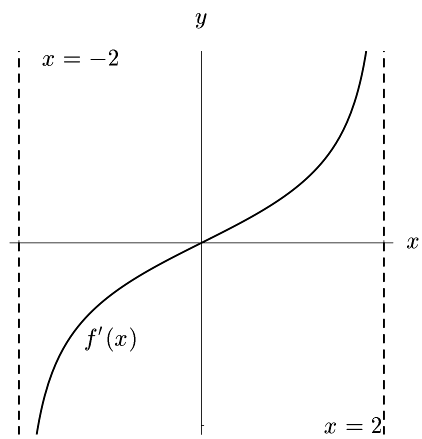
ANSWER:



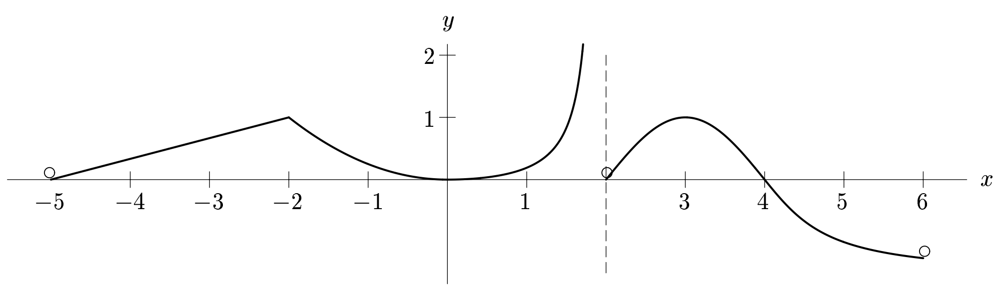
3. Sketch the graph of the derivative, $y = f'(x)$, for each of the functions $y = f(x)$ whose graphs are given below.



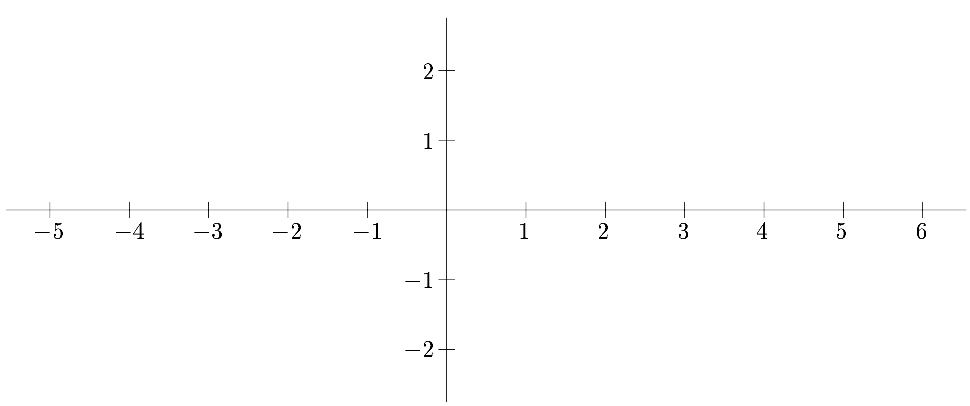
ANSWER:



4. Consider the function $y = f(x)$ graphed below. (Notice that $f(x)$ is defined for $-5 < x < 6$, *except* $x = 2$.)



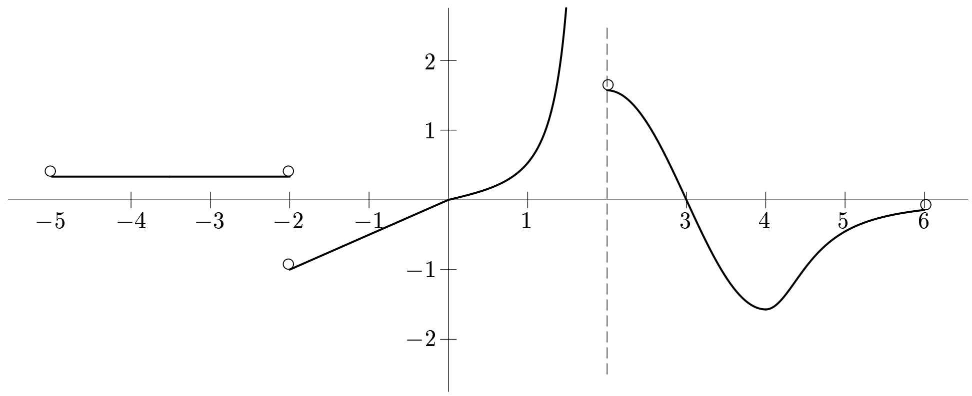
(a) For what values of x (in the domain of f) is $f'(x) = 0$?
(b) For what values of x (in the domain of f) is $f'(x)$ positive?
(c) For what values of x (in the domain of f) is $f'(x)$ negative?
(d) For what values of x (in the domain of f) is $f'(x)$ undefined?
(e) Based on your answers to the above questions, make a sketch of $y = f'(x)$ on the axes below. Make your sketch as precise as possible.



ANSWER:

(a) $x = 0, 3$
(b) $x \in (-5, -2), \quad x \in (0, 2), \quad x = (2, 3)$

(c) $x \in (-2, 0), \quad x \in (3, 6)$
(d) $x = -2$
(e)



5. Estimate the value of $f'(x)$ for the function $f(x) = 10^x$.
ANSWER:

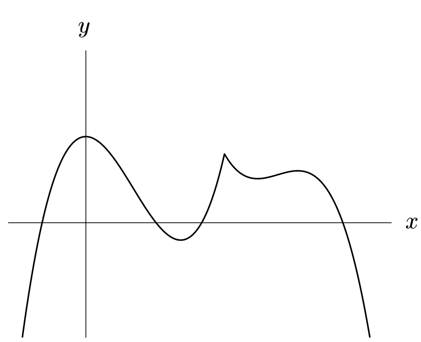
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{10^{x+h} - 10^x}{h} = 10^x \lim_{h\to 0} \frac{10^h - 1}{h}.$$

So far, our calculation is exact. We now estimate the limit by substituting small values of h;

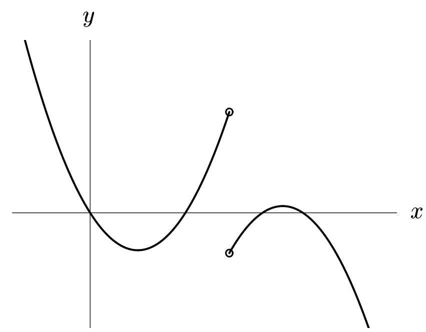
h	$\frac{10^h-1}{h}$
1	9
0.1	2.589
0.01	2.329
0.001	2.305
0.0001	2.303
0.00001	2.303

So $f'(x)$ appears to be approximately equal to $(2.303)10^x$.

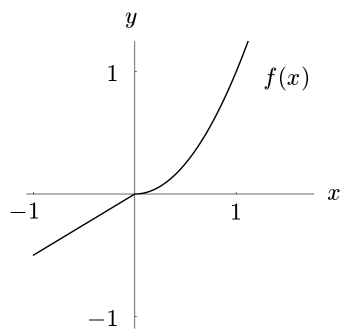
6. Sketch the graph of the derivative of the function whose graph is shown:



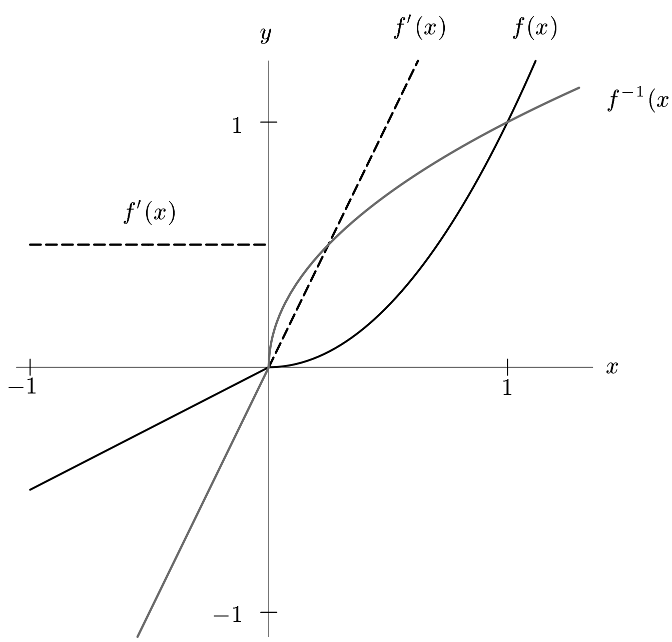
ANSWER:



7. The following graph is of $y = f(x)$. Draw $f'(x)$ and $f^{-1}(x)$ on the same axes.



ANSWER:



8. The graph of $f(x)$ is given in Figure 2.4.37.
   (a) Sketch the graph of $f'(x)$ on the same axes.
   (b) Where does $f'(x)$ change its sign?
   (c) Where does $f'(x)$ have a local maximum or minimum?

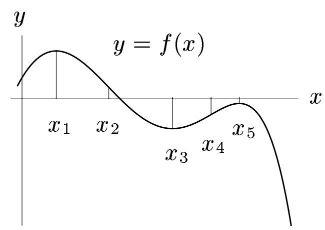
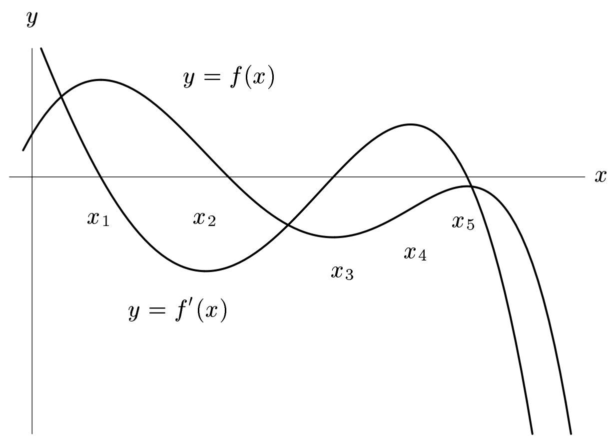


Figure 2.4.37

ANSWER:

(a)



(b) At x_1, x_3, x_5.
(c) At x_2, x_4.

9. Find the derivative of the following functions algebraically:
   (a) $g(x) = 3x^2 + 2x - 4$ at $x = 3$.
   (b) $m(x) = 2x^3$ at $x = 2$.
   (c) $p(x) = g(x) \cdot m(x)$ at $x = 2$.

   ANSWER:

   (a) Using the definition of the derivative, we have

$$\begin{aligned} g'(3) &= \lim_{h\to 0} \frac{g(3+h) - g(3)}{h} \\ &= \lim_{h\to 0} \frac{(3(3+h)^2 + 2(3+h) - 4) - (3(3)^2 + 2(3) - 4)}{h} \\ &= \lim_{h\to 0} \frac{(3h^2 + 20h + 29) - (29)}{h} \\ &= \lim_{h\to 0} \frac{3h^2 + 20h}{h} = \lim_{h\to 0} 3h + 20 = 20. \end{aligned}$$

   So $g'(3) = 20$.

   (b) Using the definition of the derivative, we have

$$\begin{aligned} m'(2) &= \lim_{h\to 0} \frac{2(h+2)^3 - 2h^3}{h} \\ &= \lim_{h\to 0} \frac{4h^3 + 12h^2 + 24h}{h} \\ &= \lim_{h\to 0} 4h^2 + 12h + 24 = 24. \end{aligned}$$

   So $m'(2) = 24$.

(c) $p(x) = (3x^2 + 2x - 4)(2x^3) = 6x^5 + 4x^4 - 8x^3$

$$p'(2) = \lim_{h \to 0} \frac{(6(h+2)^5 + 4(h+2)^4 - 8(h+2)^3) - (6(2)^5 + 4(2)^4 - 8(2)^3)}{h}$$
$$= \lim_{h \to 0} \frac{6h^5 + 64h^4 + 264h^3 + 528h^2 + 512h}{h} = 512.$$

So $p'(2) = 512$.

10. Draw the graph of a continuous function $y = g(x)$ that satisfies the following three conditions:
- $g'(x) = 0$ for $x < 0$
- $g'(x) > 0$ for $0 < x < 2$
- $g'(x) < 0$ for $x > 2$

ANSWER:

From the given information, we know that g is constant for $x < 0$, is increasing between $x = 0$ and $x = 2$, and is decreasing for $x > 2$. Figure 2.4.38 shows a possible graph—answers may vary.

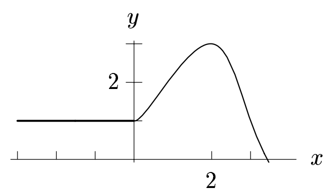


Figure 2.4.38

Questions and Solutions for Section 2.5

1. Suppose that $f(T)$ is the cost to heat my house, in dollars per day, when the outside temperature is T degrees Fahrenheit.
   (a) What does $f'(23) = -0.17$ mean?
   (b) If $f(23) = 7.54$ and $f'(23) = -0.17$, approximately what is the cost to heat my house when the outside temperature is 20°F?

   ANSWER:

   (a) $f'(23) = -0.17$ means that when the temperature outside is 23 degrees, the cost of heating the house will decrease by a rate of approximately 17 cents per day for each degree above 23. Since we know nothing about how $f(T)$ behaves at temperatures other than $T = 23$, it is impossible to know over which range of temperatures this approximation is valid. It seems reasonable to assume, however, that $f(T)$ will be relatively smooth over a range of a few degrees.
   (b) If the temperature goes down by 3° (i.e., to 20°), then the cost will increase by about $(-3)(-0.17) = 0.51$, resulting in a cost of $\$7.54 + \$0.51 = \$8.05$.

2. To study traffic flow along a major road, the city installs a device at the edge of the road at 4:00 a.m. The device counts the cars driving past, and records the total periodically. The resulting data is plotted on a graph, with time (in hours) on the horizontal axis and the number of cars on the vertical axis. The graph is shown below; it is the graph of the function

   $C(t)$ = Total number of cars that have passed by after t hours.

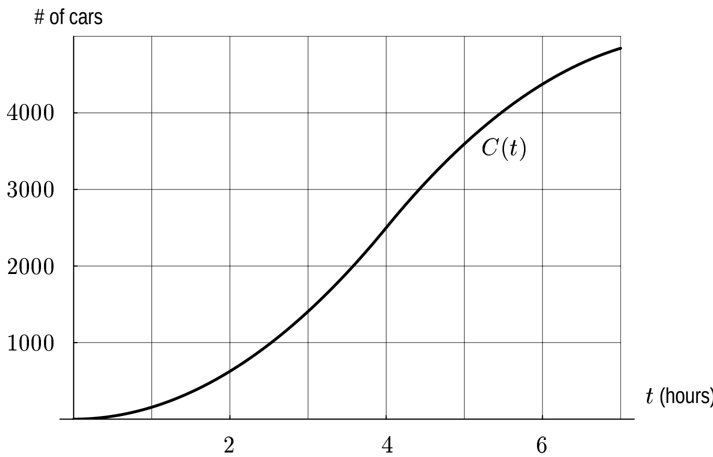


Figure 2.5.39: Traffic Along Speedway

(a) When is the traffic flow greatest?
(b) From the graph, estimate $C'(3)$.
(c) What is the meaning of $C'(3)$? What are its units? What does the value of $C'(3)$ you obtained in (b) mean in practical terms?

ANSWER:

(a) Traffic flow is greatest when the slope of $C(t)$ is greatest, which occurs at about $t = 4$. Since t is in hours past 4:00 a.m., the flow is greatest at about 8:00 a.m.
(b) $C'(3) \approx 1000$
(c) $C'(3)$ tells us how many cars per hour are flowing past at 7:00 a.m. The value we obtained above for $C'(3)$ tells us that traffic flow at that time is about 1000 cars per hour.

3. Every day the Office of Undergraduate Admissions receives inquiries from eager high school students (e.g. “Please, please send me an application”, etc.) They keep a running account of the number of inquiries received each day, along with the total number received until that point. To the right is a table of *weekly* figures from about the end of August to about the end of October of a recent year.

Week of	Inquiries That Week	Total for Year
8/28–9/01	1085	11,928
9/04–9/08	1193	13,121
9/11–9/15	1312	14,433
9/18–9/22	1443	15,876
9/25–9/29	1588	17,464
10/02–10/06	1746	19,210
10/09–10/13	1921	21,131
10/16–10/20	2113	23,244
10/23–10/27	2325	25,569

(a) One of these columns can be interpreted as a rate of change. Which one? Of what? Explain.
(b) Based on the table write a formula that gives approximately the total number of inquiries received by a given week. Explain.
(c) Using your answer in part (b), roughly how many inquiries will the admissions office receive over the entire year?
(d) The actual number of inquiries that year was about 34,000. Discuss this, using your knowledge of how people apply to college.

ANSWER:

(a) The second column – Inquiries That Week – is the weekly rate of change of the total for the year since, for example, $13{,}121 - 11{,}928 = 1193$; we see that 1193 is the difference between the total number of inquiries as of 9/04 and 9/11.
(b) We have that the ratio of consecutive entries in the second column (total applicants for the year) is always about 1.1. So if T is the total number of applicants then we can try the exponential model

$$T(t) = (11{,}928)(1.1)^t$$

with $t = 0$ corresponding to the week of $8/28$ to $9/01$.

(c) There are about 18 weeks from the start ($t = 0$) until the end of the year. Putting $t = 18$ into the formula for T above gives $T = 66{,}319$ for the total number of applications for the year.

(d) Since most students send for applications in October and November to apply by the first of January, requests should fall off in November, not continue to rise as our formula suggests. So the true figure ($34{,}000$) should be much less than the calculated figure ($66{,}319$).

4. Let $t(h)$ be the temperature in degrees Celsius at a height h (in meters) above the surface of the earth. What do each of the following quantities mean to a sky diver? Give units for the quantities.

(a) $t(1000)$
(b) $t'(20)$
(c) $t(h) + 20$
(d) h such that $t'(h) = 20$

ANSWER:

(a) The temperature in degrees Celsius at a height of 1000 meters.
(b) The rate of change of temperature with respect to height at 20 meters above the surface of the earth, in units of degrees per meter.
(c) The temperature at height h plus a temperature of 20°C.
(d) The height, in meters, at which the rate of change of temperature with respect to height is 20 degrees per meter.

5. Let $L(r)$ be the amount of lumber, in board-feet, produced from a tree of radius r (measured in inches). Interpret the following in practical terms, giving units.

(a) $L(6)$
(b) $L'(24)$
(c) r such that $L(r) = 100$
(d) r such that $L'(r) = 10$

ANSWER:

(a) The number of board-feet of lumber obtained from a tree of radius 6 inches.
(b) The rate of change in the amount of lumber, with respect to radius when radius is 24 inches, in board-feet per inch.
(c) The radius (in inches) of a tree that produces 100 board-feet of lumber.
(d) The radius (in inches) at which the rate of change in board-feet per inch is 10.

6. The noise level, N, in decibels, of a rock concert is given by $N = f(d)$, where d is the distance in meters from the concert speakers. What do the following quantities mean to someone who lives in the neighborhood near the concert? Give units for the quantities.

(a) $f'(100)$
(b) $f'(1000)$
(c) d such that $f(d) = 100$

ANSWER:

(a) The rate of change, in decibels per meter, of noise 100 meters away from the speakers.
(b) The rate of change, in decibels per meter, of noise 1000 meters away from the speakers.
(c) The distance, in meters, away from the speakers at which the noise is 100 decibels.

7. In the scenario from Exercise 6,

(a) Explain in words what it means if $f'(100) > f'(1000)$.
(b) What situation might explain the expression $f(d) + 50$ describe?

ANSWER:

(a) The rate of change in sound, in decibels, at 100 meters from the speakers is greater than the rate of change at 1000 meters from the speakers.
(b) The noise level, in decibels, d meters from the speakers plus some additional noise of 50 decibels.

8. The population of a certain town is given by the function $P(t)$ where t is the number of years since the town was incorporated.

(a) What does it mean to say $P'(175) = -50$?
(b) What does it mean to say $P'(185) = 100$?
(c) If $P'(t)$ is constant for $185 < t$, what will $P(200)$ be if $P(185) = 25,500$?

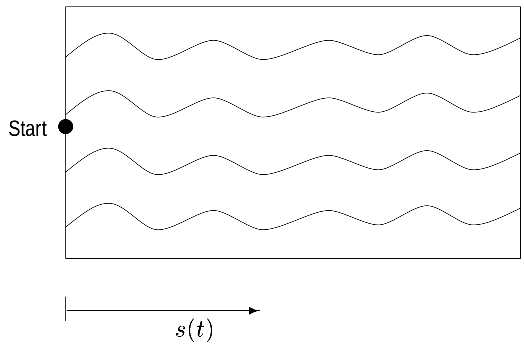
ANSWER:

(a) 175 years after incorporation, the population is decreasing at a rate of 50 people per year.
(b) 185 years after incorporation, the population is increasing at a rate of 100 people per year.
(c) From (b) we know $P'(t) = 100$ for $185 < t$. This means the population will increase by 100 people per year.

$P(200) = (200 - 185)(100) + 25,500 = 1,500 + 25,500 = 27,000$ people.

Questions and Solutions for Section 2.6

1. Esther is a swimmer who prides herself in having a smooth backstroke. Let $s(t)$ be her position in an Olympic size pool, as a function of time ($s(t)$ is measured in meters, t is seconds). (The Olympic size pool is 50 meters long.)



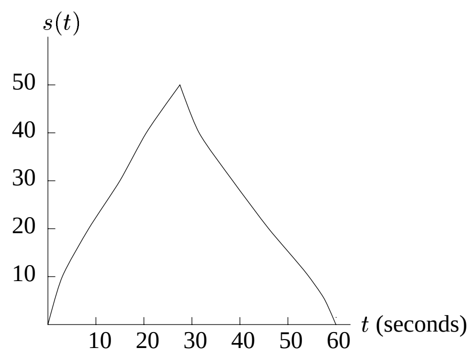
Below we list some values of $s(t)$, for a recent swim.

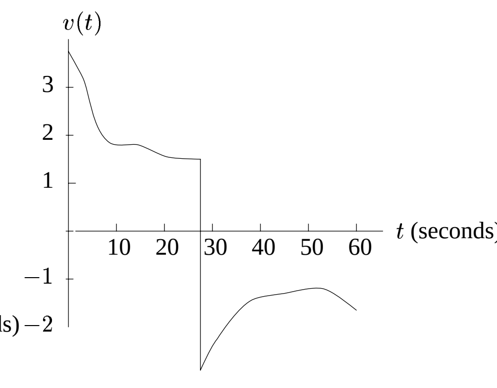
t	0	3.0	8.6	14.6	20.8	27.6	31.9	38.1	45.8	53.9	60
$s(t)$	0	10	20	30	40	50	40	30	20	10	0

(a) Sketch possible graphs for Esther's position and velocity. Put scales on your axes.
(b) Find Esther's average speed and average velocity over the whole swim.
(c) Based on the data, can you say whether or not Esther's instantaneous speed was ever greater than 3 meters/second? Why?
(d) Give a brief qualitative description of the graph of Esther's position (i.e., describe where the position is increasing, decreasing, concave up or down). Explain these qualitative features in terms of Esther's swimming behavior.

ANSWER:

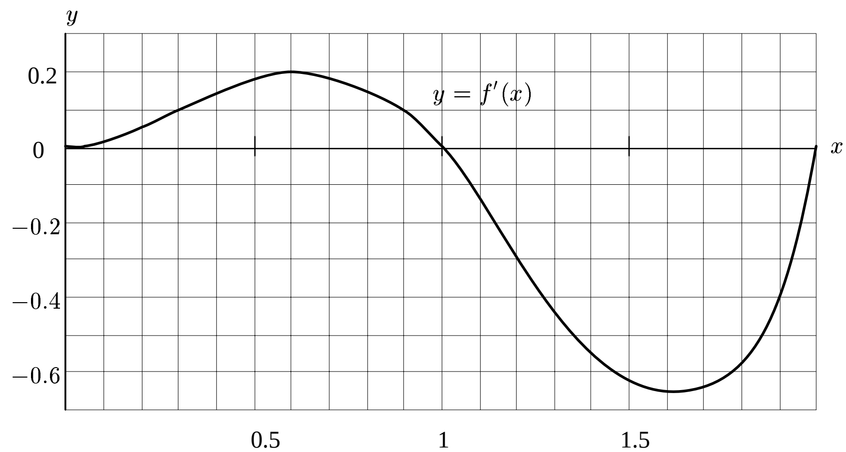
(a)





(b) Average speed $= \dfrac{\text{total distance}}{\text{total time}} = \dfrac{100}{60}$ m/s $= 1.67$ m/s
Average velocity $= \dfrac{\text{total displacement}}{\text{total time}} = \dfrac{0}{60}$ m/s $= 0$ m/s (because she finishes in the same place she started)
(c) Yes, because her average velocity over the first 3 seconds is $\frac{10}{3}$m/s > 3 m/s.
(d) Position is increasing up to the 50 meter marker, where she turns around, and then decreasing as she comes back. Position is concave down at first, because she starts out fast and then settles down to a steady speed. At steady speed, position graph is a straight line. Near the other end, the graph is concave down again. She starts fast in the opposite direction and then slows down, making the graph concave up. At the very end, she speeds up in a last minute sprint, making the graph concave down.

2. The graph below represents the *rate of change* of a function f with respect to x; i.e., it is a graph of f'.



You are told that $f(0) = 0$. On what intervals is f increasing? On what intervals is it decreasing? On what intervals is the graph of f concave up? Concave down? Is there any value $x = a$ other than $x = 0$ in the interval $0 \leq x \leq 2$ where $f(a) = 0$? If not, explain why not, and if so, give the approximate value of a.

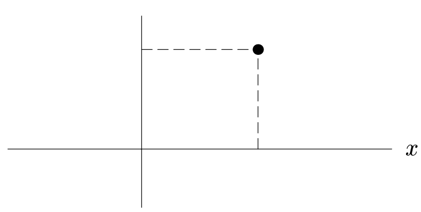
ANSWER:

f is increasing where f' is positive, namely from 0 to 1, and is decreasing where f' is negative, between 1 and 2. f is concave up where f' is increasing, namely on the intervals $[0, 0.6]$ and $[1.6, 2]$ and concave down on the interval $[0.6, 1.6]$. Finding an a, $0 \leq a \leq 2$, such that $f(a) = 0$ is equivalent to finding an a such that

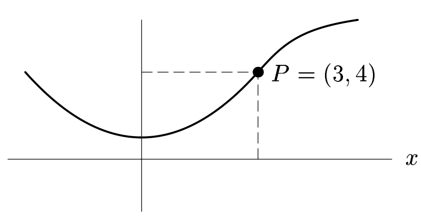
$$\int_0^a f'(x)\, dx = 0$$

We see that at $x = 1.4$, $\int_0^{1.4} f'(x)\, dx$ is approximately zero, since the area above the x-axis between 0 and 1 cancels the area below the x-axis between 1 and 1.4, so $a \approx 1.4$.

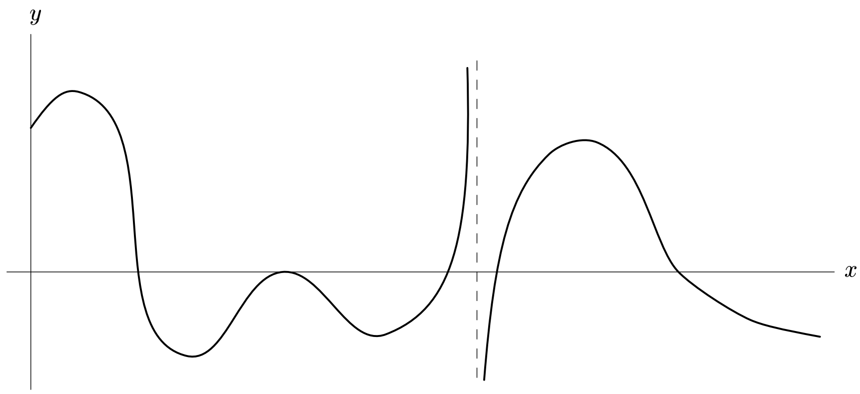
3. On the axes below, sketch a smooth, continuous curve (i.e., no sharp corners, no breaks) which passes through the point $P(3, 4)$, and which clearly satisfies the following conditions:
   - Concave up to the left of P
   - Concave down to the right of P
   - Increasing for $x > 0$
   - Decreasing for $x < 0$
   - Does *not* pass through the origin.



ANSWER:



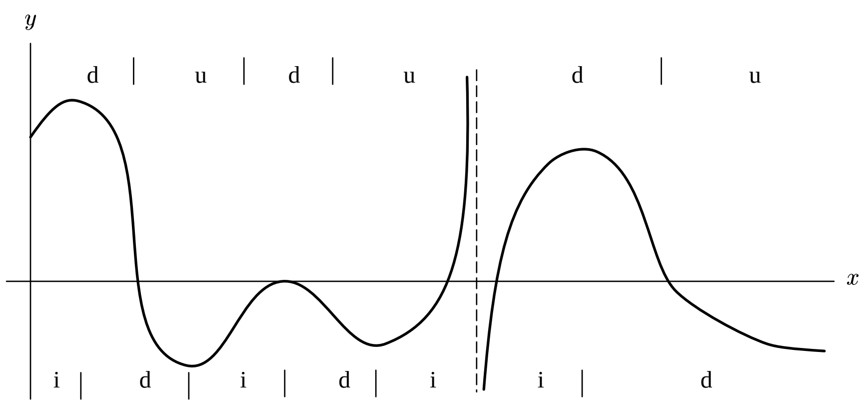
4. Given the following function:



(a) Indicate intervals where it is increasing, decreasing, concave up and concave down.
(b) Sketch the graph of the derivative function.

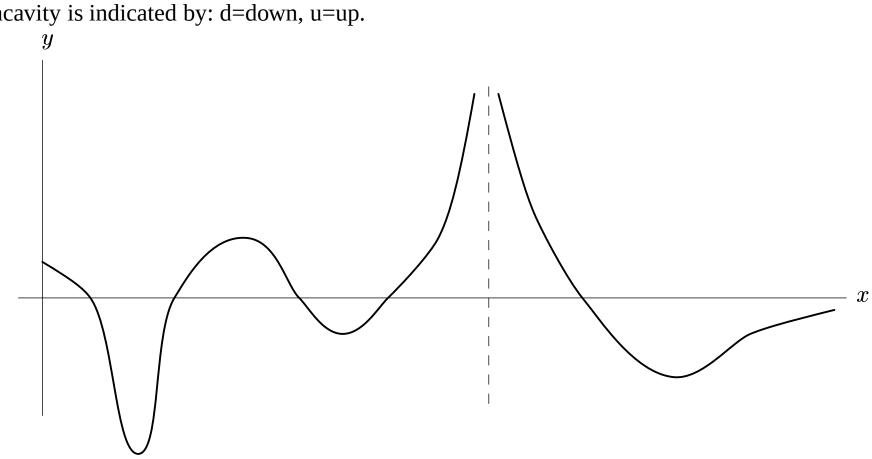
ANSWER:

(a)



For the above figure the increasing and decreasing intervals are indicated by: d=decreasing and i=increasing. The concavity is indicated by: d=down, u=up.

(b)



5. A function defined for all x has the following properties:

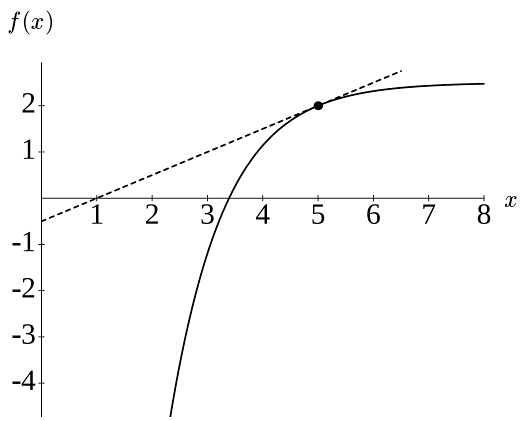
- f is increasing.

- f is concave down.
- $f(5) = 2$.
- $f'(5) = 1/2$.

(a) Sketch a possible graph for $f(x)$.
(b) How many zeros does $f(x)$ have and where are they located? Justify your answer.
(c) Is it possible that $f'(1) = \frac{1}{4}$? Justify your answer.

ANSWER:

(a)



(b) The function has exactly one zero because it is increasing everywhere. Since $f(5) = 2$, there cannot be a zero to the right of $x = 5$. The line tangent to the function at the point $(5, 2)$ crosses the x-axis at $x = 1$. Since the function is concave down, its graph must lie below such a line and thus the function must have a zero between 1 and 5.
(c) Since $f(x)$ is concave down everywhere, $f'(1) > f'(5)$. But $\frac{1}{4} < \frac{1}{2}$, so $f'(1) = \frac{1}{4}$ is impossible.

6. Assume that f and g are differentiable functions defined on all of the real line. Mark the following TRUE or FALSE.

(a) It is possible that $f > 0$ everywhere $f' > 0$, and $f'' < 0$ everywhere.
(b) f can satisfy: $f'' > 0$ everywhere, $f' < 0$ everywhere, and $f > 0$ everywhere.
(c) f and g can satisfy: $f'(x) > g'(x)$ for all x, and $f(x) < g(x)$ for all x.
(d) If $f'(x) = g'(x)$ for all x and if $f(x_0) = g(x_0)$ for some x_0, then $f(x) = g(x)$ for all x.
(e) If $f'' < 0$ everywhere and $f' < 0$ everywhere then $\lim_{x\to+\infty} f(x) = -\infty$.
(f) If $f'(x) > 0$ for all x and $f(x) > 0$ for all x then $\lim_{x\to+\infty} f(x) = \infty$.

ANSWER:

(a) FALSE.
(b) TRUE.
(c) TRUE.
(d) TRUE.
(e) TRUE.
(f) FALSE.

7. Suppose a function is given by a table of values as follows:

x	1.1	1.3	1.5	1.7	1.9	2.1
$f(x)$	12	15	21	23	24	25

(a) Estimate the instantaneous rate of change of f at $x = 1.7$.
(b) Write an equation for the tangent line to f at $x = 1.7$ using your estimate found in (a).
(c) Use your answer in (b) to predict a value for f at $x = 1.8$. Is your prediction too large or too small? Why?
(d) Is f'' positive or negative at $x = 1.7$? How can you tell? Can you estimate its value?

ANSWER:

(a) We approximate the instantaneous rate of change of $f(x)$ at $x = 1.7$ by the slope of the line joining the points $(1.7, 23)$ and $(1.9, 24)$, which is $\frac{1}{0.2} = 5$.

(b) The equation of a line with slope 5 passing through the point $(1.7, 23)$ is

$$\begin{aligned} y - 23 &= 5(x - 1.7) \\ y &= 23 + 5x - 8.5 \\ &= 14.5 + 5x. \end{aligned}$$

(c) At $x = 1.8$, we predict that

$$\begin{aligned} y &= 14.5 + 5 \cdot (1.8) \\ &= 23.5. \end{aligned}$$

Since the curve appears to be concave down over the interval $x \geq 1.3$, the line joining $(1.7, 23)$ and $(1.9, 24)$ lies *below* the curve, and hence 23.5 is an underestimate.

(d) f'' appears to be negative. To estimate the value of $f''(1.7)$, we first estimate values of $f'(1.6)$ and $f'(1.8)$:

$$\begin{aligned} f'(1.6) &\approx \frac{f(1.7) - f(1.5)}{1.7 - 1.5} = 10 \quad \text{and} \\ f'(1.8) &\approx \frac{f(1.9) - f(1.7)}{1.9 - 1.7} = 5. \end{aligned}$$

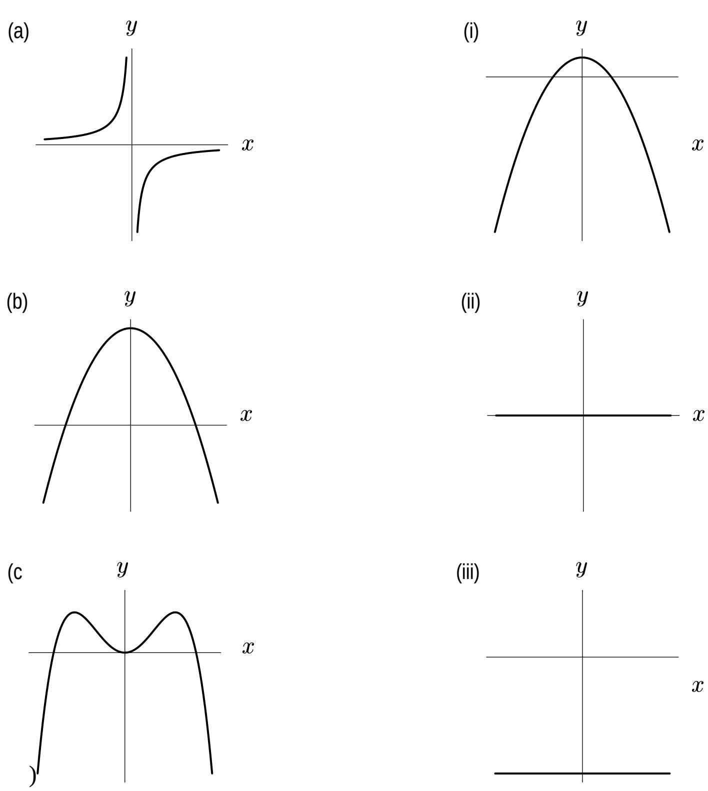
Now,

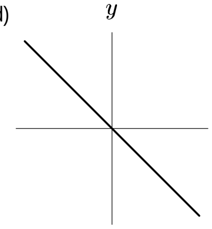
$$f''(1.7) \approx \frac{f'(1.8) - f'(1.6)}{1.8 - 1.6} = \frac{5 - 10}{0.2} = -25.$$

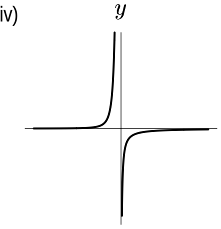
For Problems 8– 9, circle the correct answer(s) or fill in the blanks. No reasons need be given.

8. Each graph in the right-hand column below represents the *second* derivative of some function shown in the left-hand column. Match the functions and their second derivatives.

Functions | Second Derivatives







Function (a) has second derivative________________.
Function (b) has second derivative________________.
Function (c) has second derivative________________.
Function (d) has second derivative________________.

ANSWER:

(a) (iv)
(b) (iii)
(c) (i)
(d) (ii)

9. The cost of mining a ton of coal is rising faster every year. Suppose $C(t)$ is the cost of mining a ton of coal at time t.
   (a) Which of the following must be positive? (Circle those which are.)
      (i) $C(t)$
      (ii) $C'(t)$
      (iii) $C''(t)$
   (b) Which of the following must be increasing? (Circle those which are.)
      (i) $C(t)$
      (ii) $C'(t)$
      (iii) $C''(t)$
   (c) Which of the following must be concave up? (Circle those which are.)
      (i) $C(t)$
      (ii) $C'(t)$
      (iii) $C''(t)$

   ANSWER:

   (a) $C(t)$, $C'(t)$, $C''(t)$ positive
   (b) $C(t)$, $C'(t)$ increasing
   (c) $C(t)$ concave up.

10. Let $S(t)$ represent the number of students enrolled in school in the year t. What do each of the following tell us about the signs of the first and second derivatives of $S(t)$?
   (a) The number of students enrolling is increasing faster and faster.
   (b) The enrollment is getting close to reaching its maximum.
   (c) Enrollment decreased steadily.

   ANSWER:

   (a) $ds/dt > 0$ and $d^2s/dt^2 > 0$.
   (b) $ds/dt > 0$ and $d^2s/dt^2 < 0$ (but ds/dt is close to zero).
   (c) $ds/dt < 0$ and d^2s/dt^2 is constant.

11. A driver obeys the speed limit as she travels past different towns in the order A, B, C. In town A, the speed limit is 50 mph. In town B, the speed limit is 35 mph and in town C, speed limit is 65mph.
   (a) If $S(t)$ represents the driver's position at time t, compare $S'(t)$ when she is passing town A to $S'(t)$ when she is passing town C.
   (b) If it always takes her two minutes to reach the new speed limit when she passes by a new town, what can you say about $S''(t)$ for the first two minutes she travels by towns B and C?

ANSWER:

(a) $S'(t)$ represents the velocity or rate of change in position. $S'(t)$ at town A is less than $S'(t)$ at town C.
(b) $S''(t)$ represents acceleration. When she gets to town B she has to slow down from 50 mph to 35 mph. When she gets to town C she has to speed up to reach 65 mph. So $S''(t)$ at town B is negative and $S''(t)$ at town C is positive. The magnitude of $S''(t)$ at town B is less than the magnitude of $S''(t)$ at town C.

12. A company graphs $C'(t)$, the derivative of the number of pints of ice cream sold over the past ten years.

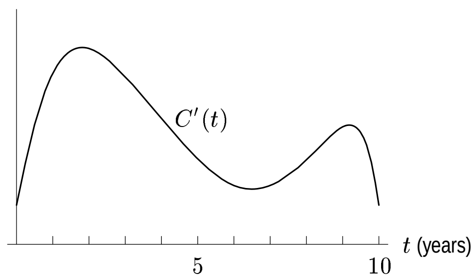


Figure 2.6.40

At what year was

(a) $C(t)$ greatest?
(b) $C'(t)$ greatest?
(c) $C''(t)$ greatest?
(d) $C(t)$ least?

ANSWER:

Since $C'(t)$ is positive everywhere, $C(t)$ is increasing everywhere. Hence $C(t)$ is greatest at year ten and least at year zero. We see $C'(t)$ is greatest around year two. $C''(t)$ is greatest where $C'(t)$ is rising most rapidly, namely before year 2.

13. Sketch a graph for the derivative of an advertising company's revenue with the following properties: First revenue increased steadily. That was followed by a short period of near zero revenue. Then revenue grew faster and faster.

ANSWER:

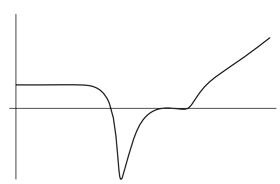


Figure 2.6.41

Questions and Solutions for Section 2.7

1. Sketch a graph $y = f(x)$ that is continuous everywhere on $-10 < x < 10$ but not differentiable at $x = -5$ or $x = 5$.

ANSWER:

Many possible.

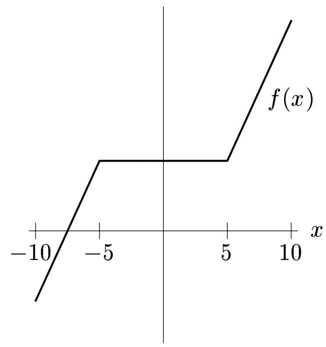


Figure 2.7.42

2. Sketch a graph of a continuous function $f(x)$ with the following properties:
   (a) $f''(x) < 0$ for $x < 4$
   (b) $f''(x) > 0$ for $x > 4$
   (c) $f''(4)$ is undefined

   ANSWER:
   Many possible.

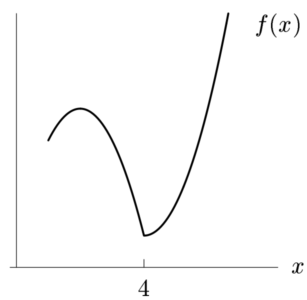


Figure 2.7.43

3. Discuss the continuity and differentiability of
   (a) $f(x) = |x+3|$
   (b) $g(x) = \dfrac{1}{x+3}$

   ANSWER:
   (a) $f(x)$ is continuous everywhere but not differentiable at $x = -3$.
   (b) $g(x)$ is neither continuous nor differentiable at $x = -3$.

4. Given the function

$$h(r) = \begin{cases} 1 - \sin(\pi r/2) & \text{for } -1 \leq \text{r} \leq 1 \\ 0 & \text{for r} < -1 \text{ or r} > 1 \end{cases}$$

   (a) Is $h(r)$ continuous at $r = 1$? Explain.
   (b) Do you think $h(r)$ is differentiable at $r = 1$? Explain.
   (c) Is $h(r)$ differentiable at $r = -1$?

   ANSWER:
   (a) The graph of $h(r)$ does not have a break or jump at $r = 1$, so $h(r)$ is continuous there. This is confirmed by the fact that

$$h(1) = 1 - \sin(\pi(1)/2) = 1 - 1 = 0$$

   So the value of $h(r)$ as you approach $r = 1$ from the left is the same as the value when you approach $r = 1$ form the right.

(b) The graph of $h(r)$ does not have a corner at $r = 1$, so $h(r)$ appears differentiable there. Notice also that the slope of $1 - \sin(\pi r/2)$ is zero at $r = 1$, so the slope to the left and right of $r = 1$ are the same.
(c) No. $h(-1) = 1 - \sin(-\pi/2) = 2$, so $h(r)$ is neither differentiable nor continuous at $r = -1$.

Review Questions and Solutions for Chapter 2

1. Let $f(x) = x^{\sin(x)}$.
(a) Using your calculator, estimate $f'(2)$.
Don't forget to set your calculator to radian mode.
(b) Find the linear approximation for $f(x)$ near $x = 2$.
(c) Using the computer, graph $f(x)$ and its linear approximation together on the same screen. For what range of values do you think your approximation is reasonably accurate? Explain how you chose your answer.
(d) Now graph $f(x)$ and $g(x) = x^x$ on the same axes. Describe what you see, including any particularly interesting features. Can you explain those features?

ANSWER:

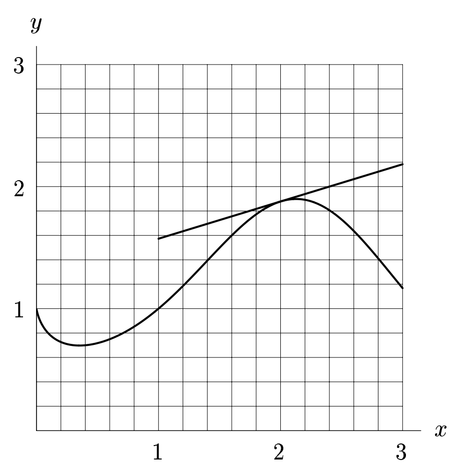
(a) We use shrinking intervals to the right of $x = 2$ to approximate $f'(2)$:

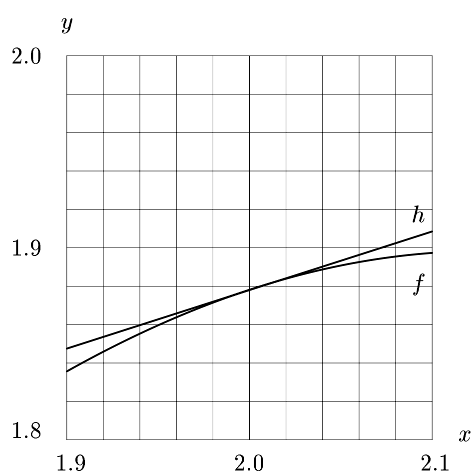
Interval size, h	$\dfrac{f(x+h) - f(x)}{h}$
0.1	$\dfrac{f(2.1) - f(2)}{0.1} \approx 0.192$
0.01	$\dfrac{f(2.01) - f(2)}{0.01} \approx 0.300$
0.001	$\dfrac{f(2.001) - f(2)}{0.001} \approx 0.311$
0.0001	$\dfrac{f(2.0001) - f(2)}{0.0001} \approx 0.312$

Hence, $f'(2)$ appears to be about 0.312.
(b) The linear approximation, $h(x)$, for $f(x)$ near $x = 2$ is given by

$$h(x) \approx f(2) + (x-2)f'(2) = 1.878 + (x-2)0.312 = 1.254 + 0.312x.$$

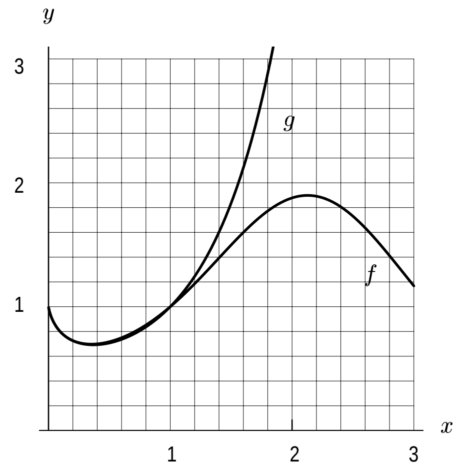
(c)





The graph on the right is an enlargement of the region $1.9 < x < 2.1$ and $1.8 < y < 2.0$. From the graph we guess that the linear approximation found is accurate only over the region $1.97 < x < 2.04$. This is because the two curves agree fairly closely over this region.

(d)



The curves x^x and $x^{\sin x}$ agree closely for $0 < x < 1$. This is because the exponents of these two functions, namely x and $\sin x$ agree fairly well over $0 < x < 1$.

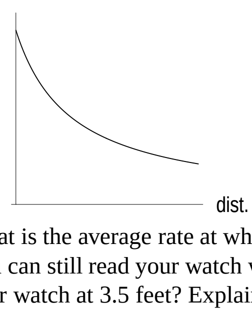
2. Alone in your dim, unheated room you light one candle rather than curse the darkness. Disgusted by the mess, you walk directly away from the candle, cursing. The temperature (in degrees Fahrenheit) and illumination (in % of one candle power) decrease as your distance (in feet) from the candle increases. In fact, you have tables showing this information!

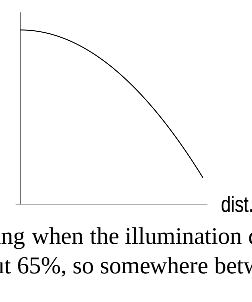
distance(feet)	Temp. (°F)
0	55
1	54.5
2	53.5
3	52
4	50
5	47
6	43.5

distance(feet)	illumination
0	100
1	85
2	75
3	67
4	60
5	56
6	53

You are cold when the temperature is below $40°$. (You are from California.) You are in the dark when the illumination is at most 50% of one candle power.

(a) Two graphs are sketched below. One is temperature as a function of distance and one is illumination as a function of distance. Which is which? Explain.





(b) What is the average rate at which the temperature is changing when the illumination drops from 75% to 56%?

(c) You can still read your watch when the illumination is about 65%, so somewhere between 3 and 4 feet. Can you read your watch at 3.5 feet? Explain.

(d) Suppose you know that at 6 feet the instantaneous rate of change of the temperature is $-4.5°$F/ft and the instantaneous rate of change of illumination is -3% candle power/ft. Estimate the temperature and the illumination at 7 feet.

(e) Are you in the dark before you are cold, or vice-versa?

ANSWER:

(a) The first graph plots illumination versus distance. We can see this in the chart because illumination drops rapidly at

first, then begins to level off.
The second graph is a plot of temperature versus distance. This can be seen on the chart as temperature drops slowly at first, then greatly with distance from the candle.

(b) $\frac{47-53.5}{5-2} = -2.17°$F/ft.

(c) If we want to find the brightness at 3.5 feet from the candle, let's average the brightness at 3 and 4 feet to get 63.5%. However, since this curve is *concave up*, 63.5% is an *overestimate*. So we cannot read the watch.

(d) If, at 6 feet from the candle, the instantaneous rate of change is $-4.5°$F/ft, then an estimate for temperature at 7 feet might be $43.5 - 4.5 = 39°$F. Since this curve is concave down, this extrapolation is an *overestimate*. If, at 6 feet from the candle, the instantaneous rate of change is -3% candle power / ft, then an estimate for the illumination at 7 feet might be $53 - 3 = 50\%$ candle power/ft. Since this concave up, this extrapolation is an *underestimate*.

(e) At 7 feet from the candle, the temperature is less than $39°$ F and the illumination is greater than 50%. You are cold, but you are not yet dark. You thus become cold before you become dark.

3. Two politicians, named A and B, carefully inspect a table of values, x versus y. A claims that the table is linear, while B claims it is exponential.

(a) You look at the table and agree with A. Explain what you saw in the table.
(b) You look at the table and agree with B. Explain what you saw in the table.
(c) You look at the table and realize that neither is *exactly* right, but *both* of them are *approximately* correct. Explain why this can be so. Referring to the derivative might be appropriate.

ANSWER:

(a) For each change in x, the corresponding change in y is proportional with the same constant of proportionality.
(b) Each time x changes by a certain amount, the ratio of the corresponding y-values is the same.
(c) The table could be approximately linear and approximately exponential if the exponential function used were approximately linear. This could happen if we used a portion of an exponential graph and had zoomed in sufficiently so that the exponential looked almost straight.

4. The graphs below are each the derivative of a function. Sketch a graph of each original function.

(a)

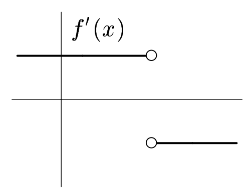


Figure 2.7.44

(b)

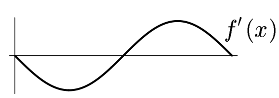


Figure 2.7.45

(c)

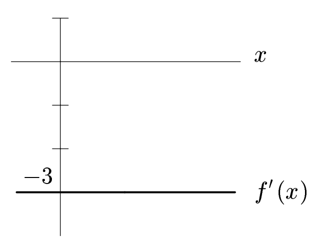


Figure 2.7.46

ANSWER:

(a)

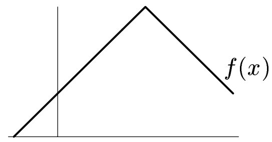


Figure 2.7.47

(b)

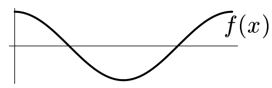


Figure 2.7.48

(c)

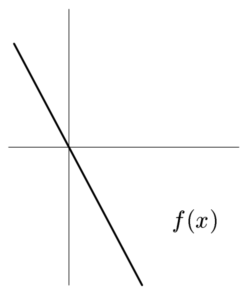


Figure 2.7.49

5. Is $f(x) = \dfrac{x^2|2x-4|}{x-2}$ continuous? Explain why or why not.
ANSWER:

$$f(x) = \frac{x^2|2x-4|}{x-2} = \begin{cases} \dfrac{x^2(2x-4)}{x-2} = 2x^2, x > 2 \\ \dfrac{x^2(4-2x)}{x-2} = -2x^2, x < 2 \end{cases}$$

$\lim\limits_{x \to 2^+} f(x) = 8$ while $\lim\limits_{x \to 2^-} f(x) = -8$, thus $\lim\limits_{x \to 2} f(x)$ does not exist.

Chapter 3 Exam Questions

Questions and Solutions for Section 3.1

1. Consider the function $f(x) = x^3 - 4x^2 + 9$.
   (a) Give the equations of the tangent line at $x = 1$ and the tangent line at $x = 2$.
   (b) Estimate $f(1.5)$ first using the tangent line at $x = 1$ and then using the tangent line at $x = 2$.
   (c) The estimate using $x = 1$ is slightly better. Explain why.

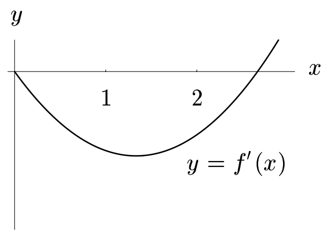
   ANSWER:

   (a) $f'(x) = 3x^2 - 8x$. Then $f'(1) = -5$ is the slope of the tangent line at $x = 1$, and $f'(2) = -4$ is the slope of the tangent line at $x = 2$. The equations of these lines are

   $$y = -5x + 11 \qquad \text{(tangent line at } (1,6))$$
   $$y = -4x + 9 \qquad \text{(tangent line at } (2,1))$$

   (b) Substituting 1.5 in for x, the first tangent line equation gives $f(1.5) \approx -5(1.5) + 11 = 3.5$. The second equation gives $f(1.5) \approx -4(1.5) + 9 = 3$. (The actual value is 3.375.)

   (c)



   The error in the local linearization depends on how much f' is changing. (There would be no error if f' were constant, for then f would itself be linear.) From the graph we can see that f' is more nearly constant between 1 and 1.5 than between 1.5 and 2, so we'd expect f to deviate more from its linearization at 2 than from its linearization at 1. Alternatively we may note that $f''(x) = 6x - 8$, whence $|f''(1)| < |f''(2)|$, so f' is changing more rapidly near 2 than near 1.

2. Given $f(x) = x^3 - 6x^2 + 9x - 5$,
   (a) Find the slope of the tangent line to the curve at $x = -2$.
   (b) What is the equation of this tangent line?
   (c) Find all points where the curve has a horizontal tangent.

   ANSWER:

   (a) $f(x) = x^3 - 6x^2 + 9x - 5$, so $f'(x) = 3x^2 - 12x + 9$. At $x = -2$, the slope is 45.
   (b) The tangent line passes through $(-2, -55)$, so its equation is

   $$y + 55 = 45(x + 2) \qquad \text{or} \qquad y = 45x + 35.$$

   (c) The line tangent to the curve $y = f(x)$ is horizontal when its slope, $f'(x)$, is zero. This happens when $3x^2 - 12x + 9 = 3(x-1)(x-3) = 0$, that is, when $x = 1$ or $x = 3$.

3. Decide whether the following statement is true or false and provide a short explanation or counterexample:
   The 10th derivative of $f(x) = x^{10}$ is 0.

   ANSWER:

   FALSE. The tenth derivative of $f(x) = x^{10}$ is 10! (10 factorial).

4. Use the definition of the derivative to justify the power rule for $n = 4$: Show $\dfrac{d}{dx}(2x^4) = 8x^3$.

   ANSWER:

$$\frac{d}{dx}(2x^4) = \lim_{h\to 0}\left(\frac{2(x+h)^4 - 2x^4}{h}\right) = \lim_{h\to 0}\frac{2}{h}\left(x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4\right)$$
$$= \lim_{h\to 0} 2\left(4x^3 + 6x^2h + 4xh^2 + h^3\right) = 8x^3$$

5. Find the derivatives of the given functions.

   (a) $y = 2x^3 - \frac{1}{2x}$

   (b) $f(x) = 2\sqrt{x} + x^3$

   (c) $g(t) = \frac{2t^3 - t^2 + 6}{t^2}$

   ANSWER:

   (a) $y = 2x^3 - \frac{1}{2}x^{-1}$

   $$y' = 6x^2 + \frac{1}{2}x^{-2} = 6x^2 + \frac{1}{2x^2}$$

   (b) $f(x) = 2x^{1/2} + x^3$

   $$f'(x) = x^{-1/2} + 3x^2 = \frac{1}{\sqrt{x}} + 3x^2$$

   (c) $g(t) = \frac{2t^3 - t^2 + 6}{t^2} = 2t - 1 + 6t^{-2}$, so
   $g'(t) = 2 - 12t^{-3}$.

6. If $g(t) = 4t^3 - t^2 + 3t$

   (a) find $g'(t)$ and $g''(t)$

   (b) If $g(t)$ represents the position of a particle at time t seconds, what do $g'(3)$ and $g''(4)$ represent?

   ANSWER:

   (a) $g'(t) = 12t^2 - 2t + 3$
   $g''(t) = 24t - 2$

   (b) $g'(3)$ represents the velocity of the particle at time 3 seconds.
   $g''(4)$ represents the acceleration of the particle at time 4 seconds.

7. Consider the function $f(x) = 2x^4 - 4x^3 + 2$. Are there values of x for which $f(x)$ has the following properties? If so, indicate the values.

   (a) Increasing

   (b) Decreasing and concave up

   ANSWER:

   (a) Increasing means $f'(x) > 0$.

   $$f'(x) = 8x^3 - 12x^2 = 4x^2(2x - 3)$$

   So $f'(x) > 0$ when $(2x - 3) > 0$ and $x \neq 0$. Therefore

   $$2x > 3$$

   $$x > 3/2$$

   (b) Decreasing means $f'(x) < 0$.
   So $f'(x) < 0$ when $(2x - 3) < 0$ and $x \neq 0$. Therefore

   $$2x < 3$$

   $$x < 3/2$$

   Concave up means $f''(x) > 0$.

   $$f''(x) = 24x^2 - 24x = 24x(x - 1)$$
   $$f''(x) > 0 \text{ when } 24x(x - 1) > 0$$
   $$x < 0 \text{ or } x > 1$$

   So both conditions hold when $x < 0$ or $1 < x < 3/2$.

8. Given a power function of the form $f(x) = ax^n$, find n and a so that $f'(2) = -1$ and $f'(4) = -1/4$, .

ANSWER:

Since $f(x) = ax^n,\ f'(x) = nax^{n-1}$. We know that $f'(2) = na(2)^{n-1} = -1$ and $f'(4) = na(4)^{n-1} = -1/4$. Therefore,

$$\frac{f'(4)}{f'(2)} = \frac{-\frac{1}{4}}{-1}$$
$$\frac{(na)4^{n-1}}{(na)2^{n-1}} = \left(\frac{4}{2}\right)^{n-1} = \frac{1}{4}$$
$$2^{n-1} = \frac{1}{4} = 2^{-2},$$
$$\text{and thus } n = -1.$$

Substituting $n = -1$ into the expression for $f'(2)$, we get

$$(-1)a(2)^{-2} = -1$$
$$\frac{a}{4} = 1, a = 4$$

9. Given $f(x) = 3x^2 - x$ and $g(x) = x^3 + 3x^2 - 3$

(a) find $\frac{d}{dx}[f(x) + g(x)]$

(b) find $\frac{d}{dx}[g(x) - 2f(x)]$

ANSWER:

(a) $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[3x^2 - x + x^3 + 3x^2 - 3] = \frac{d}{dx}[x^3 + 6x^2 - x - 3] = 3x^2 + 12x - 1$

(b) $\frac{d}{dx}[g(x) - 2f(x)] = \frac{d}{dx}[x^3 + 3x^2 - 3 - 6x^2 + 2x] = \frac{d}{dx}[x^3 - 3x^2 + 2x - 3] = 3x^2 - 6x + 2$

Questions and Solutions for Section 3.2

1. Consider the graph $y = e^x$.

(a) Find the equation of the tangent line to the graph at (a, e^a).

(b) Find the x- and y-intercepts of the line in part (a).

(c) Show that the highest y-intercept of *any* tangent line is $y = 1$. (You may give a geometric argument as long as you say clearly what properties of the graph you are using.)

ANSWER:

(a) The slope of the tangent line at (a, e^a) is $\left.\frac{dy}{dx}\right|_{x=a} = e^a$, so the equation of the tangent line is

$$\frac{y - e^a}{x - a} = e^a$$
$$y - e^a = xe^a - ae^a$$
$$y = e^a(x - a + 1)$$

(b) At $x = 0$, $y = e^a(1 - a)$. At $y = 0$, $x = a - 1$.

(c) The formula of part (b) gives the y-intercept for the line tangent to any point of the graph of $y = e^x$. So the maximum y-intercept is attained wherever $e^a(1 - a)$ is largest.

To find where $e^a(1 - a)$ is maximized, we examine the derivative: $\frac{d}{da}e^a(1 - a) = -ae^a$. This clearly has only one root, at $a = 0$, and is negative for all $a > 0$; so the maximum y-intercept is $\left.e^a(1 - a)\right|_{a=0} = 1$.

You can also see this geometrically. Since the graph of $y = e^x$ is concave up, the tangent line at any point (a, e^a) lies below the curve. So the y-intercept of any tangent line will be at or below where the curve $y = e^x$ hits the y-axis, i.e. at $(0, 1)$. Thus the tangent line at $(0, 1)$ has the largest y-intercept, namely 1, among all possible tangent lines.

2. If P dollars are invested at an annual rate of $r\%$, then in t years this investment grows to F dollars, where

$$F = P\left(1 + \frac{r}{100}\right)^t.$$

(a) Assuming P and r are constant, find $\frac{dF}{dt}$. In practical terms (in terms of money), what does this derivative mean?

(b) Solve the given equation for P. Assuming F and r are constant, find $\frac{dP}{dt}$. What is its sign? Why is this sign reasonable?

ANSWER:

(a) $\frac{dF}{dt} = P(1 + \frac{r}{100})^t \ln(1 + \frac{r}{100})$, which gives the rate at which the total amount of money grows.

(b) $P = F(1 + \frac{r}{100})^{-t}$, so $\frac{dP}{dt} = -F(1 + \frac{r}{100})^{-t} \ln(1 + \frac{r}{100})$. The sign is negative, indicating that given the amount one wants to end up with, increasing the amount of time for which the money is invested decreases the amount one must put it initially to get the desired return.

3. Find the derivative of the functions below:

(a) $y = 3^x - 3$

(b) $f(x) = 6e^x - 5^x$

(c) $g(x) = 2e^{\pi x} = 2(e^\pi)^x$

ANSWER:

(a) $y' = (\ln 3)3^x$

(b) $f'(x) = 6e^x - \ln 5(5^x)$

(c) $g(x) = 2e^{\pi x} = 2(e^\pi)^x$ so
$g'(x) = 2(e^\pi)^x \ln(e^\pi) = 2\pi(e^\pi)^x = 2\pi e^{\pi x}$.

4. Find the derivative:

(a) $f(x) = (\ln 2)x^2 + (\ln 5)e^x$

(b) $g(x) = 3x - \dfrac{1}{\sqrt[3]{x}} + 3^x$

ANSWER:

(a) $f'(x) = (2\ln 2)x + (\ln 5)e^x$

(b) $g'(x) = \dfrac{d}{dx}[3x - x^{-1/3} + 3^x] = 3 + \dfrac{1}{3x^{4/3}} + \ln 3(3^x)$

5. Find the derivative:

(a) $h(t) = t^{\pi^3} + (\pi^3)^t + \pi t^3$

(b) $g(t) = (1/e)^t + e^t + e$

ANSWER:

(a) $h'(t) = \pi^3 t^{(\pi^3 - 1)} + (\pi^3)^t \ln(\pi^3) + 3\pi t^2$

(b) $g'(t) = \ln(1/e)(1/e)^t + e^t = -(1/e)^t + e^t$

6. Consider the function $g(x) = 3x^3 + 3^x$. Give the equation of the tangent lines at $x = 0$ and $x = 2$.

ANSWER:

$$g'(x) = 9x^2 + 3^x \ln 3$$
$$g'(0) = 9(0)^2 + 3^0 \ln 3 = \ln 3$$
$$g(0) = 3(0)^3 + 3^0 = 1$$
$$y = (\ln 3)x + 1.$$
$$g'(3) = 9(3)^2 + 3^3 \ln 3 = 81 + 27\ln 3$$
$$y = (81 + 27\ln 3)x + 1.$$

7. With a yearly inflation rate of 3%, prices are described by $P = P_0(1.03)^t$, where P_0 is the price in dollars when $t = 0$ and t is time in years. If $P_0 = 1.2$, how fast (in cents/year) are prices rising when $t = 15$?

ANSWER:

Since $P = (1.2)(1.03)^t$, $\frac{dP}{dt} = (1.2)\ln(1.03)(1.03)^t$. When $t = 15$, $\frac{dP}{dt} = (1.2)\ln(1.03)(1.03)^{15} \approx 0.055$ dollars or 5.5 cents/year.

8. (a) Find the slope of the graph of $g(x) = x - 2e^x$ at the point where it crosses the y-axis.
   (b) Find the equation of the tangent line to the curve at the point from part (a).
   (c) Find the equation of the line perpendicular to the line from part (a) that goes through $(1, 1)$.
   ANSWER:
   (a) $g'(x) = 1 - 2e^x$.

   The graph crosses the y-axis when $x = 0$, so the point is $(0, -2)$.

   Slope at $(0, -2)$ is $1 - 2e^0 = -1$
   (b) $y + 2 = -1(x),\ \ y = -x - 2$
   (c) $y - 1 = 1(x - 1)$

   $y = x$

9. Use the definition of the derivative to justify the formula $\dfrac{d(x^n)}{dx} = nx^{n-1}$ for $n = -2$.
   ANSWER:

$$\begin{aligned}\frac{d(x^{-2})}{dx} &= \lim_{h\to 0} \frac{(x+h)^{-2} - x^{-2}}{h} = \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{(x+h)^2} - \frac{1}{x^2}\right] \\ &= \lim_{h\to 0} \frac{1}{h}\left[\frac{x^2 - (x+h)^2}{x^2(x+h)^2}\right] = \lim_{h\to 0} \frac{1}{h}\left[\frac{-2xh - h^2}{x^2(x+h)^2}\right] \\ &= \lim_{h\to 0} \frac{-2x - h}{x^2(x+h)^2} = \frac{-2x}{x^4} = -2x^{-3}.\end{aligned}$$

10. Given $y = 4^x + 3^x$, find y', y'', and y'''.
    ANSWER:
    $y' = \ln 4(4^x) + \ln 3(3^x)$

    $y'' = (\ln 4)^2(4^x) + (\ln 3)^2(3^x)$

    $y''' = (\ln 4)^3(4^x) + (\ln 3)^3(3^x)$

Questions and Solutions for Section 3.3

1. Consider the following table of data for the function f.

x	5.0	5.1	5.2	5.3	5.4
$f(x)$	9.2	8.8	8.3	7.7	7.0

   (a) Estimate $f'(5.1)$.
   (b) Give an equation for the tangent at $x = 5.1$.
   (c) What is the sign of $f''(5.1)$? Explain your answer.
   (d) Is this the table of data linear? Exponential? Quadratic? Explain your answer.
   (e) Suppose g is a function such that $g(5.1) = 10$ and $g'(5.1) = 3$. Find $h'(5.1)$ where
      (i) $h(x) = f(x)g(x)$
      (ii) $h(x) = f(x)/g(x)$.
   ANSWER:
   (a)

$$f'(5.1) \approx \frac{f(5.2) - f(5.1)}{0.1} = \frac{8.3 - 8.8}{0.1} = -5.$$

   (b) $y - 8.8 = -5(x - 5.1)$, so $y = -5x + 34.3$ is the equation for the tangent line.

(c) The sign of $f''(5.1)$ is negative because the derivative is decreasing (becoming more negative) here. To see this, compare

$$(f(5.1) - f(5.0))/0.1 = -4$$

with

$$(f(5.2) - f(5.1))/0.1 = -5$$

.

(d) The table is quadratic because the derivative of f is linear: $f'(5) \approx -4$, $f'(5.1) \approx -5$, $f'(5.2) \approx -6$, and so on. Overall, the change in the derivative is linearly related to the change in x. Thus $f'(x) = ax + b$, for some a and b, so $f(x) = ax^2/2 + bx + C$ for some a, b, and C.

(e) (i) Since

$$h'(x) = f'(x)g(x) + f(x)g'(x),$$
$$h'(5.1) = f'(5.1)g(5.1) + f(5.1)g'(5.1) = (-5)10 + (8.8)3 = -23.6.$$

(ii) Since

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2},$$
$$h'(5.1) = \frac{(10)(-5) - (8.8)3}{(10)^2} = -0.764.$$

2. (a) Find the derivative of $f(x) = x^{-3}(2x^4 + 2)$ using the product rule and simplify the answer.
(b) Rewrite $f(x)$ as a quotient and use the quotient rule to find $f'(x)$.
ANSWER:

(a)

$$f'(x) = x^{-3}(8x^3) + (2x^4 + 2)(-3x^{-4}) = 8 - 6 - 6x^{-4} = 2 - 6x^{-4}$$

(b)

$$f(x) = \frac{2x^4 + 2}{x^3}$$
$$f'(x) = \frac{x^3(8x^3) - (2x^4 + 2)(3x^2)}{(x^3)^2} = \frac{8x^6 - 6x^6 - 6x^2}{x^6}$$
$$= \frac{2x^6 - 6x^2}{x^6} = 2 - 6x^{-4}$$

3. Find the equation of the tangent line to the graph of $g(x) = \dfrac{x^2 - 2}{x + 1}$ at the point at which $x = 1$.
ANSWER:
Since $g(1) = \dfrac{1^2 - 2}{1 + 1} = -\dfrac{1}{2}$, the tangent line passes through the point $(1, -1/2)$.
We find the derivative of $g(x)$ using the quotient rule:

$$g'(x) = \frac{(x + 1)(2x) - (x^2 - 2)(1)}{(x + 1)^2} = \frac{2x^2 + 2x - x^2 + 2}{(x + 1)^2} = \frac{x^2 + 2x + 2}{(x + 1)^2}$$

At $x = 1$, the slope of the tangent line is $m = g'(1) = \dfrac{1 + 2 + 2}{(1 + 1)^2} = \dfrac{5}{4}$. The equation of the tangent line is

$$y + \frac{1}{2} = \frac{5}{4}(x - 1)$$
$$y = \frac{5}{4}x - \frac{7}{4}$$

4. Differentiate $g(t) = e^{-t} + 2e^{-2t}$ by writing each term as a quotient before finding the derivative.
ANSWER:

$$g(t) = \frac{1}{e^t} + \frac{2}{e^{2t}}$$
$$g'(t) = \frac{e^t(0) - 1(e^t)}{(e^t)^2} + \frac{e^{2t}(0) - 2(2e^{2t})}{(e^{2t})^2}$$
$$= \frac{-e^t}{e^{2t}} - \frac{4e^{2t}}{e^{4t}} = -e^{-t} - 4e^{-2t}$$

5. Given $f(x) = e^x,\ g(x) = 2^x,\ h(x) = f(x)g(x)$, and $j(x) = \dfrac{g(x)}{f(x)}$

(a) Find $h'(x)$ and $h''(x)$.
(b) Find $j'(x)$ and $j''(x)$

ANSWER:

(a)

$$\begin{aligned} h(x) &= e^x 2^x \\ h'(x) &= e^x(\ln 2)2^x + 2^x e^x \\ h''(x) &= e^x(\ln 2)^2 2^x + (\ln 2)2^x e^x + 2^x e^x + e^x(\ln 2)2^x \\ &= 2^x e^x((\ln 2)^2 + 2\ln 2 + 1) \end{aligned}$$

(b)

$$\begin{aligned} j(x) &= \frac{2^x}{e^x} \\ j'(x) &= \frac{e^x(\ln 2)2^x - 2^x e^x}{(e^x)^2} = \frac{2^x e^x(\ln 2 - 1)}{e^{2x}} = \frac{2^x}{e^x}(\ln 2 - 1) \\ j''(x) &= (\ln 2 - 1)(\frac{2^x}{e^x})' \\ &= (\ln 2 - 1)(\ln 2 - 1)\frac{2^x}{e^x} \\ &= (\ln 2 - 1)^2\frac{2^x}{e^x}. \end{aligned}$$

6. Given $f(x) = \dfrac{x^3}{2x+1}$ and $g(x) = \dfrac{x^2+2}{3x^2}$ and $h(x) = f(x)g(x)$, find $h'(2)$.

ANSWER:

$$\begin{aligned} h'(x) &= f(x)g'(x) + g(x)f'(x) \\ &= \frac{x^3}{2x+1}\left(\frac{3x^2(2x) - (x^2+2)(6x)}{(3x^2)^2}\right) + \frac{x^2+2}{3x^2}\left(\frac{(2x+1)(3x^2) - x^3(2)}{(2x+1)^2}\right) \\ &= \frac{x^3}{2x+1}\left(\frac{6x^3 - 6x^3 - 12x}{9x^4}\right) + \frac{x^2+2}{3x^2}\left(\frac{6x^3 + 3x^2 - 2x^3}{4x^2+4x+1}\right) \\ h'(2) &= \frac{2^3}{2(2)+1}\left(\frac{-12(2)}{9(2)^4}\right) + \frac{2^2+2}{3(2)^2}\left(\frac{4(2)^3 + 3(2)^2}{4(2)^2+4(2)+1}\right) \\ &= \frac{8}{5}\left(\frac{-24}{9\cdot 2^4}\right) + \frac{6}{12}\left(\frac{32+12}{16+8+1}\right) \\ &= -\frac{4}{15} + \frac{22}{25} = \frac{46}{75} \end{aligned}$$

7. If $j(x) = g(x)h(x)$ and $j'(x) = 2x^3 + 3x^2(2x+1)$, what are $g(x)$ and $h(x)$?

ANSWER:

$$\begin{aligned} j'(x) &= g(x)h'(x) + h(x)g'(x) \\ g(x) &= x^3 \text{ and } h(x) = 2x+1 \end{aligned}$$

8. Differentiate $g(x) = 3e^{2x}$ by first writing it as a product.

ANSWER:

$$\begin{aligned} g(x) &= 3e^x \cdot e^x \\ g'(x) &= 3e^x \cdot e^x + e^x \cdot 3e^x = 3e^{2x} + 3e^{2x} = 6e^{2x} \end{aligned}$$

9. Differentiate $\dfrac{6x^2}{x^3+1}$.
ANSWER:

$$\frac{(x^3+1)(12x)-6x^2(3x^2)}{(x^3+1)^2} = \frac{12x^4+12x-18x^4}{(x^3+1)^2} = \frac{-6x^4+12x}{(x^3+1)^2} = \frac{6x(-x^3+2)}{(x^3+1)^2}$$

Questions and Solutions for Section 3.4

1. The table to the right gives values for functions f and g, and their derivatives.

x	-1	0	1	2	3
f	3	3	1	0	1
g	1	2	2.5	3	4
f'	-3	-2	-1.5	-1	1
g'	2	3	2	2.5	3

(a) Find $\dfrac{d}{dx}(f(x)g(x))$ and $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right)$ at $x=-1$.

(b) Find $\dfrac{d}{dx}f(g(x))$ and $\dfrac{d}{dx}g(f(x))$ at $x=0$.

ANSWER:

(a)

$$\frac{d}{dx}(f(x)g(x))\Big|_{x=-1} = f(x)g'(x)+f'(x)g(x)\Big|_{x=-1} = 3(2)+(-3)1 = 3.$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)\Big|_{x=-1} = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}\Big|_{x=-1} = \frac{(-3)(1)-(3)(2)}{1^2} = -9.$$

(b)

$$\frac{d}{dx}(f(g(x)))\Big|_{x=0} = f'(g(x))g'(x)\Big|_{x=0} = f'(2)(3) = -1(3) = -3.$$

$$\frac{d}{dx}(g(f(x)))\Big|_{x=0} = g'(f(x))f'(x)\Big|_{x=0} = g'(3)(-2) = 3(-2) = -6.$$

2. A table of values for a function F near $x=3$ and tables of values for a function G near $x=3$ and near $x=7$ are given below.

x	2.9	3.0	3.1
$F(x)$	6.7	7.0	7.3

x	2.9	3.0	3.1
$G(x)$	5.2	5.0	4.8

x	6.9	7.0	7.1
$G(x)$	1.95	2.0	2.05

(a) Find $F'(3)$, $G'(3)$, $G'(7)$.
(b) If $H(x) = F(x)G(x)$, find $H'(3)$.
(c) If $H(x) = F(x)/G(x)$, find $H'(3)$.
(d) If $H(x) = G(F(x))$, find $H'(3)$.

ANSWER:

(a)

$$F'(3) \approx \frac{F(3.1) - F(3.0)}{3.1 - 3.0} = \frac{7.3 - 7.0}{0.1} = 3$$
$$G'(3) \approx \frac{G(3.1) - G(3.0)}{3.1 - 3.0} = \frac{4.8 - 5.0}{0.1} = -2$$
$$G'(7) \approx \frac{G(7.1) - G(7.0)}{7.1 - 7.0} = \frac{2.05 - 2.00}{0.1} = \frac{1}{2}$$

(b) $H'(3) = F'(3)G(3) + F(3)G'(3) = 3 \cdot 5 + 7 \cdot (-2) = 1$.
(c) $H'(3) = \frac{F'(3)G(3) - G'(3)F(3)}{G^2(3)} = \frac{29}{25}$.
(d) $H'(3) = G'(F(3))F'(3) = G'(7)F'(3) = \frac{1}{2} \cdot 3 = \frac{3}{2}$.

3. The volume of a certain tree is given by $V = \frac{1}{12\pi}C^2h$, where C is the circumference of the tree at the ground level and h is the height of the tree. If C is 5 feet and growing at the rate of 0.2 feet per year, and if h is 22 feet and is growing at 4 feet per year, find the rate of growth of the volume V.

ANSWER:

We have
$$V = \frac{1}{12\pi}C^2h,$$
so
$$\frac{dV}{dt} = \frac{1}{12\pi}\left(2Ch\frac{dC}{dt} + C^2\frac{dh}{dt}\right).$$
Since we are given that $C = 5$, $\frac{dC}{dt} = 0.2$, $h = 22$ and $\frac{dh}{dt} = 4$, we get:
$$\frac{dV}{dt} = \frac{1}{12\pi}(2 \cdot 5 \cdot 22(0.2) + 25 \cdot 4) \approx 3.82 \text{ ft}^3/\text{yr}.$$

4. Let $f(x)$ and $g(x)$ be two functions. Values of $f(x)$, $f'(x)$, $g(x)$, and $g'(x)$ for $x = 0$, 1, and 2 are given in the table below. Use the information in the table to answer the questions that follow.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	1	−1	2	5
1	−1	2	4	0
2	7	3	11	0.5

(a) If $H(x) = e^{f(x)} + \pi x$, then $H'(0) =$
  (i) $\frac{1}{e} + \pi$
  (ii) $e^x + \pi x$
  (iii) $e + \pi$
  (iv) e
  (v) $\pi - e$
(b) If $J(x) = [f(x)]^2$, then $J'(1) =$
  (i) 1

(ii) -2
(iii) 4
(iv) -4
(v) 2

(c) If $K(x) = f(g(x))$, then $K'(0) =$
(i) 15
(ii) 35
(iii) -5
(iv) -1
(v) 7

ANSWER:

(a) $H'(x) = e^{f(x)} \cdot f'(x) + \pi$
$H'(0) = e^{f(0)} \cdot f'(0) + \pi = e^1 \cdot (-1) + \pi$
(e) $\pi - e$

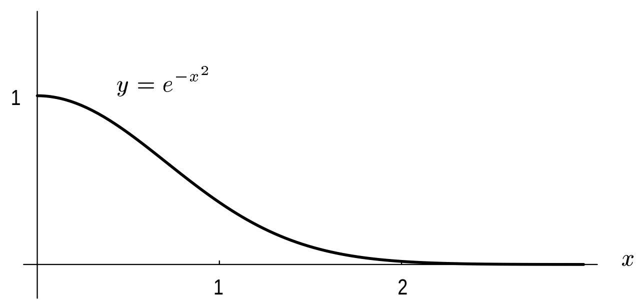
(b) $J'(x) = 2f(x) \cdot f'(x)$
$J'(1) = 2f(1) \cdot f'(1)$
(d) -4

(c) $K'(x) = f'(g(x)) \cdot g'(x)$
$K'(0) = f'(g(0)) \cdot g'(0)$
$= f'(2) \cdot g'(0)$ (a) 15
$= 3 \cdot 5 = 15$

5. (a) What is the instantaneous rate of change of the function $f(x) = e^{-x^2}$ at $x = 0$? at $x = 1$? at $x = 2$?
(b) Use the information from part (a) to sketch the graph of the function for $x \geq 0$.

ANSWER:

(a) $f'(x) = -2xe^{-x^2}$; $f'(0) = 0$; $f'(1) = -2e^{-1}$; $f'(2) = -4e^{-4}$
(b)



6. (a) Find the equation of the tangent line to $f(x) = e^{-3x}$ at $x = 2$.
(b) Use it to approximate the value of $f(2.2)$.
(c) Find the point where the tangent line crosses the x-axis.

ANSWER:

(a) Since $f'(x) = -3e^{-3x}$, $f'(2) = -3e^{-6}$. Moreover, $f(2) = e^{-6}$. The tangent line to the graph of f at $x = 2$ goes through $(2, e^{-6})$ and has slope $-3e^{-6}$. The equation of the tangent line is then

$$y - e^{-6} = -3e^{-6}(x - 2)$$
$$y = -3e^{-6}x + 7e^{-6}$$

(b) $f(2.2) \approx -3e^{-6}(2.2) + 7e^{-6} = 0.4e^{-6} = 0.00099$.
(c) The tangent line crosses the x-axis when $y = 0$; that is, when $x = 7/3$.

7. Find a point on the graph of $y = e^{3x}$ at which the tangent line passes through the origin.

ANSWER:

Call the desired point (q, e^{3q}). Then the slope of the tangent line at this point is $3e^{3q}$, and the equation of the tangent line at (q, e^{3q}) is $y - e^{3q} = 3e^{3q}(x - q)$. We know that this line passes through $(0, 0)$, so we have $-e^{3q} = 3e^{3q}(-q)$; $q = \frac{1}{3}$. So the desired point is $\left(\frac{1}{3}, e\right)$

8. Find the derivatives
   (a) $h(x) = (2x^3 + e^x)^3$
   (b) $g(x) = \sqrt{2x^3 + e^x}$
   (c) $k(x) = \dfrac{(2x^3 + e^x)^3}{\sqrt{2x^3 + e^x}}$

   ANSWER:
   (a) $h'(x) = 3(2x^3 + e^x)^2(6x^2 + e^x)$
   (b) $g(x) = (2x^3 + e^x)^{1/2}$
   $g'(x) = \dfrac{1}{2}(2x^3 + e^x)^{-1/2}(6x^2 + e^x)$
   (c) $k(x) = (2x^3 + e^x)^{5/2}$
   $k'(x) = (2x^3 + e^x)^3\left(-\dfrac{1}{2}\right)(2x^3 + e^x)^{-3/2}(6x^2 + e^x) + (2x^3 + e^x)^{-1/2}(3)(2x^3 + e^x)^2(6x + e^x)$
   $= \dfrac{5}{2}(6x^2 + e^x)(2x^3 + e^x)^{3/2}.$

9. Find the derivatives:
   (a) $f(x) = 2^{(4x-5)}$
   (b) $g(x) = \sqrt{e^x + e^{x^2}}$

   ANSWER:
   (a) $f'(x) = 4\ln 2 \cdot 2^{4x-5}$
   (b) $g(x) = \left(e^x + e^{x^2}\right)^{1/2}$
   $g'(x) = \dfrac{1}{2}\left(e^x + e^{x^2}\right)^{-1/2}\left(e^x + 2xe^{x^2}\right)$

10. Find the first three derivatives of $g(x) = (ax^2 + b)^2$. Assume that a and b are constants.
    ANSWER:

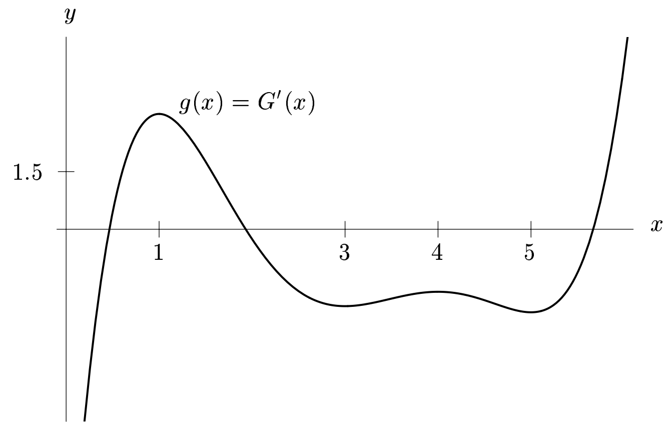
$$
\begin{aligned}
g'(x) &= 2(ax^2 + b)(2ax) = 4ax(ax^2 + b)\\
g''(x) &= 4ax(2ax) + (ax^2 + b)4a = 12a^2x^2 + 4ab\\
g'''(x) &= 24a^2x
\end{aligned}
$$

Questions and Solutions for Section 3.5

1. Find the derivatives:
   (a) $\dfrac{d}{dt}e^{2(t-1)}$
   (b) $\dfrac{d}{d\theta}\left(\theta\sin(\theta^2)\right)$

   ANSWER:
   (a) $\dfrac{d}{dt}e^{2(t-1)} = 2e^{2(t-1)}$
   (b) $\dfrac{d}{d\theta}\left(\theta\sin(\theta^2)\right) = \sin(\theta^2) + \theta\left(2\theta\cos(\theta^2)\right)\sin(\theta^2) + 2\theta^2\cos(\theta^2)$

2. (a) Find a function $F(x)$ such that $F'(x) = x^4 + \sin x$ and $F(0) = 5$.
   (b) Sketch the graph of a function $G(x)$ whose *derivative* $G'(x) = g(x)$ has the graph drawn below.

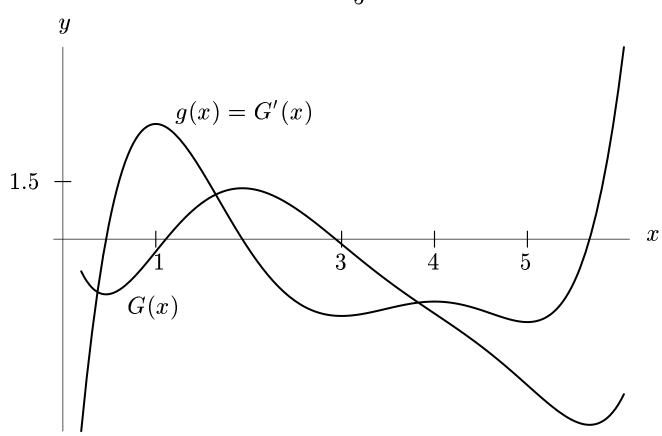


ANSWER:

(a) The anti-derivative of $F'(x) = x^4 + \sin x$ is $\frac{x^5}{5} - \cos x + C$. Since $F(0) = 5$, $\frac{0^5}{5} - \cos(0) + C = 5$, and $C = 6$. So,

$$F(x) = \frac{x^5}{5} - \cos x + 6.$$

(b)



3. A particle moves in such a way that $x(t) = 4t^2 + 7\sin t$.
   (a) What is the instantaneous rate of change at $t = 0$?
   (b) What is the instantaneous rate of change at $t = \frac{\pi}{2}$?
   (c) What is the average rate of change between $t = 0$ and $t = \frac{\pi}{2}$?

   ANSWER:

   (a) The derivative is $x'(t) = 8t + 7\cos t$, so the instantaneous rate of change at $x = 0$ is $x'(0) = 7$.
   (b) $x'(\frac{\pi}{2}) = 4\pi$.
   (c) The average rate of change over $\left[0, \frac{\pi}{2}\right]$ is given by

$$\frac{x\left(\frac{\pi}{2}\right) - x(0)}{\frac{\pi}{2} - 0} = \frac{\left(\pi^2 + 7\right) - 0}{\frac{\pi}{2}} = \frac{2\left(\pi^2 + 7\right)}{\pi} \approx 10.74.$$

4. Differentiate each of the following:
   (a) $f(x) = x^{\frac{3}{4}} - x^{\frac{4}{3}} + x^{-\frac{4}{3}}$
   (b) $f(w) = \sqrt{w^2 + 5}$
   (c) $f(z) = e^{z^2}/(z^2 - 3)$
   (d) $f(x) = xe^{-x}$
   (e) $f(\theta) = \sin(2\theta^3 + 1)$
   (f) $f(x) = (x + \sin x)^{\pi}$

   ANSWER:

   (a) $f'(x) = \frac{3}{4}x^{-\frac{1}{4}} - \frac{4}{3}x^{\frac{1}{3}} - \frac{4}{3}x^{-\frac{7}{3}}$

(b) $f'(x) = x(-e^{-x}) + e^{-x} = e^{-x}(1-x)$
(c) $f'(w) = \frac{1}{2}(w^2+5)^{-\frac{1}{2}} \cdot 2w = \dfrac{w}{\sqrt{w^2+5}}$
(d) $f'(\theta) = \cos(2\theta^3+1) \cdot 6\theta^2 = 6\theta^2\cos(2\theta^3+1)$
(e) $f'(z) = \dfrac{(z^2-3)(2ze^{z^2}) - 2z(e^{z^2})}{(z^2-3)^2} = \dfrac{2z(z^2-4)e^{z^2}}{(z^2-3)^2}$
(f) $f'(x) = \pi(x+\sin x)^{\pi-1}(1+\cos x)$

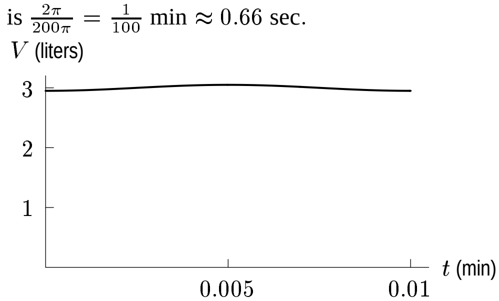
5. Find the equation of the tangent line to the curve given by $f(x) = x\sin x$ at the point $x = \pi/4$.
ANSWER:
$f'(x) = x\cos x + \sin x$; $f'(\frac{\pi}{4}) = \frac{\pi}{4} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}(\frac{\pi}{4}+1)$. Since $f(\frac{\pi}{4}) = \frac{\pi\sqrt{2}}{8}$, the equation of the line is $y - \frac{\pi\sqrt{2}}{8} = \frac{\sqrt{2}}{2}(\frac{\pi}{4}+1)(x-\frac{\pi}{4})$.

6. Consider the two functions $f(x) = -\cos^2(x)$ and $g(x) = \sin^2(x)$.
(a) Show that $f' = g'$.
(b) Use part (a) to derive the famous trigonometric identity $1 = \sin^2(x) + \cos^2(x)$.
[Hint: what can you conclude about two functions whose derivatives are the same?]

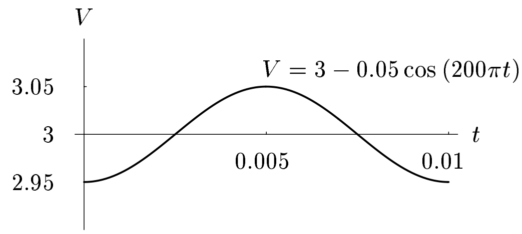
ANSWER:
(a) $\dfrac{d}{dx}(-\cos^2 x) = -2\cos x(-\sin x) = 2\cos x \sin x$, and $\dfrac{d}{dx}\sin^2 x = 2\sin x\cos x$. So,
$$f' = g'.$$
(b) Since these functions have the same derivatives, they must differ by only a constant. So $-\cos^2 x = \sin^2 x + C$. Since this holds for all values of x, set $x = 0$. Then $-\cos^2(0) = \sin^2(0) + C$ and $C = -1$. So $\sin^2 x + \cos^2 x = 1$.

7. When hyperventillating, a person breathes in and out extremely rapidly. A spirogram is a machine that draws a graph of the volume of air in a person's lungs as a function of time. During hyperventillation, the spirogram trace might be represented by
$$V = 3 - 0.05\cos(200\pi t)$$
where V is the volume of the lungs in liters and t is the time in minutes.
(a) What are the maximum and minimum volumes of air in the lungs?
(b) What is the period of this function?
(c) Sketch the graph of one period of this function, starting at $t = 0$. Put scales on the V and t axes.
(d) Find the maximum rate (in liters/minute) of flow of air during inspiration (i.e. breathing in). This is called the *peak inspiratory flow*.
(e) Find the average rate of flow of air during inspiration. This is called the *mean inspiratory flow*.

ANSWER:
(a) Since $\cos(200\pi t)$ varies between -1 and 1, V varies between 2.95 and 3.05.
(b) The period of this function is $\frac{2\pi}{200\pi} = \frac{1}{100}$ min ≈ 0.66 sec.
(c)



The following graph represents the same information, but it is shifted up and the V scale is stretched to emphasize the change in the curve with time.



(d) The rate of flow is $\frac{dV}{dt} = 10\pi \sin(200\pi t)$, whose maximum is 10π liters/minute, exactly halfway through inspiration.
(e) The *mean inspiratory flow* is

$$\frac{\int_0^{0.005} 10\pi \sin(200\pi t)\,dt}{0.005} = 20 \text{ liters/minute.}$$

8. Find two different functions $G_1(x)$ and $G_2(x)$ that satisfy $G'(x) = -\cos(3x)$.

ANSWER:
Since $G'(x)$ is of the form $-\cos u$, we can make an initial guess that

$$G(x) = -\sin(3x).$$

Then $G'(x) = -3\cos(3x)$, so we're off by a factor of 3 and get

$$G_1(x) = -\frac{1}{3}\sin(3x)$$

We can also add any constant to get $G_2(x)$. For example, $G_2(x) = -\frac{1}{3}\sin(3x) + \pi$.

9. Find the derivative of the following functions:
(a) $f(w) = \sin(2w^2) + \cos(2w^2)$
(b) $g(x) = \cos(\sin 2x)$
(c) $h(z) = e^{\cos z} + e^{\sin z}$

ANSWER:
(a) $f'(w) = 4w\cos(2w^2) - 4w\sin(2w^2)$
(b) $g'(x) = -\sin(\sin 2x)(2\cos 2x) = -2\cos 2x \sin(\sin 2x)$
(c) $h'(z) = -\sin z e^{\cos z} + \cos z e^{\sin z}$

10. Is the graph of $f(w) = \cos(w^6)$ increasing or decreasing when $w = 5$? Is it concave up or concave down?

ANSWER:
We begin by taking the derivative of $f(w)$ and evaluating at $w = 5$;

$$f'(w) = -6w^5 \sin(w^6)$$

Evaluating $\sin(5^6) = \sin(15,625)$ on a calculator, we see that $\sin(15,625) < 0$, so we know that $-6w^5\sin(w^6) > 0$ when $w = 5$, and therefore the function is increasing.

Next, we take the second derivative and evaluate it at $w = 5$;

$$f''(w) = \underbrace{-30\sin w^6 \cdot w^4}_{\text{negative}} - \underbrace{36\cos w^6 \cdot w^{10}}_{\text{negative}}$$

From this we see $f''(w) < 0$, thus the graph is concave down.

Questions and Solutions for Section 3.6

1. If P dollars is invested at an annual interest rate of $r\%$, then at t years this investment grows to F dollars, where

$$F = P\left(1 + \frac{r}{100}\right)^t.$$

Find $\frac{dF}{dr}$. (Assume P and t are constant.) In terms of money, what does this derivative tell you?

ANSWER:

$$\begin{aligned}\frac{dF}{dr} &= \frac{d}{dr}P(1+\frac{r}{100})^t \\ &= Pt(1+\frac{r}{100})^{t-1}\frac{d}{dr}\left(1+\frac{r}{100}\right) \quad \text{(using the chain rule)} \\ &= \frac{Pt}{100}\left(1+\frac{r}{100}\right)^{t-1}\end{aligned}$$

The derivative tells me how much extra money I will get for a small increase in the interest rate.

2. Find the derivatives of the following functions.

(a) $f(x) = x\ln x - x$

(b) $g(t) = \sin 3t - \cos 5t$

(c) $R(p) = \dfrac{e^p}{1+e^p}$

(d) $H(x) = \sqrt{1-x^2}$

ANSWER:

(a)

$$\begin{aligned}f(x) &= x\ln x - x \\ f'(x) &= \ln x + x\cdot\frac{1}{x} - 1 \\ &= \ln x + 1 - 1 \\ &= \ln x\end{aligned}$$

(b)

$$\begin{aligned}g(t) &= \sin 3t - \cos 5t \\ g'(t) &= 3\cos 3t + 5\sin 5t\end{aligned}$$

(c)

$$\begin{aligned}R(p) &= \frac{e^p}{1+e^p} \\ R'(p) &= \frac{e^p(1+e^p) - e^pe^p}{(1+e^p)^2} \\ &= \frac{e^p + e^{2p} - e^{2p}}{(1+e^p)^2} = \frac{e^p}{(1+e^p)^2}\end{aligned}$$

(d)

$$\begin{aligned}H(x) &= \sqrt{1-x^2} \\ H'(x) &= \frac{1}{2}(1-x^2)^{-\frac{1}{2}}\cdot(-2x) \\ &= \frac{-x}{\sqrt{1-x^2}}\end{aligned}$$

3. Find $f'(x)$

(a) $f(x) = e^{3x}\cos 5x$

(b) $f(x) = \sin\sqrt{\ln x + 7}$

(c) $f(x) = \dfrac{x+1}{x^2+3}$

ANSWER:

(a) $f'(x) = 3e^{3x}\cos 5x + 5e^{3x}(-\sin 5x) = e^{3x}(3\cos 5x - 5\sin 5x)$.

(b) $f'(x) = \cos(\sqrt{\ln x + 7})\cdot\dfrac{1}{2}\dfrac{1}{\sqrt{\ln x + 7}}\cdot\dfrac{1}{x} = \dfrac{\cos\sqrt{\ln x + 7}}{2x\sqrt{\ln x + 7}}$.

(c) $f'(x) = \dfrac{(x^2+3) - 2x(x+1)}{(x^2+3)^2} = \dfrac{-x^2 - 2x + 3}{(x^2+3)^2}$.

4. Perform the indicated differentiation. Please do not simplify your answers!

(a) $\dfrac{d}{dx}\left[\dfrac{x^2+1}{x^2-1}\right]$

(b) $\dfrac{d}{dx}[e^x\sin x]$

(c) $\dfrac{d}{dx}\left[4e^{(3x^2+7x)}\right]$

(d) $\dfrac{d}{dx}[\arctan x + x - \sqrt{x}]$

(e) $\dfrac{d}{dx}\left[(\ln x)^2 - \ln(x^2)\right]$

ANSWER:

(a) $\dfrac{d}{dx}\left[\dfrac{x^2+1}{x^2-1}\right] = \dfrac{2x(x^2-1) - 2x(x^2+1)}{(x^2-1)^2}$

(b) $\dfrac{d}{dx}[e^x\sin x] = e^x\sin x + e^x\cos x$

(c) $\dfrac{d}{dx}\left[4e^{(3x^2+7x)}\right] = 4(6x+7)e^{(3x^2+7x)}$

(d) $\dfrac{d}{dx}[\arctan x + x - \sqrt{x}] = \dfrac{1}{1+x^2} + 1 - \dfrac{1}{2\sqrt{x}}$

(e) $\dfrac{d}{dx}[(\ln x)^2 - \ln(x^2)] = \dfrac{2\ln x}{x} - \dfrac{2}{x}$

5. (a) Find an equation for the tangent line to the curve, $y = \ln x$, which passes through the origin. [Hint: Make a sketch.]
   (b) Consider the equation $\ln x = mx$ where m is some constant (positive, negative, or zero). For which values of m will this equation have no solution? For which values of m will this equation have one solution? For which values of m will this equation have two solutions?
   [Note: If you were unable to answer part (a), you may refer to the slope of the line as "m_0".]
   ANSWER:

(a) Slope of tangent $= \dfrac{y_0}{x_0}$

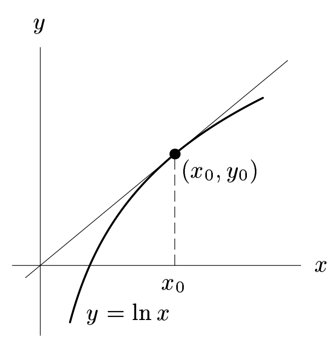
Slope of $y = \ln x$ at x_0 is $\dfrac{1}{x_0}$

These must be the same: $\dfrac{y_0}{x_0} = \dfrac{1}{x_0}$

$\underline{y_0 = 1} = \ln x_0 \quad \Rightarrow \quad \underline{x_0 = e}$

Slope of tangent $= \dfrac{y_0}{x_0} = \dfrac{1}{e}$

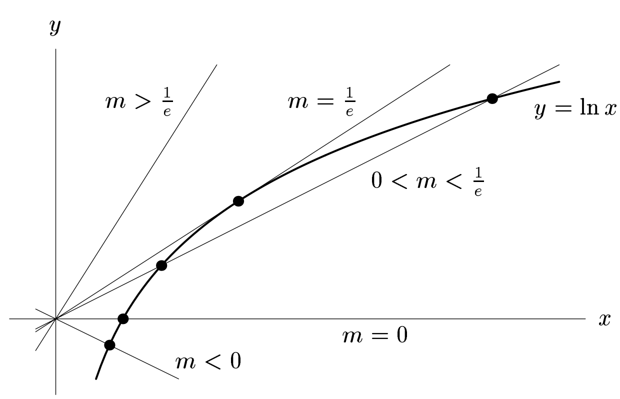
Tangent: $y = \dfrac{1}{e} \cdot x$



(b) $m > \dfrac{1}{e}$ (no solution)

$m = \dfrac{1}{e}$ and $m \le 0$ (one solution)

$0 < m < \dfrac{1}{e}$ (two solutions)



6. Find, without simplifying your answers:

(a) $\frac{d}{dx}\left(\ln(x^2e^x)\right)$ (b) $\frac{d}{d\theta}\left(\frac{\cos\theta}{\sin\theta}\right)$ (c) $\frac{d}{dz}\left(\sqrt{1+2^{3z}}\right)$

ANSWER:

(a) $\frac{d}{dx}(\ln(x^2e^x)) = \frac{d}{dx}\left(\ln(x^2)+\ln(e^x)\right) = \frac{d}{dx}(2\ln x + x) = \frac{2}{x}+1$

(b) $\frac{d}{d\theta}\left(\frac{\cos\theta}{\sin\theta}\right) = \frac{-\sin\theta\cdot\sin\theta-\cos\theta\cos\theta}{\sin^2\theta} = -\frac{1}{\sin^2\theta}$

(c) $\frac{d}{dz}\left(\sqrt{1+2^{3z}}\right) = \frac{d}{dz}\left(\sqrt{1+e^{3z\ln 2}}\right) = \frac{3\cdot\ln 2\cdot 2^{3z}}{2\sqrt{1+2^{3z}}}$

7. Let a be a positive constant (i.e., $a>0$). The equation

$$a^x = 1+x$$

has the solution $x=0$, for all a. Are there any solutions for $x>0$? How does your answer depend on the value of a? You may explore with the computer or calculator by trying various different values of a to help answer this question, and you will get partial credit for an answer which simply reports on the results of this exploration, but to receive full credit you must include an exact answer with justification.

ANSWER:

The derivative of a^x is $(\ln a)a^x$, so the slope of a^x at $x=0$ is $\ln a$, and the slope increases without bound after that. The graph of $y=1+x$ is a straight line with slope 1. Hence if $\ln a \geq 1$, the graph of a^x is always above the graph of $1+x$, since it is always increasing at a greater rate (see Figure 3.6.50). If $\ln a < 1$, then a^x starts out with slope less than the slope of $y=1+x$ but eventually overtakes it (see Figure 3.6.51), so there is a solution to $a^x = 1+x$ with $x>0$.

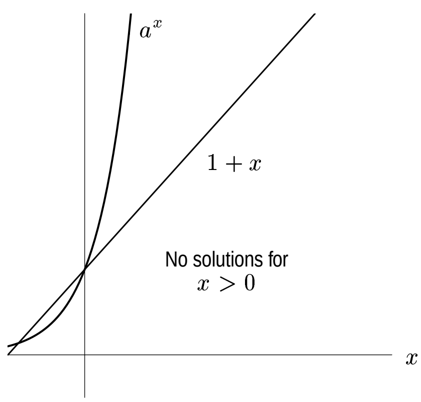


Figure 3.6.50: $a=10$

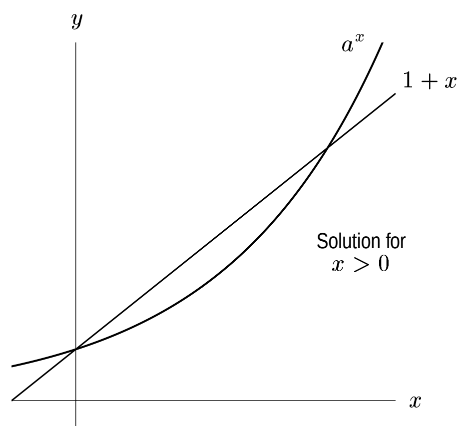


Figure 3.6.51: $a=1.5$

Now, $\ln a > 1$ if $a>e$ and $\ln a < 1$ if $a<e$. Thus the equation $a^x = 1+x$ has a solution for $x>0$ whenever $a<e$.

8. Find the derivatives of each of the following functions.

(a) $f(x) = \sin(2x)\cdot\sin(3x)$

(b) $f(x) = e^{-(1-x)^2}$

(c) $f(x) = \frac{1+x}{2+3x+4x^2}$

(d) $f(x) = \ln(\cos x)$

ANSWER:

(a) $f'(x) = 3\cos(3x)\sin(2x) + 2\cos(2x)\sin(3x)$

(b) $f'(x) = 2(1-x)e^{-(1-x)^2}$

(c) $f'(x) = -\frac{1+8x+4x^2}{(2+3x+4x^2)^2}$

(d) $f'(x) = -\tan x$

9. Differentiate the following functions

(a) $f(x) = \sqrt{x} + \dfrac{1}{\sqrt[3]{x}}$.

(b) $g(t) = 2^{2(t-1)}$.

(c) $h(\theta) = \theta \sin(\theta^2)$.

(d) $f(y) = \ln \dfrac{1+y}{1-y}$.

ANSWER:

(a) $f(x) = \sqrt{x} + \dfrac{1}{\sqrt[3]{x}}$

$$\begin{aligned}\frac{df}{dx} &= \frac{d}{dx}\left(\sqrt{x} + \frac{1}{\sqrt[3]{x}}\right)\\ &= \frac{d}{dx}(x^{\frac{1}{2}}) + \frac{d}{dx}(x^{-\frac{1}{3}})\\ &= \frac{1}{2}x^{-\frac{1}{2}} + \left(-\frac{1}{3}\right)x^{-\frac{4}{3}}\\ &= \frac{1}{2\sqrt{x}} - \frac{1}{3x\sqrt[3]{x}}\end{aligned}$$

(b) $g(t) = 2^{2(t-1)} = e^{2(t-1)\ln 2}$

By the chain rule,
$$\begin{aligned}\frac{dg}{dt} &= \frac{d}{dt}e^{2(t-1)\ln 2}\\ &= (2\ln 2)e^{2(t-1)\ln 2}\\ &= (2\ln 2)2^{2(t-1)}\\ &= 2^{2t-1}\ln 2\end{aligned}$$

(c) $h(\theta) = \theta \sin(\theta^2)$

By the product rule,
$$\begin{aligned}\frac{dh}{d\theta} &= \frac{d}{d\theta}\left(\theta \sin(\theta^2)\right)\\ &= \sin(\theta^2)\frac{d}{d\theta}(\theta) + \theta\frac{d}{d\theta}(\sin(\theta^2))\\ &= \sin(\theta^2) + \theta\frac{d}{d\theta}(\sin(\theta^2)) \quad \text{(using the chain rule)}\\ &= \sin\theta^2 + \theta(\cos\theta^2)\frac{d}{d\theta}\theta^2\\ &= \sin\theta^2 + \theta(\cos\theta^2)2\theta\\ &= 2\theta^2\cos\theta^2 + \sin\theta^2\end{aligned}$$

(d) $f(y) = \ln \dfrac{1+y}{1-y}$

$$\begin{aligned}\frac{df}{dy} &= \frac{d}{dy}\ln\left(\frac{1+y}{1-y}\right)\\ &= \frac{d}{dy}\ln(1+y) - \frac{d}{dy}\ln(1-y)\\ &= \frac{1}{1+y}\frac{d}{dy}(1+y) - \frac{1}{1-y}\frac{d}{dy}(1-y)\\ &= \frac{1}{1+y} + \frac{1}{1-y} = \frac{2}{1-y^2}\end{aligned}$$

10. What is the instantaneous rate of change of the function $f(x) = x\ln x$ at $x = 1$? at $x = 2$? What do these values suggest about the concavity of the function between 1 and 2?

ANSWER:

At any point x the derivative $f'(x)$ is $1\ln x + x\frac{1}{x} = 1 + \ln x$, so $f'(1) = 1$, and $f'(2) = 1 + \ln 2 > 1$. The derivative of f appears to be increasing between 1 and 2; this observation suggests that f is concave up on this interval.

11. A certain quantity of gas occupies a volume of 40 cm^3 at a pressure of 1 atmosphere. The gas expands without the addition of heat, so, for some constant k, its pressure, P, and volume, V, satisfy the relation

$$PV^{1.4} = k$$

(a) Find the rate of change of pressure with volume. Give units.
(b) The volume is increasing at 2 cm^3/min when the volume is 60 cm^3. At that moment, is the pressure increasing or decreasing? How fast?
(c) If the rate the volume increases doubles, how does that change the answers to (a) and (b)?

ANSWER:

(a) Since $P = 1$ when $V = 40$, we have

$$k = 1 \cdot \left(40^{1.4}\right) = 174.94$$

Thus we have

$$P = 174.94V^{-1.4}.$$

Differentiating gives

$$\frac{dP}{dV} = 174.94\left(-1.4V^{-2.4}\right) = -244.92V^{-2.4}\ \text{atm/cm}^3$$

(b) We are given that $\dfrac{dV}{dt} = 2$ cm^3/min when $V = 60$ cm^3.

Using the chain rule, we have

$$\begin{aligned}\frac{dP}{dt} &= \frac{dP}{dV}\cdot\frac{dV}{dt} = \left(-244.92V^{-2.4}\frac{\text{atm}}{\text{cm}^3}\right)\left(2\frac{\text{cm}^3}{\text{min}}\right)\\ &= -244.92\left(60^{-2.4}\right)(2)\frac{\text{atm}}{\text{min}} = -0.0265\frac{\text{atm}}{\text{min}}.\end{aligned}$$

Thus, the pressure is decreasing at 0.0265 atm/min.

Questions and Solutions for Section 3.7

1. Find the indicated derivatives.

(a) $f(x) = 4x^3 - 3x^2 + 2x - 8$. Find $f'(x)$.
(b) $y = u\sqrt{u+1}$, $u = 2x^2 + 3$. Find $\dfrac{dy}{du}$, and $\dfrac{dy}{dx}$.
(c) $y = e^{\cos^2\theta}$. Find $\dfrac{dy}{d\theta}$.
(d) $p(x) = \ln((x-a)(x-b)(x-c))$. a, b, c are constants. Find $p'(x)$.
(e) $x = \dfrac{\sin t}{1+\cos t}$. Find $\dfrac{dx}{dt}$.
(f) $(x+y)^2 = (2x+1)^3$. Find $\dfrac{dy}{dx}$. (Use implicit differentiation. Your answer will involve both x and y.)

ANSWER:

(a) $f'(x) = 12x^2 - 6x + 2$.
(b) $\dfrac{dy}{du} = u\left(\dfrac{1}{2}(u+1)^{-\frac{1}{2}}\right) + (u+1)^{\frac{1}{2}} = (u+1)^{-\frac{1}{2}}\left(\dfrac{3}{2}u + 1\right)$. Since $u = 2x^2 + 3$, and $\dfrac{du}{dx} = 4x$,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (2x^2+4)^{-\frac{1}{2}}\left(\frac{3}{2}(2x^2+3)+1\right)(4x).$$

(c) $\dfrac{dy}{d\theta} = -2\cos\theta\sin\theta e^{\cos^2\theta}$.

(d) $p(x) = \ln((x-a)(x-b)(x-c)) = \ln(x-a) + \ln(x-b) + \ln(x-c)$. Hence,

$$p'(x) = \frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c}$$

(e) $\dfrac{dx}{dt} = \dfrac{\cos t(1+\cos t) - \sin t(-\sin t)}{(1+\cos t)^2} = \dfrac{\cos t + \cos^2 t + \sin^2 t}{(1+\cos t)^2} = \dfrac{1}{1+\cos t}$.

(f)

$$\begin{aligned} 2(x+y)\left(1+\frac{dy}{dx}\right) &= 3(2x+1)^2(2) \\ (x+y) + \frac{dy}{dx}(x+y) &= 3(2x+1)^2 \\ \frac{dy}{dx} &= \frac{3(2x+1)^2 - x - y}{x+y} \end{aligned}$$

2. Suppose that $xy^2 + \sin y + x^3 = 8$.

(a) Find $\dfrac{dy}{dx}$.

(b) Give a table of estimates of values of y for x near 2. (Use $x = 1.98$, 1.99, 2.00, 2.01.)

ANSWER:

(a) We differentiate implicitly:

$$y^2 + x\left(2y\frac{dy}{dx}\right) + \cos y\frac{dy}{dx} + 3x^2 = 0$$

so

$$\frac{dy}{dx} = -\frac{3x^2 + y^2}{2xy + \cos y}$$

(b) At $(2,0)$, $\frac{dy}{dx} = -12$. Hence the line $y = -12x + 24$ is tangent to the given curve at $(2,0)$. Using this approximation we obtain the following table:

x	1.98	1.99	2.00	2.01
y	0.24	0.12	0.00	-0.12

These aren't very good estimates however; the actual values (rounded) are $0.18, 0.10, 0.00$ and -0.22.

3. Consider the following three equations:

(a) $y^2 - 2\cos x = 2$,

(b) $x\sin y + y = 2$,

(c) $\ln|y/(1-y)| = 0.71x + \ln 2$.

Assuming each of the above equations implicitly defines y as a function of x, find $\dfrac{dy}{dx}$ for each equation.

ANSWER:

(a)

$$\begin{aligned} y^2 - 2\cos x &= 2 \\ 2y\frac{dy}{dx} + 2\sin x &= 0 \\ \frac{dy}{dx} &= -\frac{\sin x}{y}. \end{aligned}$$

(b)

$$\begin{aligned} x\sin y + y &= 2 \\ \sin y + x\cos y\frac{dy}{dx} + \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{\sin y}{x\cos y + 1}. \end{aligned}$$

(c)

$$\begin{aligned}\ln\left|\frac{y}{1-y}\right| &= 0.71x+\ln 2\\ \ln|y|-\ln|1-y| &= 0.71x+\ln 2\\ \left(\frac{1}{y}+\frac{1}{1-y}\right)\frac{dy}{dx} &= 0.71\\ \frac{dy}{dx} &= 0.71y(1-y).\end{aligned}$$

4. The part of the graph of $\sin(x^2+y)=x$ that is near $(0,\pi)$ defines y as a function of x implicitly.
   (a) Is this function increasing or decreasing near 0? Explain how you know.
   (b) Does the graph of this function lie above or below its tangent line at $(0,\pi)$? Explain how you know.

   ANSWER:

   (a) To check whether the function increases or decreases near 0 we simply check the sign of $\frac{dy}{dx}$ near 0. Implicit differentiation of the function with respect to x gives

$$\begin{aligned}\cos(x^2+y)\left(2x+\frac{dy}{dx}\right) &= 1\\ \frac{dy}{dx} &= \frac{1}{\cos(x^2+y)}-2x\\ \left.\frac{dy}{dx}\right|_{x=0\ y=\pi} &= \frac{1}{\cos(0^2+\pi)}-2(0)=\frac{1}{-1}-0=-1.\end{aligned}$$

   In fact $\frac{dy}{dx}$ is negative near 0, so the function decreases near $(0,\pi)$.

   (b) We check concavity of the function at 0 by finding

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{1}{\cos(x^2+y)}-2x\right)\\ &= -\frac{1}{\cos^2(x^2+y)}\cdot(-\sin(x^2+y))\cdot\left(2x+\frac{dy}{dx}\right)-2\\ &= \frac{\sin(x^2+y)}{\cos^2(x^2+y)}\left(2x+\frac{1}{\cos(x^2+y)}-2x\right)-2\\ &= \frac{\sin(x^2+y)}{\cos^3(x^2+y)}-2\end{aligned}$$

   At $(0,\pi)$, $\frac{d^2y}{dx^2}=-2$, so the function is concave down here, and hence lies *below* its tangent line.

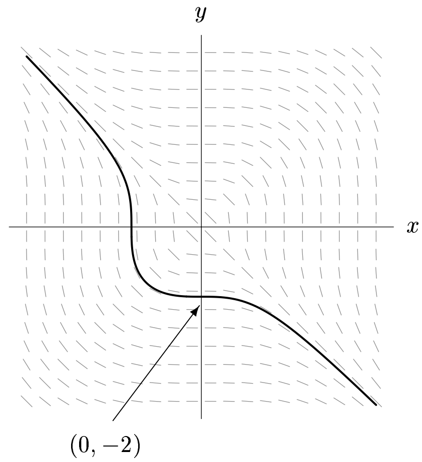
5. The purpose of this problem is to sketch the implicit function $x^3+y^3=-8$.
   (a) Find all the y-values for which $x=0$.
   (b) Calculate dy/dx.
   (c) Use parts (a) and (b), and a computer or calculator, to sketch this implicit function. Explain what program you used, and what you did, and copy the sketch of the implicit function from the computer or calculator to your paper.

   ANSWER:

   (a) If $x=0$, $y^3=-8$, so $y=-2$.

   (b) Using implicit differentiation, we find that $3x^2+3y^2\dfrac{dy}{dx}=0$, so $\dfrac{dy}{dx}=-\dfrac{x^2}{y^2}$.

(c)



The curve above was found by entering $\frac{dy}{dx} = -\frac{x^2}{y^2}$ into a program that draws slope fields and then getting the program to draw the solution curve passing through the point $(0, -2)$.

6. (a) Find the equations of the tangent lines to the circle $(x-2)^2 + (y+2)^2 = 20$ at the points where $x = 0$.
(b) Repeat part (a) for the points where $x = 6$.
ANSWER:
(a) If $x = 0$, then $(-2)^2 + (y+2)^2 = 20$

$$(y+2)^2 = 16$$

$$y = 2 \text{ or } y = -6$$

We find $\frac{dy}{dx}$ implicitly:

$$2(x-2) + 2(y+2)\frac{dy}{dx} = 0$$

$$2(y+2)\frac{dy}{dx} = -2(x-2)$$

$$\frac{dy}{dx} = \frac{-(x-2)}{y+2}$$

So the slope at (0,2) is $1/2$ and at (0,-6) is $-1/2$.

The tangent lines are $y = 0$.
(b) If $x = 6$, then $(6)^2 + (y+2)^2 = 20$

$$(y+2)^2 = 4$$

$$y = 0 \text{ or } y = -4$$

We find $\frac{dy}{dx}$ implicitly:

$$\frac{dy}{dx} = \frac{-(x-2)}{y+2}$$

So the slope at (6,0) is $\frac{-(6-2)}{0+2} = -2$ and at (6,-4) is $\frac{-(6-2)}{-4+2} = \frac{-4}{-2} = 2$.

The tangent lines are $y = -2(x-6)$ so $y = -2x + 12$ and $y + 4 = 2(x-6)$ so $y = 2x - 16$.

7. Find the equation of one line perpendicular to the tangent to the ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$ where $x=2$.
ANSWER:
Taking the derivatives implicitly, we get

$$\frac{2}{16}x+\frac{2}{25}y\frac{dy}{dx}=0$$
$$\frac{2}{25}y\frac{dy}{dx}=-\frac{2}{16}x$$
$$\frac{dy}{dx}=-\frac{25x}{16y}$$

At $x=2$,

$$\frac{(2)^2}{16}+\frac{y^2}{25}=1$$
$$\frac{y^2}{25}=\frac{3}{4}$$
$$y^2=\frac{75}{4}$$
$$y=\frac{\pm 5\sqrt{3}}{2}$$

At $\left(2,\frac{5\sqrt{3}}{2}\right)$, $\frac{dy}{dx}=\frac{-25(2)}{16(\frac{5\sqrt{3}}{2})}=-\frac{5}{4\sqrt{3}}$

The line perpendicular has slope $\frac{4\sqrt{3}}{5}$ and at $\left(2,\frac{5\sqrt{3}}{2}\right)$, the equation is

$$y+\frac{5\sqrt{3}}{2}=\frac{4\sqrt{3}}{5}(x-2)$$

8. Find the equation of the tangent line to $2x^2y=1$ at $(4,4)$.
ANSWER:
Taking the derivative implicitly, we get:

$$2x^2\frac{dy}{dx}+y\cdot 4x=0$$
$$\frac{dy}{dx}=\frac{-4xy}{2x^2}=\frac{-2y}{x}$$

At (4,4) the slope is -2 and the equation of the tangent line is $y-4=-2(x-4)$ or $y=-2x+12$.

9. Find the equation of the tangent line to $x^2=3y^2+y$ at $(2,-3)$.
ANSWER:
Taking the derivative implicitly, we get:

$$2x=6y\frac{dy}{dx}+\frac{dy}{dx}$$
$$2x=\frac{dy}{dx}(6y+1)$$
$$\frac{dy}{dx}=\frac{2x}{6y+1}$$

At (2,-3) the slope is $\frac{2(2)}{6(-3)+1}=\frac{-4}{17}$ and the equation of the tangent line is $y+3=-\frac{4}{17}(x-2)$ or $y=-\frac{4}{17}x-\frac{46}{17}$.

Questions and Solutions for Section 3.8

1. True or false? Give a reason for your answer. The equations $x=\cos(t^3)$, $y=\sin(t^3)$ parameterize a circle.
ANSWER:
True. Since $x^2+y^2=(\cos t^3)^2+(\sin t^3)^2=1$, all such points lie on a circle.

2. Which of the following diagrams represents the parametric curve $x = \sin t$, $y = \cos t$, $0 \le t \le \pi$?

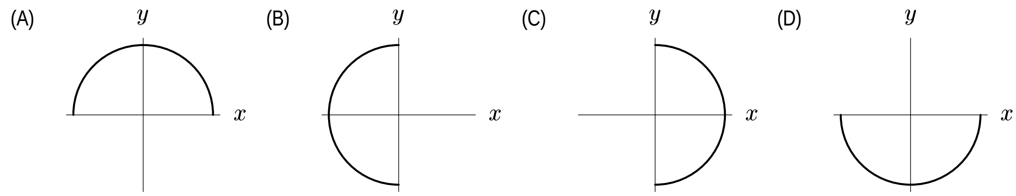


Figure 3.8.52

Justify your answer.
ANSWER:
The curve starts at $(\sin 0, \cos 0) = (0,1)$ and ends at $(\sin \pi, \cos \pi) = (0,-1)$, so it's either (B) or (C). Also, since $\sin t \ge 0$ for $0 \le t \le \pi$, $x \ge 0$ on the curve, so it must be (C).

3. Find parametric equations for a line through the points, $A = (-1,1)$ and $B = (2,3)$ so that the point A corresponds to $t = 0$ and the point B to $t = 6$.
ANSWER:

$$x = -1 + \frac{t}{2}$$
$$y = 1 + \frac{t}{3}$$

for $-\infty < t < \infty$.

4. A lady bug moves on the xy-plane according the the equations

$$x = 2t(t-6), \quad y = 1 - t,$$

(a) Does the lady bug ever stop moving? If so, when and where?
(b) Is the lady bug ever moving straight up or down? If so, when and where?
(c) Suppose that the temperature at a point (x, y) in the plane depends only on the y coordinate of the point and is equal to $4y^2$. Find the rate of change of the temperature of the lady bug at time t.
ANSWER:
(a) Differentiating gives
$$\frac{dx}{dt} = 4t - 12, \quad \frac{dy}{dt} = -1.$$
The lady bug comes to a stop whenever $dx/dt = dy/dt = 0$. Since dy/dt is never 0, the lady bug never comes to a stop.
(b) The lady bug is moving straight up or down if $dx/dt = 0$ but $dy/dt \ne 0$. Since $dx/dt = 0$ only when $t = 3$ and dy/dt is always negative, the lady buy is moving straight *down* when $t = 3$, or at the point $(-18, -2)$ in the xy-plane.
(c) Substituting $y = 1 - t$ into the expression for the temperature gives us $T(t) = 4(1-t)^2$. To find the rate of change of the lady bug's temperature at time t, we differentiate T with respect to t:
$$\text{Rate of change of temperature} = \frac{dT}{dt} = (-1)(8)(1-t) = 8(t-1).$$

5. The equations $x = \frac{4}{\pi}\cos(\pi t/180)$, $y = \frac{4}{\pi}\sin(\pi t/180)$ describe the motion of a particle moving on a circle. Assume x and y are in miles and t is in days.
(a) What is the radius of the circle?
(b) What is the period of the circular motion?
(c) What is the speed of the particle when it passes through the point $(-4/\pi, 0)$?

ANSWER:

(a)
$$r = \sqrt{x^2 + y^2} = \frac{4}{\pi} \text{ miles}$$

(b)

$$\text{period } T = \frac{2\pi}{\pi/180} = 360 \text{ days}$$

(c) Since

$$\frac{dx}{dt} = -\frac{\pi}{180}(\frac{4}{\pi}\sin(\frac{\pi t}{180})) = -\frac{\pi}{180}y$$

and

$$\frac{dy}{dt} = \frac{\pi}{180}(\frac{4}{\pi}\cos(\frac{\pi t}{180})) = \frac{\pi}{180}x$$

then at point $(-4/\pi, 0)$ the particle's speed is

$$v = \sqrt{(-\frac{\pi}{180}(0))^2 + (\frac{\pi}{180}(-\frac{4}{\pi}))^2}$$
$$= \frac{1}{45} \text{ miles per day.}$$

6. Find two parametric equations for the line passing through the points (4,1) and (-2,-6).
ANSWER:

$$\frac{(-2-4)}{1} = -6 \text{ and } \frac{(-6-1)}{1} = -7$$

So the equations are

$$x = 4 - 6t \quad y = 1 - 7t$$
$$\text{and } x = -2 - 6t \quad y = -6 - 7t$$

7. Describe in words the curve represented by the parametric equations $x = 2 - t^3$ and $y = 3 + t^3$.
ANSWER:
$t^3 = 2 - x$, so $y = 3 + 2 - x$ or $y = -x + 5$.
It is a straight line through the point (3,2) with slope -1.

8. Motion of a particle is given by

$$x = t^2 - 3t \quad y = 2t^3 - 3t$$

where t is time in minutes. Find the instantaneous speed of the particle.
ANSWER:
Instantaneous speed is defined to be

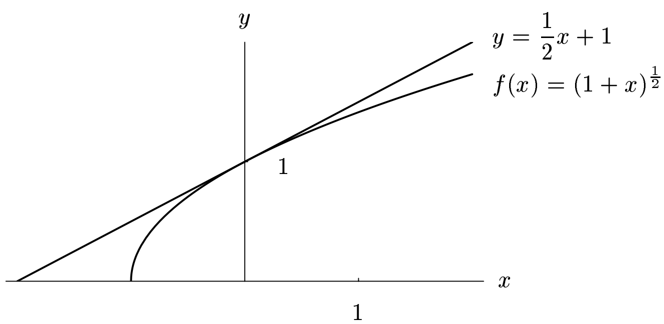
$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$
$$\frac{dx}{dt} = 2t - 3 \text{ and } \frac{dy}{dt} = 6t^2 - 3$$
$$\text{so } v = \sqrt{(2t-3)^2 + (6t^2-3)^2} = \sqrt{-12t + 36t^4 - 32t^2 + 18}$$

Questions and Solutions for Section 3.9

1. (a) Find the best linear approximation to $f(x) = (1+x)^{\frac{1}{2}}$ at $x = 0$.
(b) Use a calculator or a computer to plot $f(x)$ and your linear approximation. Describe what program you used, what you did, and how this confirms that your linear approximation is reasonable.
ANSWER:

(a) We know that $f(0) = 1$, and $f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$, so $f'(0) = \frac{1}{2}$. The best linear approximation to $f(x)$ at $x = 0$ is the tangent line with slope $\frac{1}{2}$ through the point $(0, 1)$, which is the line $y = \frac{1}{2}x + 1$.

(b) We plot the functions $f(x) = (1+x)^{1/2}$ and $g(x) = \dfrac{1}{2}x + 1$ to get the graph below. From the graph, we see that these two functions are very close near $x = 0$.



2. The graph of $f(x)$ is shown in Figure 3.9.53.

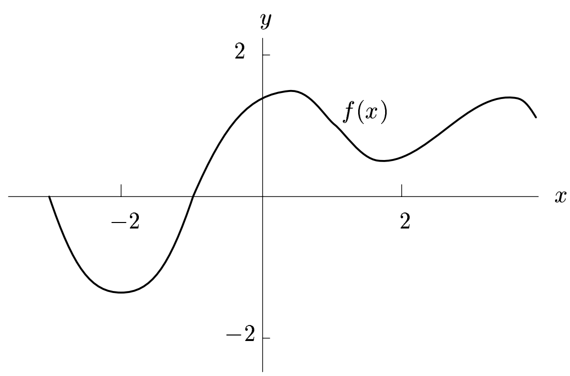


Figure 3.9.53

(a) List in increasing order (from smallest to largest)

$$f'(2), f(0), f'(-0.9), \text{ the number } 1, f''(0), f''(1).$$

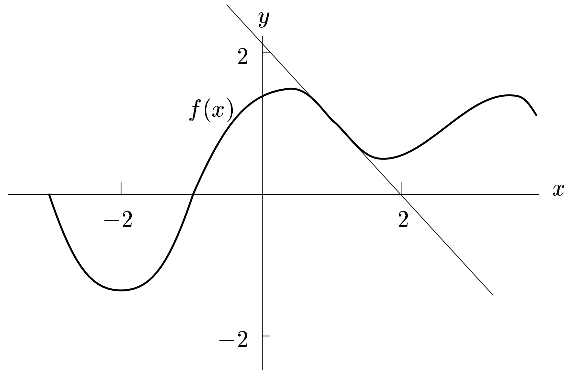
(b) Suppose we want to estimate $f(1.5)$ by using tangent line approximations at $x = 0, 1$ and 2. Which tangent line yields the best approximation?

ANSWER:

(a) $f'(2)$ is certainly positive, and appears to be about $\frac{1}{2}$. $f(0)$ is greater than 1, but less than 2. $f'(-0.9)$ is approximately 2. $f''(0)$ is negative. $f''(1)$ looks like it is about 0, since there is an inflection point near $x = 1$. So we have:

$$f''(0) < f''(1) < f'(2) < 1 < f(0) < f'(-0.9)$$

(b)



From the graph we see that $f(1.5)$ can be best approximated using the tangent line at $(1, f(1))$ because the tangent line at $(1, f(1))$ appears to pass through $(1.5, f(1.5))$. (The graph between $(1, f(1))$ and $(1.5, f(1.5))$ is almost linear.)

Questions and Solutions for Section 3.10

1. Find the limit:

$$\lim_{x\to 0} \frac{\sin x \cos x}{x}$$

ANSWER:
Since both the numerator and denominator have limit 0, we use L'Hopital's rule:

$$\lim_{x\to 0} \frac{\sin x \cos x}{x} = \lim_{x\to 0} \frac{\sin x(-\sin x) + \cos x(\cos x)}{1}$$
$$= \lim_{x\to 0} -\sin^2 x + \cos^2 x = 0 + 1 = 1$$

2. Find the limit:

$$\lim_{x\to\infty} \frac{5e^{-x} + 5x}{x}$$

ANSWER:
Since both the numerator and denominator have limit ∞, we use L'Hopital's rule:

$$\lim_{x\to\infty} \frac{5e^{-x} + 5x}{x} = \lim_{x\to\infty} \frac{-5e^{-x} + 5}{1} = 5$$

3. Which function dominates as $x \to \infty$, $e^{\frac{x}{100}}$ or x^{100}?
ANSWER:
The exponential dominates. After 100 applications of L'Hopital's rule

$$\lim_{x\to\infty} \frac{x^{100}}{e^{.01x}} = \frac{100x^{99}}{.01e^{.01x}} = \cdots = \lim_{x\to\infty} \frac{100!}{(.01)^{100}e^{.01x}} = 0$$

Review Questions and Solutions for Chapter 3

1. Differentiate:

(a) $f(x) = \dfrac{x^2+1}{x^2-1}$

(b) $f(x) = 2^x \tan x$

(c) $f(x) = \sqrt{(\ln x)^2 + 5}$

ANSWER:

(a) Use the quotient rule,

$$\frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right) = \frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2} = \frac{2x^3 - 2x - 2x^3 - 2x}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}.$$

(b) Since $\frac{d}{dx}(2^x) = 2^x \ln 2$, we can use the product rule:

$$\frac{d}{dx}(2^x \tan x) = 2^x \frac{1}{\cos^2 x} + 2^x(\ln 2)\tan x.$$

(c) Repeated application of the chain rule yields

$$\frac{d}{dx}\sqrt{(\ln x)^2 + 5} = \frac{1}{2\sqrt{(\ln x)^2+5}}\frac{d}{dx}\left((\ln x)^2 + 5\right) = \frac{(2\ln x)/x}{2\sqrt{(\ln x)^2+5}} = \frac{\ln x}{x\sqrt{(\ln x)^2+5}}$$

For Problems 2– 4, circle the correct answer(s) or fill in the blanks. No reasons need be given.

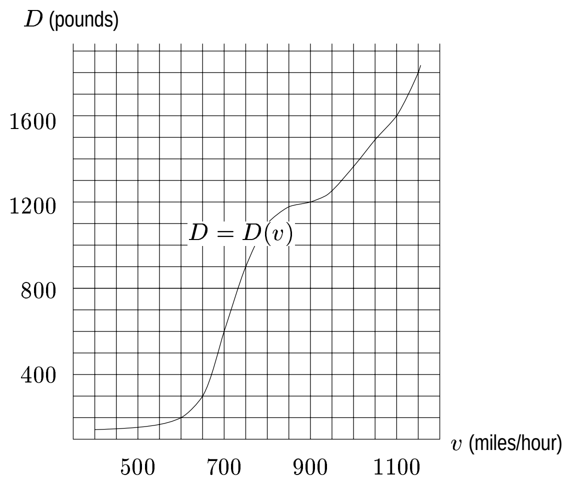
2. If $f(x) = (\ln 2)^x$, then $f'(2)$ is

(a) $2 \ln 2$
(b) $(\ln 2)^3$
(c) approximately -0.176
(d) approximately -0.173

ANSWER:
(c) $f'(x) = (\ln 2)^x \ln \ln 2$ so $f'(2) = (\ln 2)^2 \cdot \ln \ln 2 = -0.176$

3. The function $D(v)$ in the figure below gives the air resistance, or drag (in pounds), on a blunt object as a function of its velocity. (Notice that the curve rises sharply near $v = 700$ miles/hour, the speed of sound. This represents the "sound barrier.") If a blunt object traveling 725 miles/hour is accelerating at the constant rate of 7000 miles/hour^2, at approximately what rate (in pounds/hour) is the drag increasing at that moment? Circle the *closest* answer.

(a) 6000 lb/hr
(b) 8 lb/hr
(c) 700 lb/hr
(d) 60,000 lb/hr



ANSWER:
(d) Slope of curve at $V = 725$ is $\dfrac{dD}{dV} \approx 8$ lb/mph.

By chain rule $\dfrac{dD}{dt} = \dfrac{dD}{dV} \cdot \dfrac{dV}{dt} = 8 \cdot 7000 \approx 56,000.$

4. The slope of the tangent line to the curve $y \cos x + e^y = 7$ at $\left(\frac{\pi}{2}, \ln 7\right)$ is

(a) 0
(b) 7
(c) $\dfrac{(\ln 7)}{7}$
(d) $\dfrac{(7 + \ln 7)}{7}$
(e) None of the above.

ANSWER:
(c) $y' \cos x - y \sin x + e^y y' = 0$ so $y' = \dfrac{y \sin x}{\cos x + e^y} = \dfrac{\ln 7 \cdot \sin\left(\frac{\pi}{2}\right)}{\cos\frac{\pi}{2} + e^{\ln 7}} = \dfrac{\ln 7}{7}$

5. Find derivatives of the following functions. Do not simplify your answers.

(a) $\dfrac{(z^2+1)^2}{z}$
(b) $\sin(\theta \cos\theta)$
(c) $\dfrac{t}{e^t} + e^2$
(d) $\dfrac{\sqrt{x}}{x} - \sqrt{x^2+1}$

(e) $\arctan\left(\frac{92}{x}\right)$

ANSWER:

(a) $\frac{d}{dz}\left(\frac{z^4+2z^2+1}{z}\right) = \frac{d}{dz}\left(z^3+2z+\frac{1}{z}\right) = 3z^2+2-\frac{1}{z^2}$

(b) $\frac{d}{d\theta}(\sin(\theta\,\cos\theta)) = \cos(\theta\,\cos\theta)\cdot[\cos\theta - \theta\,\sin\theta]$

(c) $\frac{d}{dt}(te^{-t}+e^2) = e^{-t} - te^{-t}$

(d) $\frac{d}{dx}\left(x^{-\frac{1}{2}} - (x^2+1)^{\frac{1}{2}}\right) = -\frac{1}{2}x^{-\frac{3}{2}} - \frac{1}{2}(x^2+1)^{-\frac{1}{2}}\cdot 2x$

(e) $\frac{d}{dx}\left(\arctan\left(\frac{92}{x}\right)\right) = \frac{1}{1+\left(\frac{92}{x}\right)^2}\cdot\left(\frac{-92}{x^2}\right)$

6. Find derivatives of the following functions. You need not simplify:

(a) $\cos(x^2+1)$ (b) $\frac{e^{-x}}{1+x^3}$ (c) $2^e - \frac{2}{x} + \sqrt{1-x}$

ANSWER:

(a) $\frac{d}{dx}\cos(x^2+1) = -2x\sin(x^2+1)$.

(b)

$$\frac{d}{dx}\left(\frac{e^{-x}}{1+x^3}\right) = \frac{(1+x^3)\frac{d}{dx}(e^{-x}) - (e^{-x})\frac{d}{dx}(1+x^3)}{(1+x^3)^2}$$
$$= \frac{-e^{-x}(1+3x^2+x^3)}{(1+x^3)^2}.$$

(c) $\frac{d}{dx}\left(2^e - \frac{2}{x} + \sqrt{1-x}\right) = 2x^{-2} - \frac{1}{2}(1-x)^{-\frac{1}{2}}$.

7. Differentiate each of the following:

(a) $f(x) = (x+\sin x)^e$

(b) $g(x) = e^{(x+\sin x)}$

(c) $F(z) = \frac{\tan z}{\ln z}$

(d) $P(w) = 5^w w^{-\frac{5}{2}}$

(e) $R(\theta) = \sqrt{\theta}\cos(\theta^2)$.

ANSWER:

(a) By the chain rule, $f'(x) = e(x+\sin x)^{e-1}(1+\cos x)$.

(b) $g'(x) = e^{(x+\sin x)}(1+\cos x)$.

(c) Using the quotient rule,

$$F'(z) = \frac{\frac{1}{\cos^2 z}\ln z - (\tan z)\frac{1}{z}}{(\ln z)^2}.$$

(d) $P'(w) = 5^w \ln 5 w^{-\frac{5}{2}} + 5^w\left(-\frac{5}{2}w^{-\frac{7}{2}}\right)$

(e) $R'(\theta) = \frac{1}{2\sqrt{\theta}}\cos(\theta^2) - \sqrt{\theta}\cdot 2\theta\sin(\theta^2)$.

8. Find the derivatives of the following functions:

(a) $f(x) = x^2\ln(x^2)$

(b) $g(x) = \frac{x^2+4}{x^2-4}$

(c) $h(x) = (x^2+1)\arctan x$

(d) $m(x) = \sin(\cos(e^{3x}))$

ANSWER:

(a) $\frac{d(x^2\ln(x^2))}{dx} = (x^2)\left(\frac{2x}{x^2}\right) + 2x\ln(x^2) = 2x + 2x\ln(x^2) = 2x(1+2\ln x)$.

(b) $\dfrac{d\left(\frac{x^2+4}{x^2-4}\right)}{dx} = \dfrac{(x^2-4)2x-(x^2+4)2x}{(x^2-4)^2} = \dfrac{-16x}{(x^2-4)^2}.$

(c) $\dfrac{d((x^2+1)\arctan x)}{dx} = (x^2+1)\cdot\dfrac{1}{x^2+1} + 2x\arctan x = 1 + 2x\arctan x.$

(d) $\dfrac{d(\sin(\cos(e^{3x})))}{dx} = \cos(\cos(e^{3x}))\cdot(-\sin(e^{3x}))\cdot 3e^{3x} = -3e^{3x}\sin(e^{3x})\cos(\cos(e^{3x})).$

9. Find $\dfrac{dy}{dx}$ if:

(a) $y = xe^{-3x}$

(b) $y = \cos^2(3x-1)$

(c) $y = \dfrac{1}{x^2+1}$

ANSWER:

(a) $\dfrac{d(xe^{-3x})}{dx} = x(-3e^{-3x}) + e^{-3x} = -3xe^{-3x} + e^{-3x}$

(b) $\dfrac{d(\cos^2(3x-1))}{dx} = 2\cos(3x-1)(-\sin(3x-1))(3) = -6\sin(3x-1)\cos(3x-1) = -3\sin(6x-2)$

(c) $\dfrac{d(\frac{1}{x^2+1})}{dx} = \dfrac{d((x^2+1)^{-1})}{dx} = -(x^2+1)^{-2}(2x) = \dfrac{-2x}{(x^2+1)^2}$

10. Differentiate:

(a) $x^2\sqrt{1-x}$

(b) $\sin\dfrac{1}{x}$

(c) e^{e^x}

ANSWER:

(a) $\dfrac{d}{dx}(x^2\sqrt{1-x}) = \sqrt{1-x}\dfrac{d}{dx}(x^2) + x^2\dfrac{d}{dx}(\sqrt{1-x}) = 2x\sqrt{1-x} - \dfrac{1}{2\sqrt{1-x}}\cdot x^2$

(b) $\dfrac{d}{dx}\sin\left(\dfrac{1}{x}\right) = \cos\left(\dfrac{1}{x}\right)\cdot\left(-\dfrac{1}{x^2}\right) = -\dfrac{1}{x^2}\cos\left(\dfrac{1}{x}\right)$

(c)

$$\begin{aligned}\frac{d}{dx}e^{(e^x)} &= \frac{de^{e^x}}{de^x}\cdot\frac{de^x}{dx}\\ &= e^{e^x}\cdot e^x = e^{x+e^x}\end{aligned}$$

11. Let $f(x) = x^4 - 4x + 2$.

(a) How many zeros does f have? Justify your answer.

(b) Approximate one of the zeros by first getting an initial estimate and then improving it by using Newton's Method *once*.

(c) An initial estimate of $x = 1$, or x very near to 1, doesn't work well for Newton's Method. Why?

(d) Suppose we wished to solve $f(x) = b$ instead (same f). For which values of b would there be no solutions? For which values of b would there be one solution? For which two solutions? For which more than two solutions?

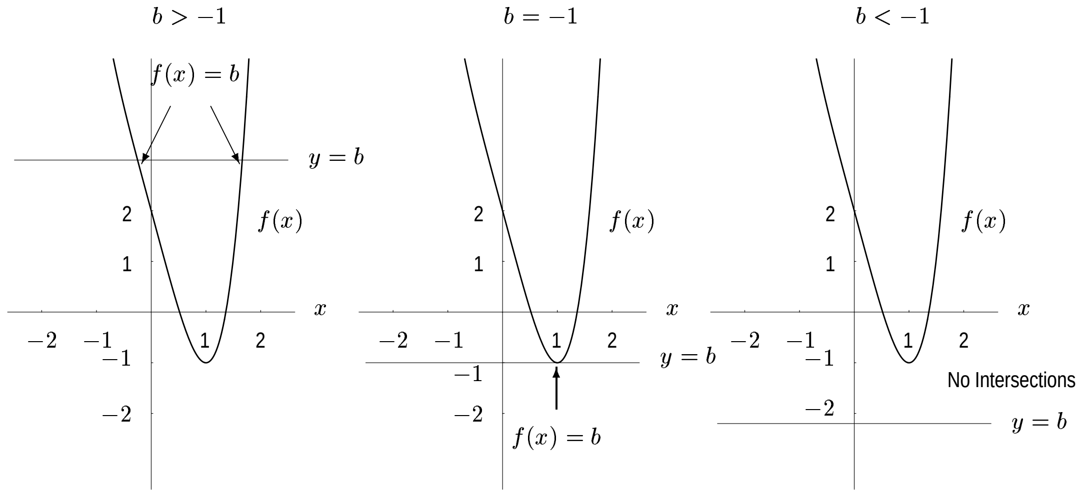
ANSWER:

(a) Since $f'(x) = 4x^3 - 4$ has only one zero, at $x = 1$, the function f has only one critical point. Therefore, f can have at most two roots. Since f changes sign ($f(0) = 2$, $f(1) = -1$ and $f(2) = 10$) it must cross the x axis twice, so it has two zeros, one between 0 and 1, the other between 1 and 2.

(b) Try the initial estimate $x_0 = 0$. Use the improved value of the root, $x_1 = x_0 - f(x_0)/f'(x_0)$, with $f(0) = 2$, $f'(0) = -4$ to get $x_1 = 0 - \frac{2}{-4} = \frac{1}{2}$, a better estimate.

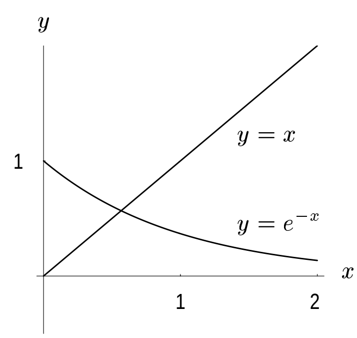
(c) If we try the above method with x_0 close to 1, $f'(x_0)$ will be close to 0, so the fraction $f(x_0)/f'(x_0)$ will be very large indeed. So $x_1 = x_0 - f(x_0)/f'(x_0)$ will be far from the true value of the root. Geometrically, the slope of the tangent to the graph of f is very flat near $x = 1$, so the point at which such a tangent line intersects the x-axis is far away.

(d) Since $f(x)$ has only one critical point, a minimum at $(1,-1)$, and $f(x)\to\infty$ when $x\to\pm\infty$, we see that f will attain the value b twice if $b > -1$, once for $b = -1$, and not at all for $b < -1$. There can never be more than two roots.



12. Sketch the graphs of $g(x) = x$ and $h(x) = e^{-x}$ for $x \geq 0$. Use Newton's method to estimate the location of the point of intersection of the two curves correct to four decimal places.

ANSWER:



We want to find the point where $x = e^{-x}$. This is really the same as finding the roots of $f(x) = x - e^{-x}$. The iteration formula for Newton's method is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

where x_n is the n^{th} iterated approximation of the root of $f(x)$. We use an initial estimate of $x_0 = 0.5$. $f'(x) = 1 + e^{-x}$, so

$$\begin{aligned} x_1 &= 0.5 - \frac{0.5 - e^{-0.5}}{1 + e^{-0.5}} \\ &= 0.5 - \frac{-0.10653}{1.60653} \\ &= 0.56631. \end{aligned}$$

Similarly,

$$\begin{aligned} x_2 &= 0.56714, \\ x_3 &= 0.56714. \end{aligned}$$

This is correct to four decimal places. The point of intersection is thus $(0.56714, 0.56714)$.

13. The purpose of this problem is to find all the roots of $f(x) = 3x^3 + 2x^2 - 4x + 1$ *exactly*.

(a) Use a computer or a calculator to obtain all the roots of $f(x)$ to 5 decimal places. Do this using both the bisection method and Newton's method. Explain what program you used, and what you did, paying particular attention to the number of iterations, bracketing and initial values. How do you *know* you have found *all* the roots?

(b) If you have done part (a) correctly, then one of the approximate roots you have obtained should suggest an exact root. Which one, and how did you confirm that the root was exact?

(c) Based on the information you have obtained in part (b), find exact expressions for all the roots.

ANSWER:

(a) Using a computer or calculator root-finding program, we find that the roots are at 0.33333, 0.61803, and -1.61803. Since a cubic can have at most three roots, and we have found three roots, we must have found all of them.

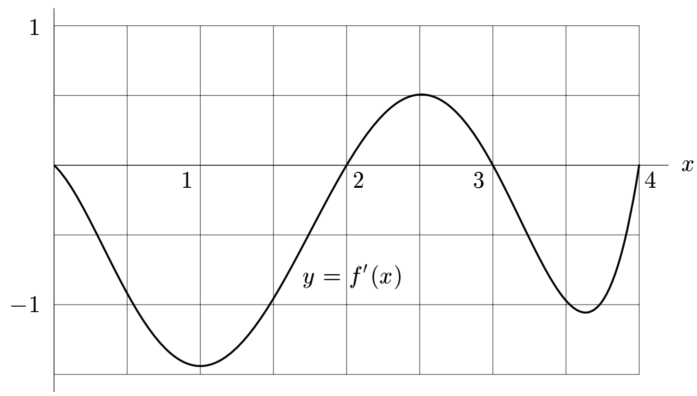
(b) It seems likely that 0.33333 is, in fact, $\frac{1}{3}$. Substituting $x = \frac{1}{3}$ into the function, we obtain 0, so $x = \frac{1}{3}$ is indeed an exact root of the polynomial.

(c) Since we know that $\frac{1}{3}$ is an exact root, we know that $(3x - 1)$ is a factor of $f(x)$. Thus $f = (3x - 1)(x^2 + x - 1)$. We then use the quadratic formula to find the roots of the second factor to give us the other two roots of $f(x)$, which turn out to be $\frac{-1+\sqrt{5}}{2}$ and $\frac{-1-\sqrt{5}}{2}$.

Chapter 4 Exam Questions

Questions and Solutions for Section 4.1

1. Below is the graph of the *derivative* of a function f, i.e., it is a graph of $y = f'(x)$.



(a) State the intervals on which f is increasing and on which it is decreasing.
(b) Say where the local maxima and minima of f occur, and for each one say whether it is a local maximum or a local minimum.
(c) Where in the interval $0 \leq x \leq 4$ does f achieve its global maximum?
(d) Suppose that you are told that $f(0) = 1$. Estimate $f(2)$.
(e) Still assuming that $f(0) = 1$, write down an exact expression for $f(2)$.

ANSWER:

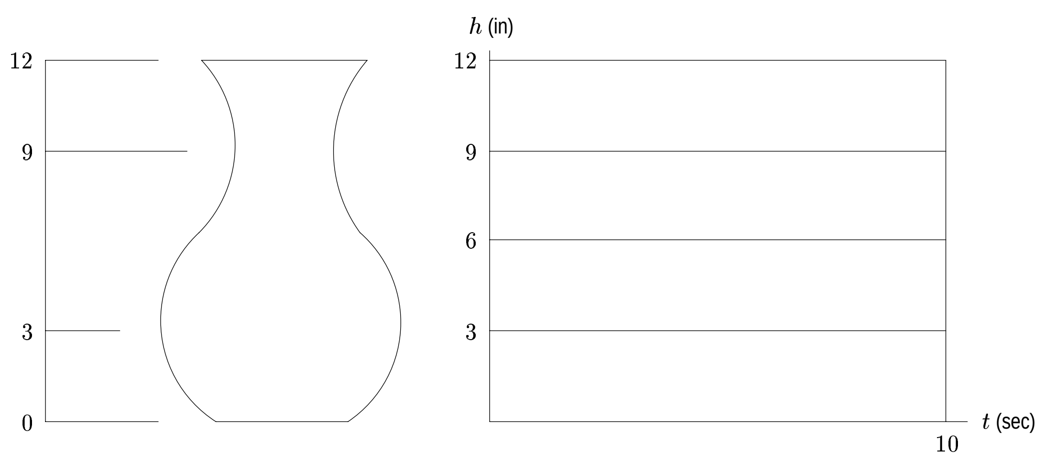
(a) The function f is increasing between 2 and 3, since the derivative is positive there. It is decreasing everywhere else.
(b) The point $x = 0$ is a local maximum, $x = 2$ is a local minimum, $x = 3$ is a local maximum, $x = 4$ is a local minimum.
(c) The global maximum occurs at $x = 0$.
(d) $f(2) \approx -0.5$; this is calculated as

$$1 - (\text{the area between the } x\text{-axis and the curve, from 0 to 2}).$$

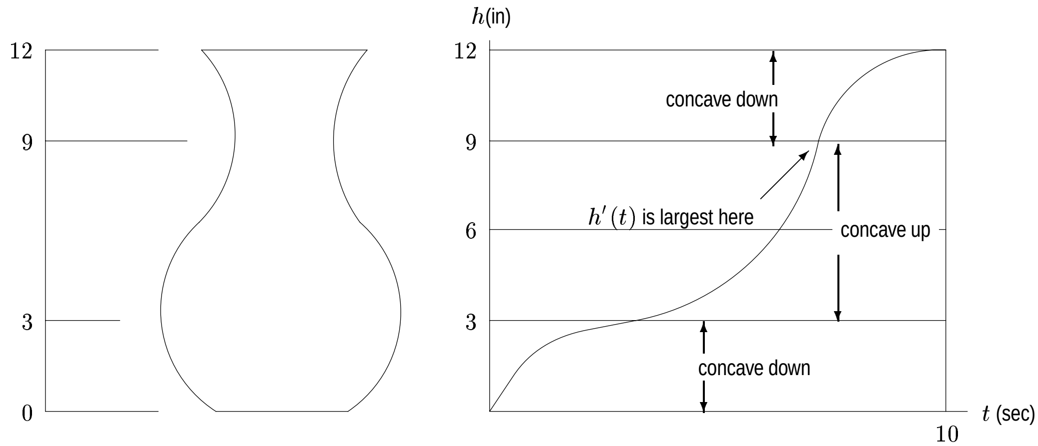
Note that each grid square has area 0.25.

(e) $f(2) = \int_0^2 f'(x)dx + 1$.

2. Starting at time $t = 0$, water is poured *at a constant rate* into an empty vase (pictured below). It takes ten seconds for the vase to be filled completely to the top. Let $h = f(t)$ be the depth of the water in the vase at time t. On the axes provided, sketch a graph of $h = f(t)$. On your graph, indicate the region(s) where the function is concave up, and where it is concave down. Finally, label the point (on the curve) at which $f'(t)$ is largest.



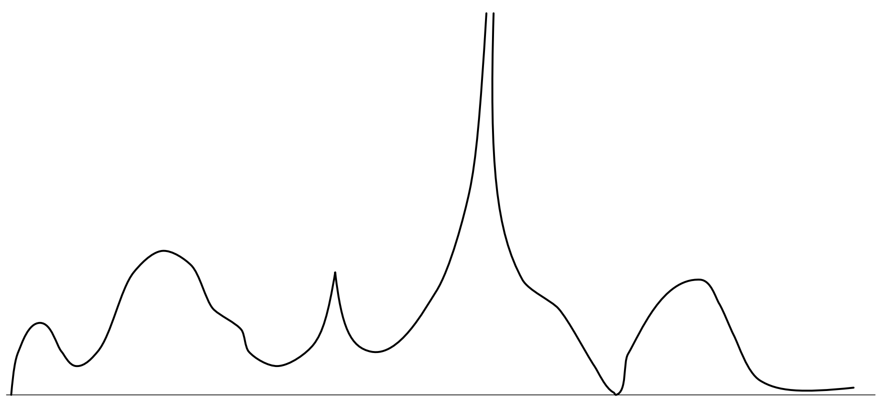
ANSWER:



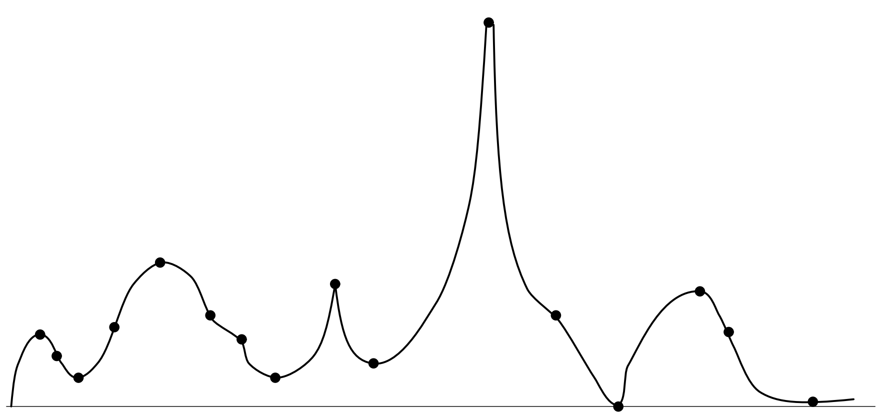
The rate of increase of h is greatest when the cross sectional area of the vase at length h is the smallest. Since the vase gets wider as h increases from 0 to 3, the value of $h'(t)$ gets smaller. This means that the graph is concave down between $h = 0$ and $h = 3$. For $3 < h < 9$, the vase gets narrower as we go up, so $h'(t)$ increases over this interval, so the graph is concave up and then the vase widens from $h = 0$ to $h = 12$, so it's concave down there.

Finally, $h'(t)$ is greatest when the vase is narrowest. This occurs at $h = 9$.

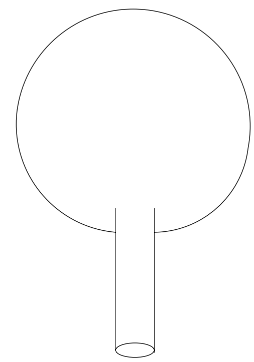
3. Given the following graph, label all points where the function has a maximum, a minimum, a horizontal tangent, a vertical tangent, a point of inflection or a point where f is not defined.



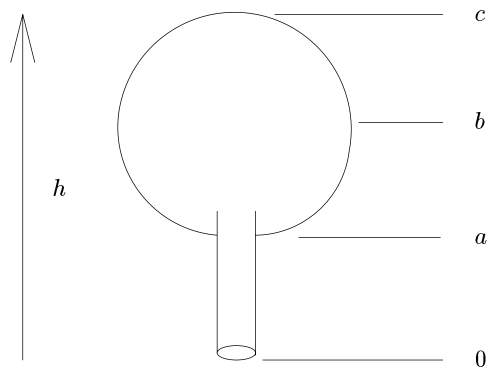
ANSWER:



4. A water tank is constructed in the shape of a sphere seated atop a circular cylinder. If water is being pumped into the tank at a constant rate, sketch the graph of the height of the water as a function of time. Be sure to indicate the location of all "interesting" points, including any critical points where the function has horizontal slope or is not differentiable, and points of inflection.

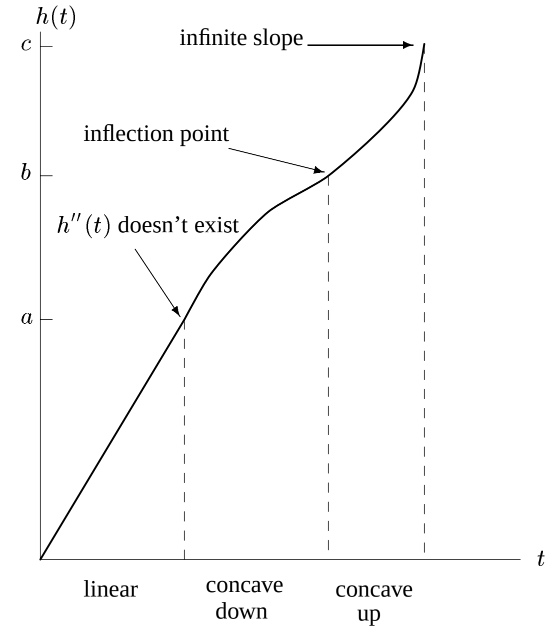


ANSWER:
Mark the points a, b, c in the picture as shown:



From $h = 0$ to $h = a$ the height increases linearly with time since the radius, and hence the cross-sectional area of the cylinder, does not change over this interval. From a to b, height will be an increasing but concave down function of t; between a and b, the radius grows and $\frac{dh}{dt}$ must slow down accordingly. Similarly, over the interval $[b, c]$, $h(t)$ will be increasing, but concave up.

The points a, b, and c are of interest. At $h = b$, the slope $\frac{dh}{dt}$ is a minimum, since the radius is at its largest, so concavity changes. (We also knew from the discussion above that concavity changes on either side of b.) Hence, we have an inflection point when $h = b$. At $h = c$, the radius has shrunk to zero; hence $\frac{dh}{dt}$ must be infinite at that point only. When $h = a$, $\frac{d^2h}{dt^2}$ does not exist. This is true because $\frac{d^2h}{dt^2} = 0$ over $[0, a]$, and beginning at $h = a$, $\frac{d^2h}{dt^2}$ immediately takes on negative values that are not infinitely close to 0 (due to downward concavity). This discontinuity in the second derivative is reflected in a "kink" in the graph of $\frac{dh}{dt}$ vs. t, but cannot be seen in the graph of h vs. t. Here is the complete picture:



5. (a) Sketch the graph of a continuous function with the following properties:
   - $f(0) = 1$,
   - $|f'(x)| < 0.5$,
   - $f''(x) < 0$, for $x < 0$,
   - $f'(2) = 0$.

   (There are infinitely many possible graphs.)

   (b) Does your graph in (a) have a local maximum for $x < 0$?

   (c) Could the graph of f have a local maximum for $x < 0$ and still satisfy the given four conditions? If so, draw such a graph. If not, explain why not.

   (d) Which of the following are inconsistent with the four conditions in part (a)? (Explain each answer.)

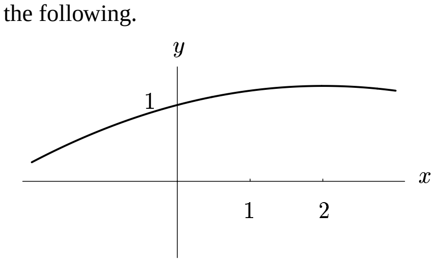
   (i) $\lim_{x \to -\infty} f(x) = 0$,

   (ii) $f(2) = 3$,

   (iii) $f''(2) = 0$.

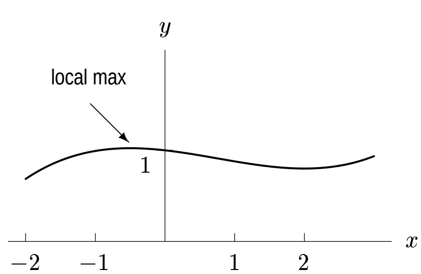
   ANSWER:

(a) One possible graph would be the following.



(b) No.

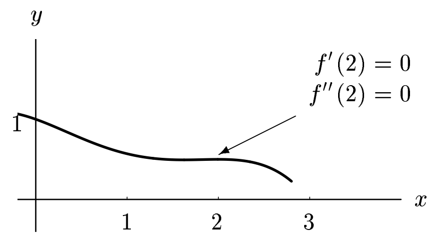
(c) Yes.



(d) (i) Inconsistent. Since $f''(x) < 0$ for $x < 0$, $f(x)$ can either cross the x-axis to the left of $x = 0$ and approach $-\infty$, or approach ∞ or a positive constant as $x \to -\infty$; it cannot approach 0.

(ii) Inconsistent. Since $f(0) = 1$ and $|f'(x)| < 0.5$, the most that $f(3)$ could be is $1 + 2(0.5) = 2$.

(iii) Consistent. We could have a picture like the following:



6. Given below are the graphs of two functions $f(x)$ and $g(x)$.

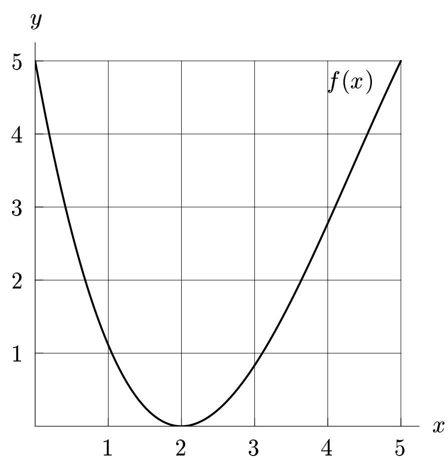


Figure 4.1.54

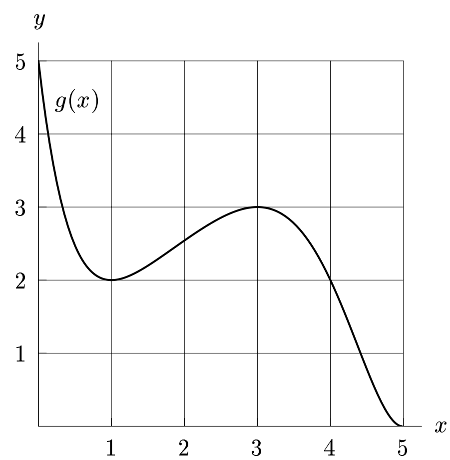


Figure 4.1.55

Let $h(x) = f(g(x))$. Use the graphs to answer the following questions about h.

(a) Find (approximately) the critical points of h and classify them.

(b) Where is h increasing? Decreasing?

(c) On the axes below sketch a graph of h.

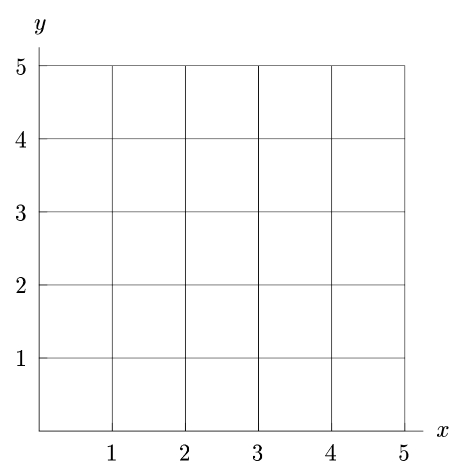


Figure 4.1.56

ANSWER:

(a) Note that $h'(x) = f'(g(x))g'(x)$. So $h'(x) = 0$ whenever $f'(g(x)) = 0$ or $g'(x) = 0$. But $g'(x) = 0$ for $x = 3$ or $x = 1$ and $f'(x) = 0$ when $x = 2$, so $f'(g(x)) = 0$ when $g(x) = 2$, i.e., when $x = 4$. Thus the critical points of h are at $x = 1$, $x = 3$ and $x = 4$.

The critical point of h at $x = 3$ is a maximum since, for x slightly greater or less than 3, $g(x) < g(3)$, and since f is increasing near $g(3) = 3$, $f(g(x)) < f(g(3))$.

The critical point of h at $x = 4$ is a global, hence local, minimum because $h(4) = 0$ and $h(x) \geq 0$ for all x (since $f(x) \geq 0$ for all x).

The critical point of h at $x = 1$ is a minimum, since $g(x) \geq g(1)$ for x near 1, and $f(g(x)) \geq f(g(1))$ for x near 1 also, since f is increasing near $g(1) = 3$.

(b) For $0 < x < 1$, $g'(x) < 0$ and $3 < g(x) < 5$, so $f'(g(x)) > 0$ and $h'(x) = f'(g(x))g'(x) < 0$. Hence h is decreasing on $[0, 1]$.
For $1 < x < 3$, $g'(x) > 0$ and $3 < g(x) < 5$, so $f'(g(x)) > 0$ and $h'(x) = f'(g(x))g'(x) > 0$. Hence h is increasing on $[1, 3]$.
For $3 < x < 4$, $g'(x) < 0$ and $2 < g(x) < 5$, so $f'(g(x)) > 0$ and $h'(x) = f'(g(x))g'(x) < 0$. Hence h is decreasing on $[3, 4]$.
For $4 < x < 5$, $g'(x) < 0$ and $0 < g(x) < 2$, so $f'(g(x)) < 0$ and $h'(x) = f'(g(x))g'(x) > 0$. Hence h is increasing on $[4, 5]$.

(c)

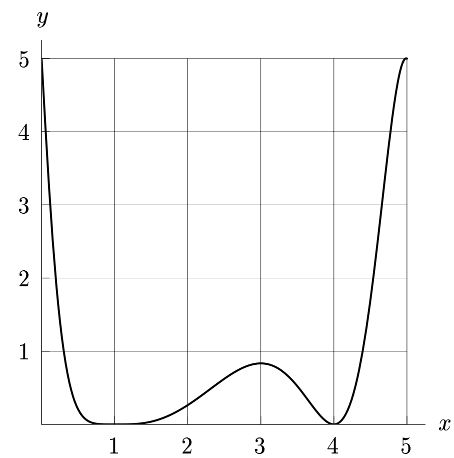


Figure 4.1.57: Graph of $h(x) = f(g(x))$

7. Let $f(x)$ be a function with positive values and let $g = \sqrt{f}$.
   (a) If f is increasing at $x = x_0$, what about g?
   (b) If f is concave down at $x = x_1$, what about g?
   (c) If f has a local maximum at $x = x_2$, what about g?

ANSWER:

(a) $\dfrac{dg}{dx} = \dfrac{d(\sqrt{f})}{dx} = \dfrac{f'}{2\sqrt{f}}$. If f is increasing at $x = x_0$, then $f'(x_0) > 0$. Thus, $g'(x_0) = \frac{f'(x_0)}{2\sqrt{f(x_0)}} > 0$, and so g is increasing at $x = x_0$.

(b) $\dfrac{d^2g}{dx^2} = \dfrac{d\left(\frac{f'}{2\sqrt{f}}\right)}{dx} = \dfrac{f''}{2\sqrt{f}} + f' \cdot \left(-\dfrac{1}{4}f^{-\frac{3}{2}} \cdot f'\right) = \dfrac{f'' - (f')^2/(2f)}{2\sqrt{f}}$. If f is concave down at $x = x_1$, then $f''(x_1) < 0$. Since $f(x_1)$ and $(f'(x_1))^2$ are positive,
$g''(x_1) = \dfrac{f''(x_1) - (f'(x_1))^2/(2f(x_1))}{2\sqrt{f(x_1)}}$ is negative, and so g is concave down at $x = x_1$.

(c) If f has a local maximum at $x = x_2$, then $f'(x_2) = 0$, f' is positive for x slightly smaller than x_2, and f' is negative for x slightly larger than x_2. Recall that $g'(x) = \dfrac{f'(x)}{2\sqrt{f(x)}}$, and so $g'(x)$ has the same sign as $f'(x)$ for all x (since $2\sqrt{f(x)}$ is always positive). Consequently, $g'(x_2) = 0$, g' is positive for x slightly smaller than x_2, and g' is negative for x slightly larger than x_2. Therefore, g has a local maximum at $x = x_2$.

8. Let f be a function. Is the following sentence true or false?

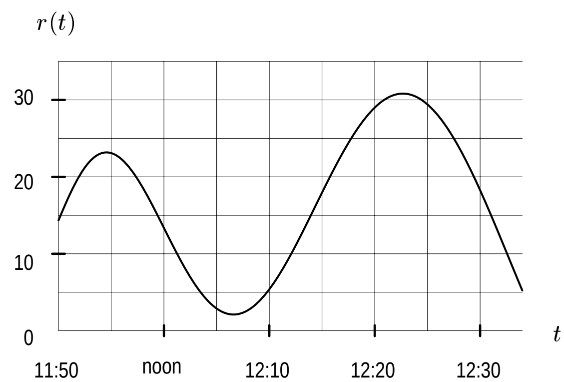
The inflection points of f are the local extrema of f'.

Explain your answer in a couple of short, clear sentences. You may assume that the second derivative of f is defined and continuous everywhere.

ANSWER:

This is true. If $x = p$ is an inflection point, then f'' changes sign at $x = p$, so since f'' is defined and continuous everywhere, we have $f''(p) = 0$. Hence p is a critical point of the derivative function f'. Since f'' changes sign at p, an inflection point is a local extremum of f', by the first derivative test applied to f'.

9. **Lunch at Cafeteria Charlotte**
Below is the graph of the rate r at which people arrive for lunch at Charlotte.



Checkers start at 12:00 noon and can pass people through at a constant rate of 5 people/minute. Let $f(t)$ be the length of the line (i.e. the number of people) at time t. Suppose that at 11:50 there are already 150 people lined up. Using the graph together with the information above, answer the following. Explain your answers.

(a) Find and classify all critical points of f.
(b) When is f increasing? decreasing?
(c) When is f concave up? concave down?
(d) Sketch the graph of f. Label the important points.
(e) When is the line longest? shortest?

ANSWER:

(a) This problem compares two rates, the rate of people arriving in line, shown by the graph of $r(t)$ and the rate at which they are checked through, a constant rate of 5 people/minute starting at 12:00 noon. To compare these rates draw the line $r = 5$ starting from 12:00. Since $f'(t) = r(t) = -5$ and the line $r = 5$ cuts the graph of $r(t)$ at approximately $12:03$, $12:10$ and $12:34$, these are the critical points of $f(t)$. The point $t = 12:03$ is a local maximum since f' changes from positive to negative there. Likewise, $t = 12:10$ is a local minimum and $t = 12:34$ is a local maximum.
Symbolically, we could write

$$f(t) = \int_0^t r(x)\,dx + 150, \qquad 0 \le t \le 10,$$

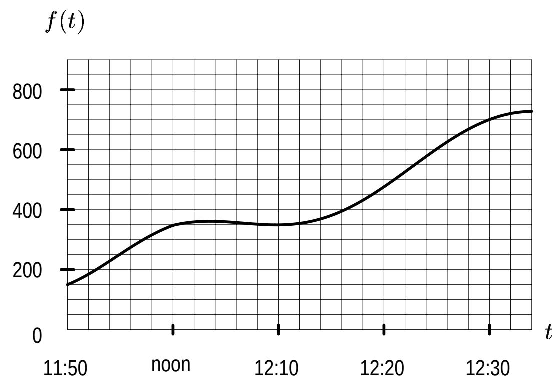
where $t = 0$ is $11:50$ and t is measured in minutes, and for $t \ge 10$

$$\begin{aligned} f(t) &= 150 + \int_0^{10} r(x)\,dx + \int_{10}^t (r(x) - 5)\ dx \\ &= 150 + \int_0^t r(x)\,dx - \int_{10}^t 5\,dx \\ &= 150 + \int_0^t r(x)\,dx - (5t - 50) \\ &= \int_0^t r(x)\,dx - 5t + 200 \end{aligned}$$

(b) f is increasing when $r(t) > 5$, and that happens before 12:03 and from 12:10 until 12:34. f is decreasing when $r(t) < 5$, which is the case between 12:03 and 12:10.

(c) We get the concavity of f by finding where f'' is positive and where it's negative. Since $f'' = r'$, f is concave up when $r' > 0$, which happens before 11:55 and between 12:07 and 12:23. f is concave down when $r' < 0$, which takes place between 11:55 and 12:07 and between 12:23 and 12:34.

(d)



(e) To answer the question, we check the critical points and the endpoints of f. $f(\text{11:50}) = 150$; $f(\text{12:03}) = 361$; $f(\text{12:10}) = 349$; $f(\text{12:34}) = 727$, so the line is longest at 12:34 and shortest at 11:50.

10. Consider $f(x) = x^2e^{-x}$ for $-1 \le x \le 3$.

(a) Show that $f'(x) = e^{-x}(2x - x^2)$ and that $f''(x) = e^{-x}(x^2 - 4x + 2)$.
(b) For which x is f increasing? For which is f decreasing?
(c) Find the values of x where $f(x)$ is the greatest; where $f(x)$ is the least.
(d) Find all values of x where there is a point of inflection.
(e) Find the values of x where f is increasing most rapidly; where f is decreasing most rapidly.
(f) Sketch the graph of f.

ANSWER:

(a)

$$\begin{aligned} f'(x) &= x^2\frac{d(e^{-x})}{dx} + \frac{d(x^2)}{dx}e^{-x} = x^2(-e^{-x}) + 2xe^{-x} \\ &= (2x - x^2)e^{-x}. \\ f''(x) &= (2x - x^2)\frac{d(e^{-x})}{dx} + \frac{d(2x - x^2)}{dx}e^{-x} = (x^2 - 2x)e^{-x} + (2 - 2x)e^{-x} \\ &= (x^2 - 4x + 2)e^{-x}. \end{aligned}$$

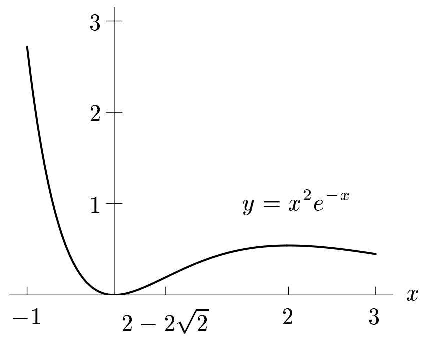
(b) f is increasing when f' is positive: that is, when $(2x - x^2)e^{-x} > 0$. Since $e^{-x} > 0$, we just need the condition $2x - x^2 > 0$, which is equivalent to $x(2 - x) > 0$. This holds for $0 < x < 2$. f' will be negative, and consequently f will be decreasing, when $x < 0$ or $x > 2$.

(c) The maximum and minimum of f will occur at critical points or the endpoints of our interval $[-1, 3]$. By part (b), $f'(x) = 0$ when $x = 0$ or $x = 2$. Since $f(0) = 0$, $f(2) = 4e^{-2}$, $f(-1) = e$, $f(3) = 9e^{-3}$, we can see that $f(x)$ is the greatest for $x = -1$ and the least for $f(0) = 0$.

(d) f can have a point of inflection only where $f''(x) = e^{-x}(x^2 - 4x + 2) = 0$, that is, when $x^2 - 4x + 2 = 0$ (since $e^{-x} > 0$). The solutions to this equation are $x = 2 \pm \sqrt{2}$. Only $x = 2 - \sqrt{2}$ lies in the interval $-1 \le x \le 3$. Since the sign of the second derivative changes from positive to negative across $x = 2-\sqrt{2}$, $(2-\sqrt{2}, (2-\sqrt{2})^2e^{-(2-\sqrt{2})})$ is an inflection point.

(e) The function f increases (or decreases) most rapidly where f' is greatest (or least). The only critical point of f' on $-1 \le x \le 3$ occurs at $x = 2 - \sqrt{2}$. It is a maximum (since f'' changes sign from positive to negative across $x = 2 - \sqrt{2}$). Now, $f'(-1) = -3e$, $f'(3) = -3e^{-3}$, so $f'(-1)$ is the global minimum for f'. Hence f is increasing most rapidly at $x = 2 - \sqrt{2}$, and decreasing most rapidly at $x = -1$.

(f)



11. Given $f(x) = x^4 - 4x^3 - 8x^2 + 1$ on the interval $[-5, 5]$, find all the maxima and minima and points of inflection. Use this information to sketch the curve.

ANSWER:

To find the maxima and minima of f on $[-5, 5]$, we need to check critical points and endpoints. To find the critical points, set $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 4x^3 - 12x^2 - 16x \\ &= 4x(x^2 - 3x - 4) \\ &= 4x(x-4)(x+1). \end{aligned}$$

So critical points occur where $x = 0, 4$ and -1.

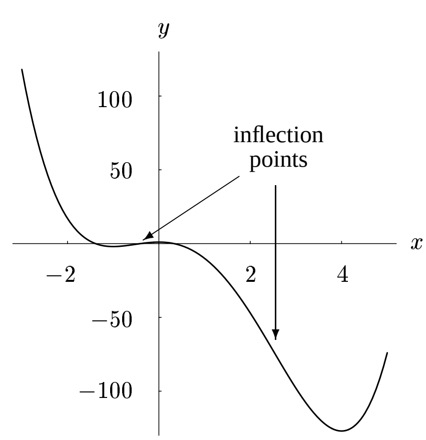
$$f''(x) = 12x^2 - 24x - 16$$

Since $f''(0) = -16 < 0$, $(0, 1)$ is a local maximum point.
Since $f''(-1) = 20 > 0$, $(-1, -2)$ is a local minimum point.
Since $f''(4) = 80 > 0$, $(4, -127)$ is a local minimum point.
As for the end points, we have $f(-5) = 926$ and $f(5) = -74$. Therefore, $(-5, 926)$ is a global maximum and $(4, -127)$ is a global minimum.

To find the inflection points, set $f''(x) = 0$. Then $3x^2 - 6x - 4 = 0$, so

$$\begin{aligned} x &= \frac{6 \pm \sqrt{36 - 4(-4)(3)}}{6} \\ &= 1 \pm \frac{\sqrt{21}}{3} \\ &\approx 1 \pm 1.528 \\ &= 2.528 \text{ or } -0.528 \end{aligned}$$

Since the second derivative changes sign across these x-values (we can see this because $f''(-1) = 20, f''(0) = -16$ and $f''(3) = 20$), they are the x-coordinates of inflection points.



12. Consider the function $f(x) = x + 2\cos x$, for $0 \le x \le 2\pi$.
   (a) Find where f is increasing and where f is decreasing.
   (b) Find the largest and smallest values of f in decimal form.
   (c) Find all points of inflection.
   (d) Find where f is increasing most rapidly.
   (e) Sketch the graph of f.
   (f) How many roots are there for $f(x) = 1$ in the given interval $0 \le x \le 2\pi$? How many for $f(x) = 2$? How many for $f(x) = 3$? Explain your answers with pictures.

ANSWER:

(a) $f(x) = x + 2\cos x$, and $f'(x) = 1 - 2\sin x$. f increases when $f' > 0$, i.e. when $1 - 2\sin x > 0$, or when $\sin x < \frac{1}{2}$. This is true for $0 \le x < \frac{\pi}{6}$, and $\frac{5\pi}{6} < x \le 2\pi$. f decreases when $f' < 0$, i.e. when $1 - 2\sin x < 0$, or when $\sin x > \frac{1}{2}$. This is true for $\frac{\pi}{6} < x < \frac{5\pi}{6}$.

(b) The largest and smallest values of any function f over an interval occur either at the critical points of f or at the endpoints of the interval. $f'(x) = 0$ when $\sin x = \frac{1}{2}$, namely when $x = \frac{\pi}{6}, \frac{5\pi}{6}$.

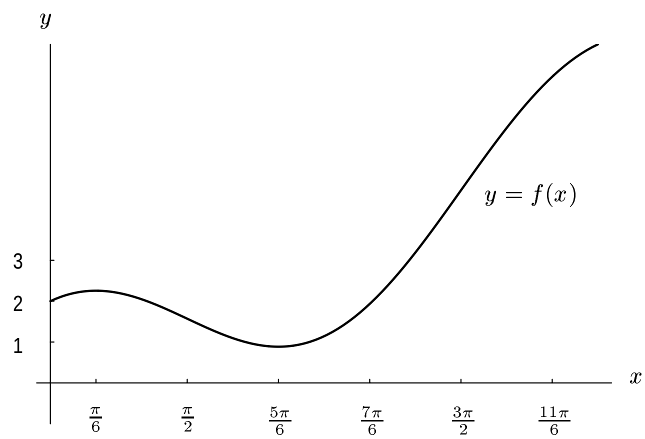
$$f\left(\frac{\pi}{6}\right) = \frac{\pi}{6} + 2\left(\frac{\sqrt{3}}{2}\right) \approx 2.26$$
$$f\left(\frac{5\pi}{6}\right) = \frac{5\pi}{6} + 2\left(-\frac{\sqrt{3}}{2}\right) \approx 0.89$$

Now check endpoints: $f(0) = 2$, while $f(2\pi) = 2 + 2\pi \approx 8.28$. Thus the largest value of f is 8.28 and the smallest value is 0.89.

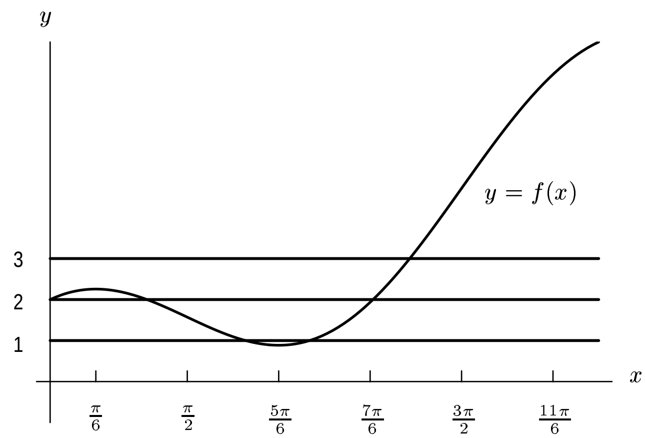
(c) Inflection points can only occur when $f''(x) = -2\cos x = 0$. This is true for $x = \pi/2, 3\pi/2$. Since $f''(x)$ changes sign at both $x = \pi/2$ and $x = 3\pi/2$, these are in fact inflection points.

(d) The maxima of f' will occur at the critical points of f' (namely $x = \pi/2$ and $3\pi/2$) or at the endpoints (namely $x = 0$ and 2π). Now, $f'(0) = 1$, $f'(\pi/2) = -1$, $f'(3\pi/2) = 3$, and $f'(2\pi) = 1$, so f increases most rapidly at $x = 3\pi/2$.

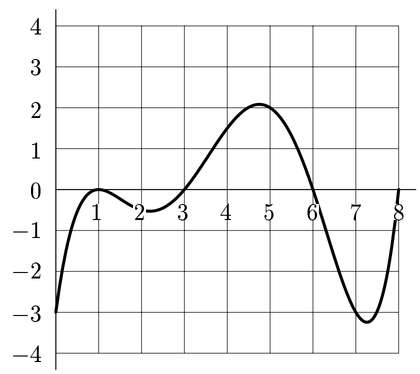
(e)



(f) A solution to $f(x) =$ constant will occur whenever the graph of $y = f(x)$ intersects the graph $y =$ constant. As can be seen below, $f(x) = 1$ has 2 solutions, $f(x) = 2$ has 3 solutions, and $f(x) = 3$ has 1 solution.



13. The graph of f' is shown below. (The graph of f is not shown.) Use the graph of f' to answer the following questions.

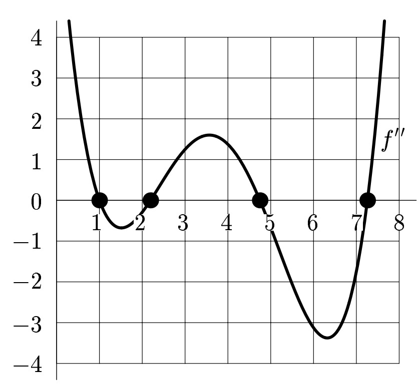


(a) On which intervals, if any, is f increasing?
(b) At which values of x does f have a local maximum? A local minimum?
(c) On which intervals, if any, is f concave up?
(d) Which values of x, if any, correspond to inflection points on the graph of f?
(e) Sketch a graph of f''. (Your graph need only have the right general shape. You do not need to put units on the vertical axis.)
(f) Assume that $f(0) = 0$. Sketch a graph of f. (Your graph need only have the right general shape. You do not need to put units on the vertical axis.)

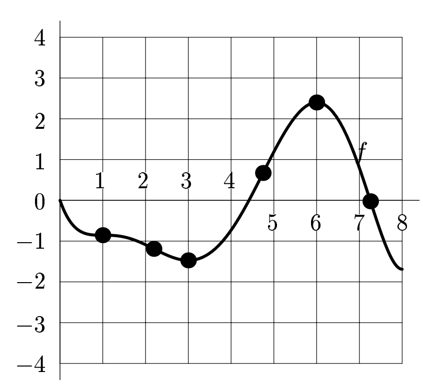
ANSWER:

(a) $3 < x < 6$
(b) Local max at $x = 6$; local min at $x = 3$.
(c) $0 < x < 1$ and $2.1 < x < 4.9$ and $7.2 < x < 8$.
(d) $x = 1, 2.1, 4.9, 7.2$

(e)



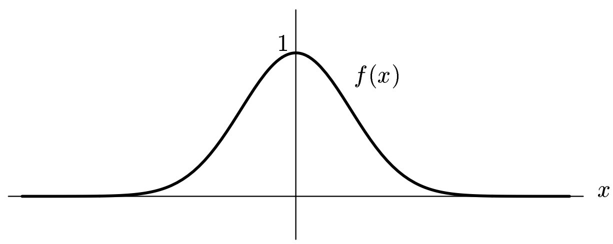
(f)



Questions and Solutions for Section 4.2

1. Let
$$f(x) = e^{-\frac{x^2}{b}},$$
where b is a positive constant.
   (a) Using a computer or a calculator, sketch a graph of f. By choosing various different values of b, observe how the shape of the graph changes when b is made larger or smaller. Describe your observation in a clear, concise sentence that would make sense to someone who can not see your graph.
   (b) Find the inflection points of f, in terms of b.
   (c) Use your answer to (b) to explain mathematically the effect of varying b that you observed in (a).

   ANSWER:
   (a) The graph, which is a bell shaped curve with height 1 and centered at $x = 0$, becomes more spread out horizontally as b is increased, while retaining the same central height of 1.



   (b) To find the inflection points, we need to find the second derivative.
$$\begin{aligned} f'(x) &= e^{-\frac{x^2}{b}} \cdot \left(-\frac{2x}{b}\right) \\ f''(x) &= \left(e^{-\frac{x^2}{b}} \cdot \frac{-2x}{b}\right) \cdot \left(-\frac{2x}{b}\right) + e^{-\frac{x^2}{b}} \cdot \left(-\frac{2}{b}\right) \\ &= e^{-\frac{x^2}{b}} \left(\frac{4x^2}{b^2} - \frac{2}{b}\right) \end{aligned}$$
   Setting this equal to 0, we get
$$\begin{aligned} e^{-\frac{x^2}{b}} \left(\frac{4x^2}{b^2} - \frac{2}{b}\right) &= 0 \\ \text{since } e^{-\frac{x^2}{b}} \text{ is never 0.} \quad \left(\frac{4x^2}{b^2} - \frac{2}{b}\right) &= 0 \\ \frac{4x^2}{b^2} &= \frac{2}{b} \\ 2x^2 &= b \\ x &= \pm\sqrt{\frac{b}{2}} \end{aligned}$$
   Since $f''(x)$ changes sign on either side of the points $x = \sqrt{\frac{b}{2}}$ and $x = -\sqrt{\frac{b}{2}}$, these are indeed inflection points.
   (c) As b increases, the inflection points move farther out from the y axis. This explains the horizontal spreading observed in (a).

2. Consider the two-parameter family of curves
$$y = ax + \frac{b}{x}.$$
Assume that $a > 0, b > 0$.
   (a) For three (reasonable) choices of a and b with $a < b$, $a = b$, $a > b$, respectively, sketch the three curves. (You may use your calculator. Label your choices of a and b.)

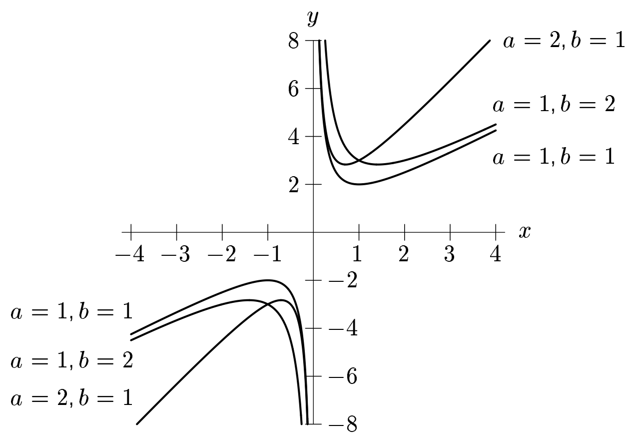
(b) For the family
$$y = ax + \frac{b}{x},$$
determine the critical points, critical values, local and global maxima or minima and concavity in terms of the parameters a and b. (In the general case, not just for your curves in part (a)).

(c) In words and in sketches explain how the sizes of a and b influence the shape of $y = ax + \frac{b}{x}$.

(d) From part (b) deduce the famous inequality between the arithmetic and the geometric mean:
$$\text{If } a \text{ and } b \text{ are positive numbers, then: } \sqrt{ab} \leq \frac{a+b}{2}.$$

ANSWER:

(a) Graphs for $a < b$, $a = b$, $a > b$



(b) Solving $y' = a - \frac{b}{x^2} = 0$, we get only two critical points: $x = \sqrt{\frac{b}{a}}$ and $x = -\sqrt{\frac{b}{a}}$. The coordinates of the critical points are thus $(\sqrt{\frac{b}{a}}, 2\sqrt{ab})$, and $(-\sqrt{\frac{b}{a}}, -2\sqrt{ab})$. Now, $y'' = 2\frac{b}{x^3}$. Since $y''(\sqrt{\frac{b}{a}}) > 0$, the graph has a local minimum at $(\sqrt{\frac{b}{a}}, 2\sqrt{ab})$. Since $y''(-\sqrt{\frac{b}{a}}) < 0$, the graph has a local maximum at $(-\sqrt{\frac{b}{a}}, -2\sqrt{ab})$. As $x \to 0^+$, $y \to +\infty$ and as $x \to 0^-$, $y \to -\infty$. Therefore, we see that $x = 0$ is an asymptote and that there are no global maxima or minima. Finally, since $y'' > 0$ for $x > 0$ and $y'' < 0$ for $x < 0$, the graph is concave up for $x > 0$ and concave down for $x < 0$. No inflection point.

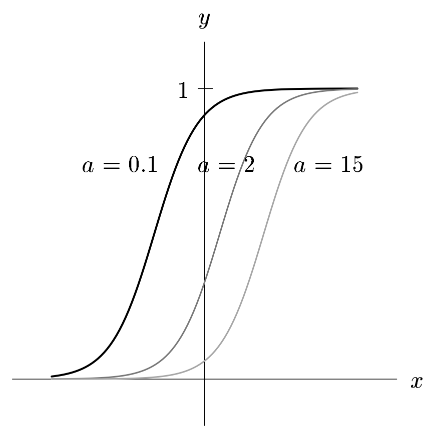
(c) a and b do not influence the concavity of $y = ax + \frac{b}{x}$. Let's consider the effect on the critical points at $(\sqrt{\frac{b}{a}}, 2\sqrt{ab})$ and $(-\sqrt{\frac{b}{a}}, -2\sqrt{ab})$ of varying a and b. If a is held fixed, increasing b causes the graph to move farther from the x-axis and the critical points to move farther from the y-axis. If b is held fixed, increasing a causes the graph to move farther from the x-axis and the critical points to move closer to the y-axis. Or, the larger the *ratio* of b to a, the further out from the y-axis the critical points will be, and the larger the *product* of a and b, the further from the x-axis the critical points will be.

(d) From Part (b) we know that the curve $y = ax + \frac{b}{x}$ attains a local minimum $y = 2\sqrt{ab}$ at $x = \sqrt{\frac{b}{a}}$. Since the curve has no other critical points for $x > 0$ and as $x \to 0^+$ or $x \to +\infty$, $y \to +\infty$, we see that $2\sqrt{ab}$ is a global minimum of $y = ax + \frac{b}{x}$ for $x > 0$. We therefore have:
$$2\sqrt{ab} \leq ax + \frac{b}{x} \quad \text{for } x > 0.$$
Let $x = 1$. We get:
$$2\sqrt{ab} \leq a + b$$
$$\sqrt{ab} \leq \frac{a+b}{2}.$$

3. The purpose of this problem is to explain the graph of $f(x) = \dfrac{1}{1+ae^{-x}}$ for $a > 0$.
   (a) Use the computer to graph the function $f(x)$ for $a > 0$. Do this for various values of a. Based on what you see, write down what you think happens to the graph of $f(x)$ as a increases. Pay particular attention to what happens to any asymptotes, maxima, minima, and points of inflection.
   (b) It is claimed that if x represents time, then $f(x)$ is related to the number of people on Earth. Briefly explain why this might be reasonable.
   (c) Forget what you did in parts (a) and (b). Calculate $f'(x)$ and $f''(x)$.
   (d) Using the results you obtained in part (c) confirm that the statements you made in part (a) are accurate.

   ANSWER:

   (a) There are no vertical asymptotes; there are horizontal asymptotes at $y = 1$ and $y = 0$ for all values of a. $f(x)$ approaches 1 as x approaches infinity, and $f(x)$ approaches 0 as x approaches negative infinity. The graph is strictly increasing, so there are no maxima or minima. The point of inflection, which always has a y-value of $\frac{1}{2}$, moves to the right as a increases. It seems that all curves in the family $\dfrac{1}{1+ae^{-x}}$, $a > 0$ are just horizontally shifted versions of one another.



   (b) At first, $f(x)$ increases very slowly, but its rate of increase is growing. This continues until its rate of increase is quite large, at which point it starts slowing down. Here, $f(x)$ is still increasing, but more and more slowly, until the graph is practically level. Similarly, the world's population has always been increasing, but at different rates. When the population was small, so was the rate of increase. Over time, the population has grown larger and so has the rate of increase of population. Theoretically, if the population gets too large, the rate at which it is increasing will start decreasing, as the effects of overpopulation become significant. The asymptote as x approaches infinity corresponds to the maximum population the Earth can support.
   (c) $f' = \dfrac{ae^{-x}}{(1+ae^{-x})^2}, \qquad f'' = \dfrac{ae^{-x}(ae^{-x}-1)}{(1+ae^{-x})^3}.$
   (d) For all x, f' is positive, so f is always increasing. The second derivative of f is zero when $x = \ln a$, which gives a y value of $\frac{1}{2}$, so the inflection point is $(\ln a, \frac{1}{2})$, and does move to the right as a increases.

4. Consider the one-parameter family of functions

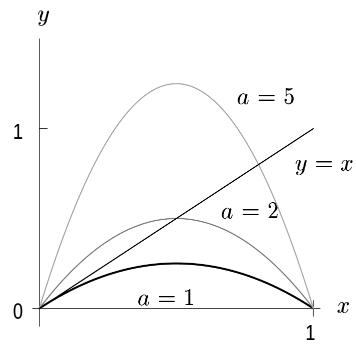
$$f(x) = ax(1-x), \qquad a > 0,$$

for $0 \le x \le 1$. As simple as this family is, it exhibits many remarkable properties which have been studied intensively over the last several years.

   (a) Sketch several members of the family for a in the range $1 < a < 5$. Also sketch the line $y = x$. Label your choices of a. (Remember, take $0 \le x \le 1$).
   (b) Find the local and global maxima and minima of $f(x)$ in terms of a.
   (c) What is the largest value of a such that we have $f(x) \le 1$ for all $0 \le x \le 1$?
   (d) In terms of a, find all points x in $0 \le x \le 1$ where $f(x) = x$. These are called the "fixed points" of the function. How can you spot the fixed points from your sketches in Part (a)?
   (e) If x_0 is a fixed point of $f(x) = ax(1-x)$ and $x_0 \ne 0$, show that $f(x) < x$ for $x > x_0$. Prove this using calculus and also relate it to your sketches in Part (a).

ANSWER:

(a)



(b)

$$\begin{aligned} f'(x) &= \frac{d}{dx}(ax(1-x)) \\ &= \frac{d}{dx}(ax - ax^2) \\ &= a - 2ax. \end{aligned}$$

So if $f'(x) = 0, a - 2ax = 0$. Then $2x = 1$ (since a cannot be 0) and $x = \frac{1}{2}$. Thus $\frac{1}{2}$ is the only critical point. Checking endpoints, $f(0) = f(1) = 0$ are global minima, and $f(\frac{1}{2}) = \frac{a}{4}$ is a maximum, since $f''(x) = -2a < 0$.

(c) Since $\frac{a}{4}$ is the maximum of f, f is less than or equal to 1 for all $0 < x < 1$, provided $a \leq 4$. The largest value occurs when $a = 4$.

(d) $f(x) = x$ means that $ax(1-x) = x$, so $x(a(1-x) - 1) = 0$, and thus $x = 0$ or $x = 1 - \frac{1}{a}$. The fixed points are where each curve intersects the line $y = x$.

(e) From Part (d), we see that $x_0 = 1 - \dfrac{1}{a}$. For $x > x_0 = 1 - \frac{1}{a}$,

$$\begin{aligned} f(x) &= ax(1-x) \\ &< ax(1-x_0) \quad \text{because we are considering } x > x_0 \\ &= ax(\frac{1}{a}) \quad \text{because } x_0 = 1 - \tfrac{1}{a} \\ &= x. \end{aligned}$$

In Part (a), the curve lies below the line $y = x$ after the point of intersection.
Another way of doing this is as follows:

Note that $f'(x) = a - 2ax$. If $x > x_0$, we have

$$\begin{aligned} x &> 1 - \frac{1}{a} \\ -2ax &< -2a(1 - \frac{1}{a}) \\ a - 2ax &< a - 2a + 2 \\ f'(x) &< -a + 2 \end{aligned}$$

But since a is at least 1, $f'(x)$ can be at most 1, when $a = 1$. So the slope of f is less than the slope of the line $y = x$ for $x > x_0$, and hence $f(x) < x$ for $x > x_0$. This can also be seen on the graph.

5. Given the following data about the second derivative of a function f,

x	0	1	2	3
$f''(x)$	1	-1	-3	-5

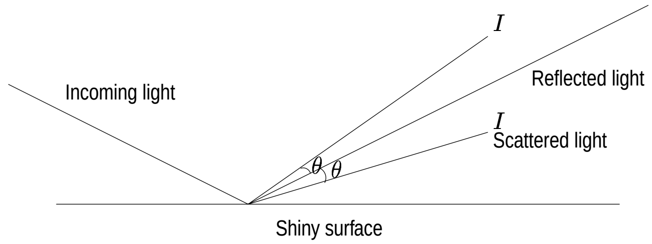
which of the following types of function could f be? (Circle all that apply.) Assume $b > 0$. The other constants can be positive or negative.

(a) ae^{bx}

(b) $e^{-x^2/b}$
(c) A quadratic (i.e., $ax^2 + bx + c$).
(d) A cubic (i.e., $ax^3 + bx^2 + cx + d$)
(e) $\sin(bx)$.

ANSWER:
The data in the table suggests that $f'(x)$ is a linear function. We expect $f(x)$ to be a cubic.

6.



When light strikes a shiny surface, much of it is reflected in the direction shown. However some of it may be scattered on either side of the reflected light. If the intensity (brightness) of the scattered light at the angle θ (shown in the picture) is I, the Phong model says that

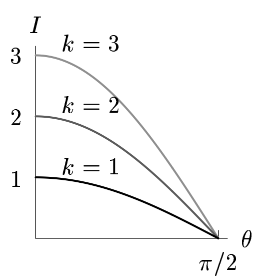
$$I = k\cos^n(\theta)$$

where k and n are positive constants depending on the surface. Thus this function gives an idea of how "spread-out" the scattered light is.

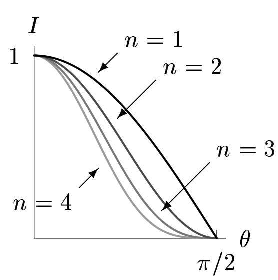
(a) Sketch graphs I against θ for $0 \le \theta \le \frac{\pi}{2}$ for various values of k and n.
(b) Explain in words what effect the parameter k has on the shape of the graph.
(c) Explain in words what effect the parameter n has on the shape of the graph. In particular, what is the difference between the ways in which surfaces with small n and surfaces with large n scatter light?
(d) If k remains fixed, what happens to the graph of I against θ as $n \to \infty$? What does this tell you about how I depends on θ in this case?

ANSWER:

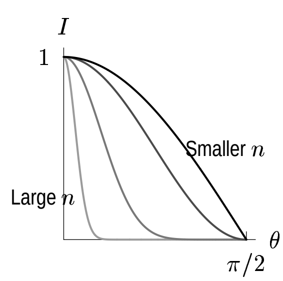
(a) $n = 1$, various k's



$k = 1$, various n's.



(b) Stretches graph upward ($k > 1$), shrinks ($k < 1$).
(c) Value of I drops to 0 more quickly as n increases. Surfaces with small n scatter light more, those with large n scatter less light.
(d) It drops more and more sharply from the vertical intercept:



As $n \to \infty$, less light is scattered. In the limit, $I = k$ when $\theta = 0$ and $I = 0$ elsewhere. This mans that no light is scattered–we have a perfectly reflecting surface.

7. Consider the one-parameter family of functions given by

$$e^{Ax} + e^{-Ax}, \text{ where } A > 0.$$

Use calculus to draw a graph of members of the family. Show what happens as A gets very small and very large. Be sure to label critical points and points of inflection, if any.

ANSWER:

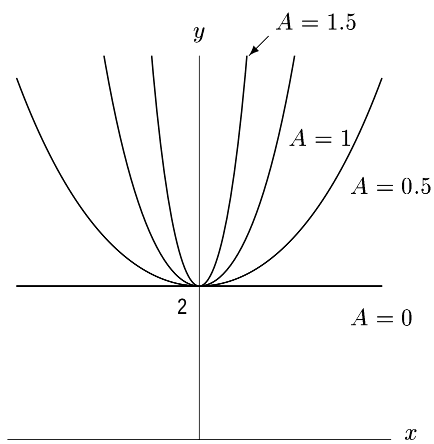
Consider the general curve $f(x) = e^{Ax} + e^{-Ax}$. To find the maxima and minima of this function, set

$$\begin{aligned} f'(x) = \frac{d}{dx}\left(e^{Ax} + e^{-Ax}\right) &= 0 \\ Ae^{Ax} - Ae^{-Ax} &= 0 \\ e^{Ax} &= e^{-Ax}. \end{aligned}$$

This is true only when $x = 0$. To find concavity, take the second derivative:

$$f''(x) = \frac{d}{dx}\left(Ae^{Ax} - Ae^{-Ax}\right) = A^2\left(e^{Ax} + e^{-Ax}\right).$$

Since this quantity is positive for all x and any A, any curve in this family is concave up and has a minimum at the point $(0, 2)$. Since there is no change in the concavity, there are no inflection points. This function is also even, as we can see by replacing x with $-x$ in the original function: $f(-x) = e^{-Ax} + e^{Ax} = f(x)$. We also know that the greater A is, the greater is the curvature or concavity. We obtain the following family of curves:



8. For the function $y = axe^{-bx}$, choose a and b so that y has a critical point at $x = 2$ and a maximum value of 7.

ANSWER:

To find the critical points of $y = axe^{-bx}$, set

$$\begin{aligned} y' &= 0 \\ ae^{-bx} - abxe^{-bx} &= 0 \\ a(1 - bx)e^{-bx} &= 0 \end{aligned}$$

There is thus a critical point at $x = \frac{1}{b}$. Since

$$\begin{aligned} y'' &= -abe^{-bx} - ab(1 - bx)e^{-bx} \\ &= -ab(2 - bx)e^{-bx}, \end{aligned}$$

$y''(\frac{1}{b}) = -abe^{-1} < 0$, and so there is indeed a local maximum at $x = \frac{1}{b}$. b must then be $\frac{1}{2}$, and since $y(2) = 2ae^{-1} = 7$, $a = \frac{7}{2}e$.

9. (a) Find the family of all quadratic functions that have zeros at $x = 1$ and $x = 5$. (Your answer will contain one arbitrary constant.)

(b) Use your answer to (a) to find the family of all cubic functions, f, that have critical points at $x = 1$ and $x = 5$.

(c) For all cubics, f, in this family, find

(i) p, the x-coordinate of the point of inflection;

(ii) $f''(p)$, where p is the x-coordinate of the point of inflection;

(iii) $f'(1)$.

(d) From the list below, check off the data you would like to be told in order to specify the cubic f uniquely. Don't ask for more or less information than you need. (There are many possible answers. Just give one choice or one set of choices.) Briefly explain your answer.

___$f(0)$

___$f(p)$ where p is the point of inflection

___$f'(p)$ where p is the point of inflection

___$f'(0)$

ANSWER:

(a) $y = D(x-1)(x-5)$, where D is an arbitrary constant, is the family of all quadratic functions having zeros at $x = 1$ and $x = 5$.

(b) The derivative of a cubic with critical points at 1 and 5 is a quadratic with zeros at 1 and 5 . So a cubic with this kind of derivative is an antiderivative of $D(x-1)(x-5)$.

$$\begin{aligned} f(x) &= \int D(x-1)(x-5)\,dx \\ &= D\int x^2 - 6x + 5\,dx \\ &= \frac{Dx^3}{3} - 3Dx^2 + 5Dx + C \end{aligned}$$

(c) (i) $f''(x) = 2Dx - 6D$. This is zero when $D(x-3) = 0$, so $p = 3$ is the x-coordinate of the point of inflection.

(ii) $f''(3) = 0$

(iii) $f'(1) = 0$, since f has a critical point at $x = 1$.

(d) Since the expression found for f in (b) has two arbitrary constants, we need two conditions to fix f uniquely. Any two conditions from $f(0)$, $f'(0)$, $f(p)$ and $f'(p)$ will work, except for the two $f'(0)$ and $f'(p)$, since neither of these two will fix C.

10. Find all critical points of $f(x) = 4x^3 + 7x^2 + 4x$.

ANSWER:

$$\begin{aligned} f'(x) &= 12x^2 + 14x + 4 \\ &= 2(6x^2 + 7x + 2) \\ &= 2(2x+1)(3x+2) \end{aligned}$$

$$\begin{aligned} 2x + 1 = 0,\ \ 3x + 2 = 0 \\ x = -\frac{1}{2} \text{ or } \text{x} = -\frac{2}{3} \end{aligned}$$

Questions and Solutions for Section 4.3

1. Find the best possible upper and lower bounds for the function $f(x) = xe^{-x}$ for $x \geq 0$, i.e., find numbers A and B such that

$$A \leq xe^{-x} \leq B, \quad x \geq 0.$$

The numbers A and B should be as close together as possible.

ANSWER:

First find the critical points. Set $f'(x) = 0$ to obtain:

$$\begin{aligned} f'(x) = e^{-x} - xe^{-x} &= 0, \\ e^{-x}(1-x) &= 0. \end{aligned}$$

Since e^{-x} never equals 0, $x = 1$ is a critical point. Since f' is positive to the left of $x = 1$ and is negative to the right of $x = 1$, there is a local maximum at $x = 1$. $f(0) = 0$ and as $x \to \infty$, $f(x) \to 0$. So the lower bound of $f(x)$ is 0 and the upper bound is $f(1) = e^{-1}$. So $0 \leq xe^{-x} \leq e^{-1}$ for $x \geq 0$.

2. Find two positive numbers whose sum is 8 such that the sum of the cube of the first and the square of the second is a minimum.

   ANSWER:

   Let x, y be the two positive numbers such that $x + y = 8$; we wish to minimize $x^3 + y^2$. Since $y = 8 - x$, we need to minimize $x^3 + (8 - x)^2$. Differentiating and setting this derivative equal to zero,

$$3x^2 + 2(8 - x)(-1) = 0$$
$$3x^2 - 16 + 2x = 0.$$

   Using the quadratic formula, we obtain:

$$x = \frac{-2 \pm \sqrt{4 - 4(3)(-16)}}{6}$$
$$= -\frac{1}{3} \pm \frac{7}{3} = -\frac{8}{3},\ 2.$$

   But x must be positive, so $x = 2$, $y = 6$. Comparing values of $x^3 + y^2$ at the critical points, namely $x = 2$, $y = 6$, and the endpoints, namely $x = 0$, $y = 8$ and $x = 8$, $y = 0$, we see that $x = 2$, $y = 6$ is in fact the global minimum.

3. A landscape architect plans to enclose a 3000 square-foot rectangular region in a botanical garden. She will use shrubs costing $25 per foot along three sides and fencing costing $20 per foot along the fourth side. Find the dimensions that minimize the total cost.

   ANSWER:

   A rectangle with area 3000 ft^2 has dimensions of x ft by $\frac{3000}{x}$ ft. The cost of lining three sides of the rectangle with shrubs will be $25(2x + \frac{3000}{x})$ dollars, and the cost of lining the remaining side with a fence is $20 \cdot \frac{3000}{x}$ dollars. Therefore, the total cost incurred will be $C(x) = 50x + \frac{75000}{x} + \frac{60000}{x} = 50x + \frac{135000}{x}$ dollars. To minimize the cost, set $C'(x) = 0$, to obtain:

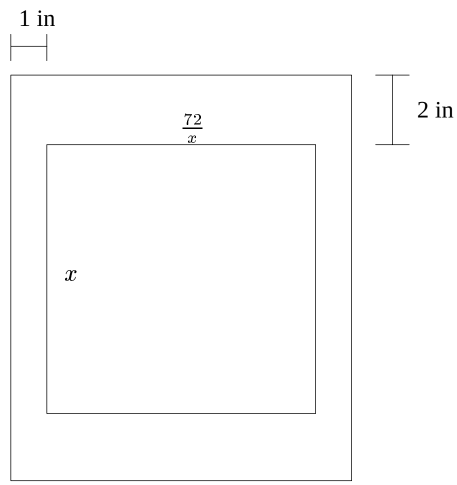
$$C'(x) = 50 - \frac{135000}{x^2} = 0$$
$$x^2 = \frac{135000}{50}$$
$$x = \sqrt{2700} \approx \pm 52.0 \text{ ft}.$$

   Since we are only interested in positive distances, we discard $x \approx -52.0$ ft. Now, $x \approx 52.0$ ft is a possible minimum. Since $C''(52.0) = \frac{2 \cdot 135000}{52.0^3} > 0$, $x \approx 52.0$ ft is indeed a minimum. The dimensions of the plot are then ≈ 52.0 ft by $\frac{3000}{52.0} \approx 57.7$ ft.

4. A rectangular sheet of paper is to contain 72 square inches of printed matter with 2 inch margins at top and bottom and 1 inch margins on the sides. What dimensions for the sheet will use the least paper?

   ANSWER:

   Let x be the length of the side of the printed matter. Then $\frac{72}{x}$ is the length of the top edge of the printed matter.



The area of the sheet is given by $A(x) = (\frac{72}{x} + 2)(x + 4)$. To minimize $A(x)$, set $A'(x) = -\frac{288}{x^2} + 2 = 0$. Since $x = 12$ is the only positive (and hence reasonable) solution to this equation and since $A''(12)$ is positive, $x = 12$ must minimize the area. Thus, the piece of paper must be 16 inches high and 8 inches wide.

5. What point of the parabola whose equation is $y = x^2$ is nearest to the point $(6, 3)$?

ANSWER:

We wish to minimize the distance between a point (x, x^2) on the parabola $y = x^2$ and the point $(6, 3)$. Using the distance formula we note that $d(x)$, the distance from the parabola to $(6, 3)$, equals $\sqrt{(x-6)^2 + (x^2-3)^2}$. This will be minimized if and only if its square is also minimized. We will call this $f(x)$.

$$
\begin{aligned}
f(x) &= (x-6)^2 + (x^2-3)^2 \\
&= (x^2 - 12x + 36) + (x^4 - 6x^2 + 9) \\
&= x^4 - 5x^2 - 12x + 45. \\
f'(x) &= 4x^3 - 10x - 12
\end{aligned}
$$

To find critical points, set $f'(x) = 0$,

$$
\begin{aligned}
4x^3 - 10x - 12 &= 0 \\
2x^3 - 5x - 6 &= 0 \\
(x-2)(2x^2 + 4x + 3) &= 0.
\end{aligned}
$$

There is only one real solution, $x = 2$; the other two roots are complex. Thus $(2, 4)$ is a critical point since the derivative is zero at $x = 2$. Since points on the parabola move away from $(6, 3)$ for large x, this critical point minimizes the distance. Alternatively, $f''(x) = 12x^2 - 10$, so $f''(2) = 48 - 10 = 38 > 0$, so $(2, 4)$ is a minimum. $(2, 4)$ is closest to $(6, 3)$ since $x = 2$ is the only critical point.

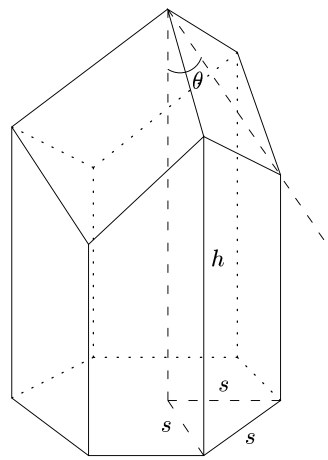
6. A single cell of a bee's honey comb has the shape shown to the right. The surface area of this cell is given by

$$A = 6hs + \frac{3}{2}s^2\left(\frac{-\cos\theta}{\sin\theta} + \frac{\sqrt{3}}{\sin\theta}\right)$$

where h, s, θ are as shown in the picture.

(a) Keeping h and s fixed, for what angle, θ, is the surface area minimal?

(b) Measurements on bee's cells have shown that the angle actually used by bees is about $\theta = 55^\circ$. Comment.



ANSWER:

(a) To minimize A with h and s fixed we have to find $\dfrac{dA}{d\theta}$.

$$\begin{aligned}\frac{dA}{d\theta} &= \frac{3}{2}s^2\frac{d}{d\theta}\left(\frac{\sqrt{3}-\cos\theta}{\sin\theta}\right)\\ &= \frac{3}{2}s^2\left(\frac{\sin^2\theta-\cos\theta(\sqrt{3}-\cos\theta)}{\sin^2\theta}\right)\\ &= \frac{3}{2}s^2\left(\frac{\sin^2\theta-\sqrt{3}\cos\theta+\cos^2\theta}{\sin^2\theta}\right)\\ &= \frac{3}{2}s^2\left(\frac{1-\sqrt{3}\cos\theta}{\sin^2\theta}\right)\end{aligned}$$

Set $\dfrac{dA}{d\theta} = \dfrac{3}{2}s^2\left(\dfrac{1-\sqrt{3}\cos\theta}{\sin^2\theta}\right) = 0$. Then $1-\sqrt{3}\cos\theta = 0$ and $\cos\theta = \frac{1}{\sqrt{3}}$, so $\theta \approx 54.7^\circ$ is a critical point of A. Since

$$\begin{aligned}\left.\frac{d^2A}{d\theta^2}\right|_{\theta=54.7} &= \frac{3}{2}s^2\left.\left(\frac{\sqrt{3}\sin^3\theta-2\sin\theta\cos\theta(1-\sqrt{3}\cos\theta)}{\sin^4\theta}\right)\right|_{\theta=54.7}\\ &\approx \frac{3}{2}s^2(2.122) > 0,\end{aligned}$$

$\theta = 54.7$ is indeed a minimum.

(b) Since 55° is very close to 54.7°, we conclude that bees attempt to minimize the surface areas of their honey combs.

7. Find the best possible bounds for each of the following functions.

(a) $x^3+3x^2+3x,$ for $-3 \le x \le 3$

(b) $\sin 2x + 2x,$ for $0 \le x \le 2\pi$

ANSWER:

(a) Let $y = x^3+3x^2+3x$. To locate the critical points, we solve $y' = 0$:

$$\begin{aligned}y' &= 3x^2+6x+3 = 0\\ &= 3(x^2+2x+1) = 0\\ &= 3(x+1)(x+1) = 0\end{aligned}$$

The critical point is $x = -1$.

To find the global minimum and maximum on $-3 \le x \le 3$, we check the critical point and the end points.

$y(-1) = (-1)^3 + 3(-1)^2 + 3(-1) = -1 + 3 - 3 = -1$
$y(-3) = (-3)^3 + 3(-3)^2 + 3(-3) = -9$
$y(3) = (3)^3 + 3(3)^2 + 3(3) = 63$

Thus the global minimum is at $x = 1$ and the global maximum is at $x = 3$. $-9 \leq y \leq 63$.

(b) Let $y = \sin 2x + 2x$.
$y' = 2\cos 2x + 2$.
$y' = 0$ when $x = \pi/2$

Checking the critical points and the end points:
$y(\pi/2) = \pi$
$y(0) = 0$
$y(2\pi) = 4\pi$

Thus the global minimum is at $x = 0$ and the global maximum is at $x = 2\pi$. $0 \leq y \leq 4\pi$.

8. For $f(x) = 2\cos^2 x - \sin(x)$ and $0 \leq x \leq \pi$, find, to two decimal places, the value(s) of x for which $f(x)$ has a global maximum or global minimum.

ANSWER:
Find the critical points:

$$f'(x) = -4\cos x \sin x - \cos x = \cos x(1 - 4\sin x)$$

$$x = \pi/2 \text{ or } x = \arcsin(-1/4) \approx -0.25.$$

Since -0.25 is not in the domain $0 \leq x \leq \pi$, we need only check the first three answers.

$$f(\pi/2) = -1 \qquad f(0) = 2 \qquad f(\pi) = 2$$

Global maxima are at $x = 0$ and $x = \pi$.

Global minimum is at $x = \pi/2$.

9. Sketch a graph of a function with two local minima, no global maximum, but a global minimum. Indicate the domain of your function.

ANSWER:
Answers will vary. One example:

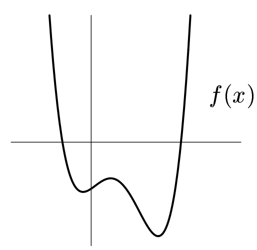
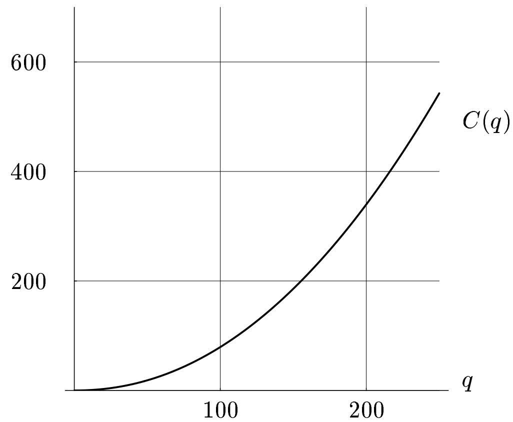


Figure 4.3.58

Questions and Solutions for Section 4.4

1. The cost $C(q)$ (in dollars) of producing a quantity of q of a certain product is shown in the graph below.



Suppose that the manufacturer can sell the product for \$2 each (regardless of how many are sold), so that the total revenue from selling a quantity q is $R(q) = 2q$. The difference

$$\pi(q) = R(q) - C(q)$$

is the total profit. Let q_0 be the quantity that will produce the maximum profit.

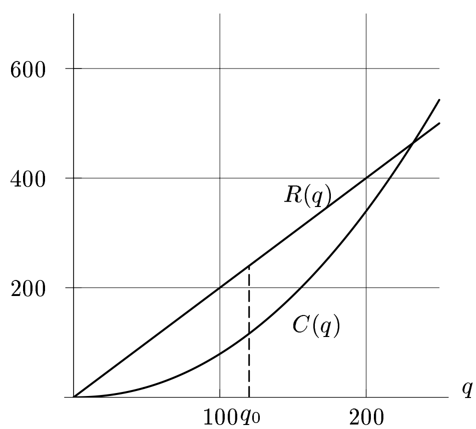
You are told that q_0 can be found by finding the point on the graph where $C'(q_0) = 2$.

(a) Draw the graph of R on the figure above and then explain why this rule makes sense graphically.

(b) Now give a mathematical explanation of the rule, using what you know about maxima and minima.

ANSWER:

(a)



Graphically this makes sense, because the point where the separation between $R(q)$ and $C(q)$ is greatest is the point where the slopes of the two lines are equal (namely 2).

(b) Mathematically, to maximize $\pi(q)$, first find the first derivative,

$$\begin{aligned}\pi'(q) &= R'(q) - C'(q) \\ &= 2 - C'(q).\end{aligned}$$

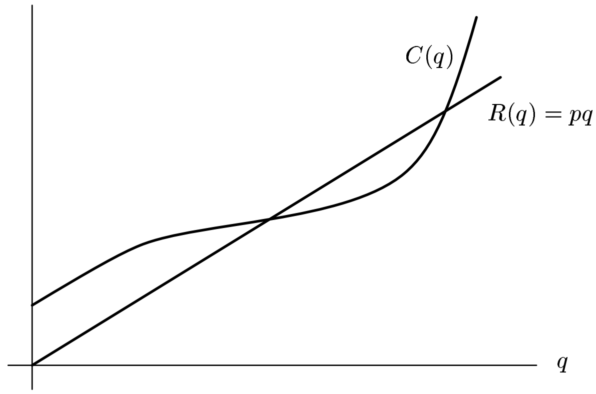
Now set $\pi'(q) = 0$. We know q_0 where $C'(q_0) = 2$ is a critical point. Since $\pi'(q) = -C''(q)$ and the graph of $C(q)$ is concave up, $\pi''(q) \leq 0$. Hence q_0 is a local maximum. Since $\pi''(q) \leq 0$ for all q, q_0 is the global maximum.

2. Given below is the graph of the cost $C(q)$ of producing the quantity q and the revenue $R(q) = pq$ of selling the quantity q, where p is the price per unit (p is a constant).

(a) Show graphically where the quantity q_0 is such that the average cost per unit, $a(q) = \frac{C(q)}{q}$, is a minimum. Explain what you are doing.

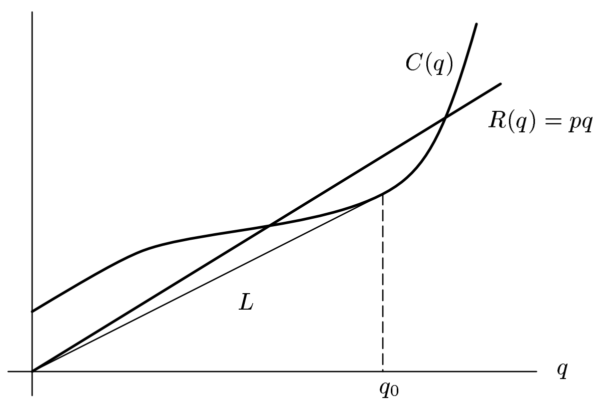
(b) Show graphically where the quantity q_1 is such that your profit $R(q) - C(q)$ is a maximum. Explain the relationship between $C'(q_1)$ and p that should be evident in your picture.

(c) Which is larger, $C'(q_0)$ or $C'(q_1)$? If you want to maximize profit, should you minimize your average cost? Explain your answer.



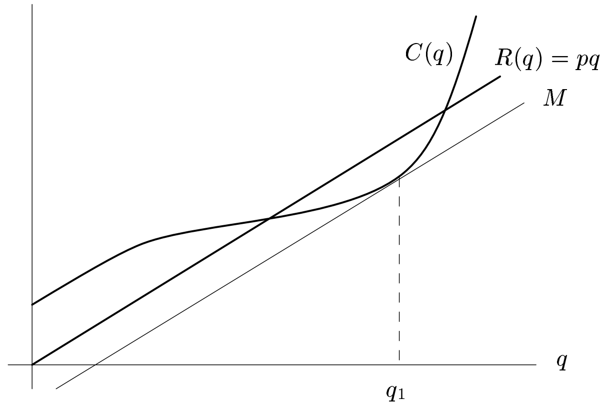
ANSWER:

(a) The quantity $a(q) = C(q)/q$ is the slope of a line joining the point $(q, C(q))$ to the origin. So to minimize $a(q)$ we must search among lines that join the origin to a point on the graph of C and find the one L of least slope. From the figure, we can see that the line L will be tangent to C at the point $(q_0, C(q_0))$. The quantity q_0 is the one which minimizes $a(q)$. (We can exclude the possibility of an end-point minimum by looking at the given figure.) Also, notice that C will be concave up near q_0 because it lies above its tangent L.

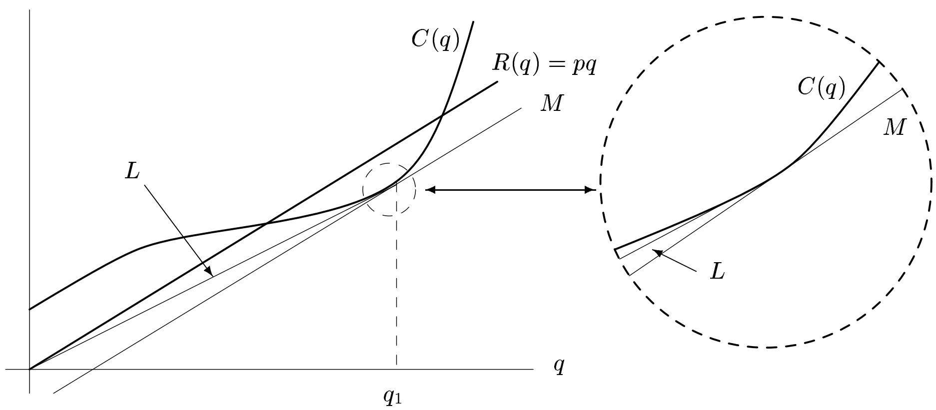


This result can also be obtained analytically. We find that $a'(q) = (qC'(q) - C(q))/q^2$. If q_0 is a critical point for a, then $a'(q_0) = 0$, so $q_0 C'(q_0) - C(q_0) = 0$, which leads to $C'(q_0) = C(q_0)/q_0$; i.e. the slope of the tangent at $(q_0, C(q_0))$ is the same as the slope of the line joining $(q_0, C(q_0))$ to the origin, which is exactly what we found above. This will be a minimum point if $a''(q_0) > 0$. Since $a''(q) = C''(q)/q - 2(qC'(q) - C(q))/q^3$, we find $a''(q_0) = C''(q_0)/q_0$. Since $q_0 > 0$, we shall have $a''(q_0) > 0$ if and only if $C''(q_0) > 0$; i.e., C is concave up at q_0. The geometric argument gives much clearer insight into the problem.

(b) The profit π will be maximized at a point q_1 where $\pi'(q_1) = 0$. (Again we rely on the picture to exclude the possibility of an end-point maximum.) So $\pi'(q_1) = R'(q_1) - C'(q_1) = 0$, giving $p = C'(q_1)$. Hence the profit π is maximized at the point q_1 where the tangent M to C is parallel to the graph of $R(q)$.

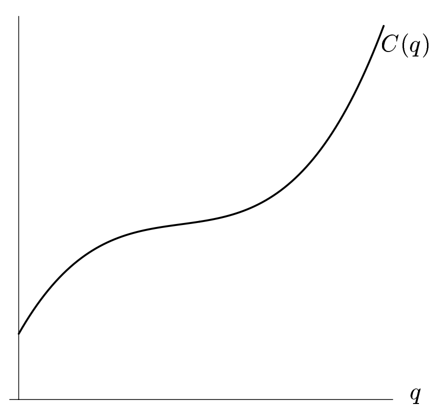


(c) The picture shows that the slope of L is less than the slope of M, so $C'(q_0) < C'(q_1)$. Thus if we decide to minimize average cost (by setting $q = q_0$), we will *not* in general have maximized profit (which requires $q = q_1$).



The enlargement shows that since C is concave up in this region, $q_0 < q_1$.

3. The cost $C(q)$ (in dollars) of producing a quantity q of a certain product is shown in the graph below.



The *average cost* is $a(q) = \frac{C(q)}{q}$.

(a) Interpret $a(q)$ graphically, as the slope of a line in the sketch above.
(b) Based on the graphical interpretation in (a), find on the graph the quantity q_0 where $a(q)$ is minimal.
(c) Now suppose that the fixed costs (i.e., the costs of setting up before production starts) are doubled. How does this affect the cost function? Sketch the new cost function on the same set of axes as the original one.

(d) Let q_1 be the quantity where the new $a(q)$ is minimal. Where is q_1 in relation to q_0? Does your answer make sense in terms of economics?

ANSWER:

(a) The function $a(q)$ is the slope of the line through $(0,0)$ and $(q, C(q))$.

(b) The average cost is minimized when the slope of the line joining $(0,0)$ to $(q, C(q))$is minimal. This occurs when the line L is tangent to the graph of $C(q)$, as drawn in Figure 4.4.59. Thus $(q_0, C(q_0))$ is the point where L touches the graph of $C(q)$.

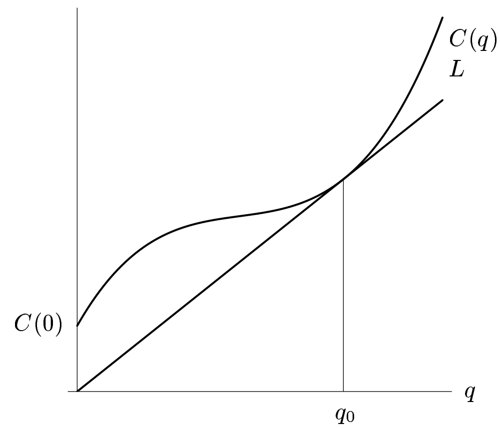
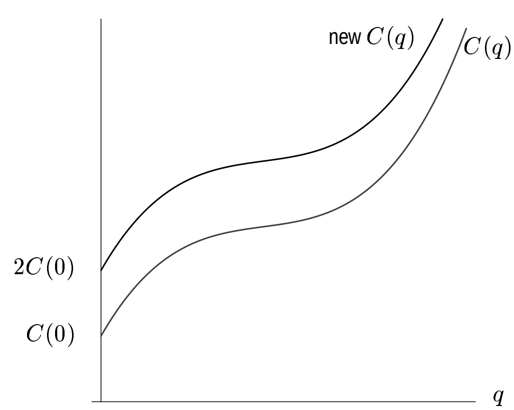


Figure 4.4.59

(c) The cost function is shifted up by $C(0)$.



(d) Suppose M is the tangent line which gives the new minimal value of $a(q)$. From the figure, we can see that M is steeper than L. Since C is concave up, M touches the cost graph to the right of q_0. Thus $q_1 > q_0$. In practical terms, this means that if the start up costs are double, the manufacturer must make more of the product to minimize the average cost. This makes economic sense.

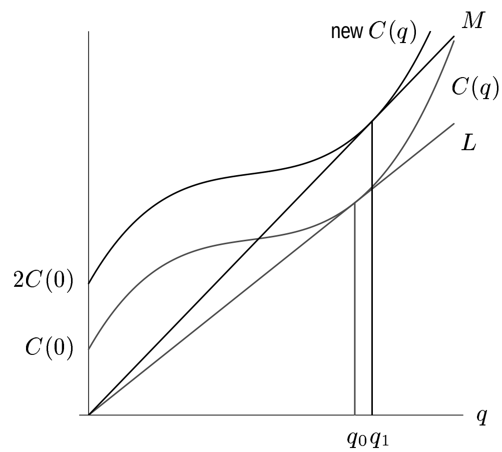


Figure 4.4.60

4. Total cost and revenue are approximated by the functions $C = 1500 + 3.7q$ and $R = 5q$, both in dollars. Identify the fixed cost, marginal cost per item, and profit.
   ANSWER:
   Fixed cost =1500, marginal cost per item =3.7, and profit $= R - C = 5q - (3.7q + 1500) = 1.3q - 1500$.
5. Write a formula for total cost as a function of quantity r when fixed costs are $\$50,000$ and variable costs are $\$1,200$ per item.
   ANSWER:
   Total cost $=50,000 + 1,200r$.
6. The revenue for selling q items is $R(q) = 400q - 2q^2$ and the total cost is $C(q) = 100 + 40q$. Write a function that gives the total profit earned, and find the quantity which maximizes profit.
   ANSWER:

$$\begin{aligned} P &= 400q - 2q^2 - (100 + 40q) \\ &= 360q - 2q^2 - 100 \end{aligned}$$

   To find the maximum profit we find the critical points of P :

$$P'(q) = 360 - 4q$$

$$P'(q) = 0 \text{ when } q = 90.$$

   $P''(q) = -4$, so profit will be a maximum when $q = 90$.
7. The function $C(r) = 15r^2 - 50$ gives cost in dollars of producing r items. What is the marginal cost of increasing r by 1 item from the current production level of $r = 5$?
   ANSWER:
   Marginal cost $=C'(r) = 30r$. For $r = 5, C'(r) = 150$ dollars.
8. Find the marginal cost and marginal revenue for $q = 100$ when the fixed costs in dollars are 3,000 and the variable costs are 200 per item and each sells for $\$400$.
   ANSWER:
   Marginal cost $= 200$

   Marginal revenue $= 400$.
9. Find the quantity q which maximizes profit if the total revenue, $R(q)$, and the total cost, $C(q)$, are given in dollars by

   $R(q) = 7q - 0.02q^2$

   $C(q) = 400 + 1.5q$, where $0 \le q \le 600$ units.

ANSWER:
We look for production levels that give marginal revenue=marginal cost:

$$MR = R'(q) = 7 - 0.04q$$

$$MC = C'(q) = 1.5$$

so

$$7 - 0.04q = 1.5$$

$$q = 138 \text{ units}$$

Does this value of q represent a local maximum or minimum of profit? We can tell by looking at production levels of 137 units and 139 units.

When $q = 137$ we have $MR = 1.52$.

When $q = 139$, $MR = 1.44$.

Therefore, $q = 138$ is a local maximum for the profit function.

Questions and Solutions for Section 4.5

1. A wire of length L is cut into two pieces. [We allow the possibility that one of the pieces has zero length.] The first piece is bent into a circle, the second into a square. How long should the piece of wire that is bent into a *circular* shape be in order to *maximize* the *sum* of the areas of the *two* shapes? How long should that piece be if you want to *minimize* the sum of the areas? [For full points, you must explain in detail *why* the length you suggest will indeed result in a maximum or minimum for the sum of the areas of the two shapes.]

ANSWER:
$x =$ length bent into the circle.
$L - x =$ length bent into the square.
$x =$ circle of circumference $x \Rightarrow x = 2\pi r$

$r = \dfrac{x}{2\pi}$, so area of circle $= \dfrac{\pi}{4\pi^2}x^2 = \dfrac{x^2}{4\pi}$

$x \Rightarrow$ Area $= \left(\dfrac{L-x}{4}\right)^2$

Let $A(x) =$ sum of areas.

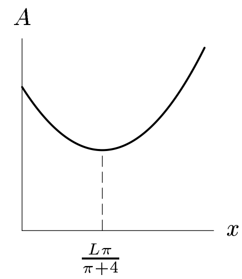
$$A(x) = \frac{\pi}{4\pi^2}x^2 + \frac{1}{16}(L-x)^2 = \frac{1}{4\pi}x^2 + \frac{1}{16}(L-x)^2, x \in [0, L]$$

Now find a max/min of $A(x)$ on $[0, L]$:

$$\frac{dA}{dx} = \frac{2}{4\pi}x + \frac{2}{16}(L-x)(-1) = \frac{1}{2\pi}x - \frac{1}{8}(L-x) = \left(\frac{1}{2\pi} + \frac{1}{8}\right)x - \frac{1}{8}L$$

$$\frac{dA}{dx} = 0 \text{ when } \frac{1}{2\pi}x + \frac{1}{8}x = \frac{1}{8}L \Rightarrow x = \frac{L}{1 + \frac{4}{\pi}} = \frac{L\pi}{\pi + 4}$$

$\dfrac{d^2A}{dx^2} = \left(\dfrac{1}{2\pi} + \dfrac{1}{8}\right) > 0$ for all x, i.e. the length $L\pi/(\pi + 4)$ will give a minimum area. Consider the endpoints: $A(0) = L^2/16$, $A(L) = L^2/4\pi > A(0)$. Therefore, the absolute maximum area occurs when $x = L$.



2. A rectangular building is to cover $20{,}000$ square feet. Zoning regulations require 20 foot frontages at the front and the rear and 10 feet of space on either side. Find the dimensions of the smallest piece of property on which the building can be legally constructed.

ANSWER:

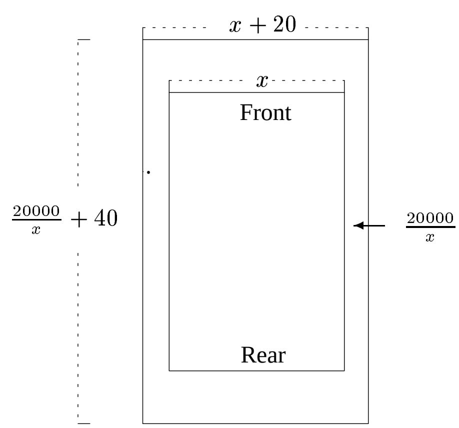
Assume that the front of the building is x feet long. Then the sides of the building must be $\frac{20000}{x}$ feet long. The dimensions of the property will thus be $x + 20$ and $\frac{20000}{x} + 40$. The area of the plot will be:

$$A(x) = (x + 20)\left(\frac{20000}{x} + 40\right) = 20000 + \frac{400000}{x} + 800 + 40x$$

Differentiating and setting this equal to zero,

$$\begin{aligned} A'(x) &= -\frac{400000}{x^2} + 40 = 0 \\ 40 &= \frac{400000}{x^2} \\ x^2 &= 10000 \\ x &= 100 \end{aligned}$$

(No negative roots allowed!) The property will thus have dimensions $120' \times 240'$.



3. The regular air fare between Boston and San Francisco is \$500. An airline flying 747s with a capacity of 380 on this route observes that they fly with an average of 300 passengers. Market research tells the airlines' managers that each \$20 fare reduction would attract, on average, 20 more passengers for each flight. How should they set the fare to maximize their revenue? Why?

ANSWER:

Suppose we lower the fare by x dollars. Then the revenue generated at this fare will be:

$$r(x) = (500 - x)(300 + x),$$

where x is the discount in dollars. (For every dollar decrease in fare, another person will fly.) To maximize revenue, we examine the critical points of r:

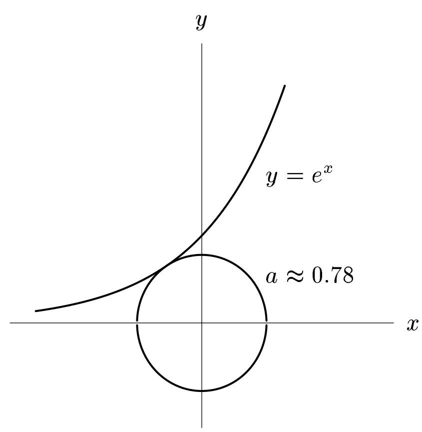
$$\begin{aligned}\frac{dr}{dx} &= \frac{d}{dx}\left((500-x)(300+x)\right)\\ &= -(300+x)+(500-x)\\ &= 200-2x.\end{aligned}$$

So, $\frac{dr}{dx} = 0$ when $x = 100$. But this is too high, since when $x = 100$, the number of people on the plane is 400. So the maximum must occur when $x = 80$ or 0 (the endpoints.) When $x = 80$, $r(x) = 420 \cdot 380 = 159{,}600$; when $x = 0$, $r(x) = 300 \cdot 500 = 150{,}000$. Hence, the airline would do best to fill up its planes at \$420.

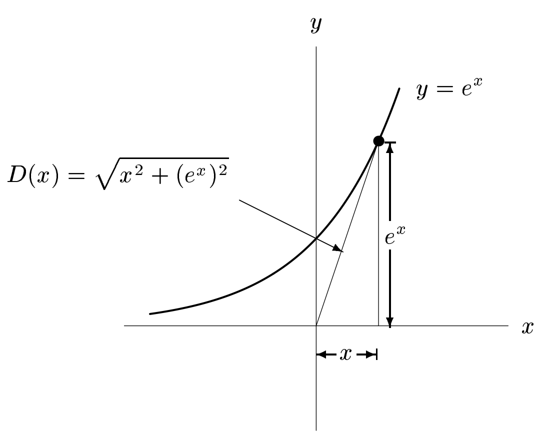
4. The purpose of this problem is to find the x-value which produces the shortest distance from the point $(0, 0)$ to the curve $y = e^x$.
   (a) Use the computer. Plot the curve $y = e^x$ and a circle of radius a centered at $(0, 0)$. By varying a, estimate the value of x which gives the shortest distance from $(0, 0)$ to the curve. What is your estimate? Explain what you did.
   (b) Using calculus, confirm that the statement you made in part (a) is accurate. (You may need to use a computer or a calculator to find roots.) What is your estimate for x, accurate to 5 decimal places? Explain what method you used to get this accuracy.

   ANSWER:

   (a) When $a = 0.78$, the circle barely touches the curve e^x. They touch at $x \approx -0.43$, so that value of x gives us the point on the curve closest to the the origin.



   (b) The distance from the point (x, e^x) to the origin is $D(x) = \sqrt{x^2 + (e^x)^2} = \sqrt{x^2 + e^{2x}}$, so, in order to find the closest e^x comes to the origin, we want to minimize $D(x)$. By taking the derivative and solving for a root (using a root-finding program), we find that $x = -0.42630$ gives the desired minimum.



5. If you throw a stone into the air at an angle of θ to the horizontal, it moves along the curve

$$y = x\tan\theta - \frac{x^2}{2k}(1+\tan^2\theta)$$

where y is the height of the stone above the ground, x is the horizontal distance, and k is a positive constant.

(a) If the angle θ is fixed, what value of x gives the maximum height? (Your answer will contain k and θ.) Explain how you know this x-value gives a maximum.

(b) Suppose the stone is to be thrown over a wall at a fixed horizontal distance ℓ away from you. If you can vary θ, what is the highest wall that the stone can go over? (Your answer will contain k and ℓ.) You do not need to justify that your answer is a maximum.

ANSWER:

(a) $\dfrac{dy}{dx} = \tan\theta - \dfrac{2x}{2k}(1+\tan^2\theta) = 0$ gives $x = \dfrac{k\tan\theta}{1+\tan^2\theta}$.

This value gives a maximum because curve is an upside down parabola.

(b) When $x = \ell$, $y = \ell\tan\theta - \dfrac{\ell^2}{2k}(1+\tan^2\theta)$

$\dfrac{dy}{d\theta} = \dfrac{\ell}{\cos^2\theta} - \dfrac{\ell^2}{2k}\left(\dfrac{2\tan\theta}{\cos^2\theta}\right) = 0$ when $\dfrac{\ell}{k}\dfrac{(k-\ell\tan\theta)}{\cos^2\theta} = 0$ so $\tan\theta = \dfrac{k}{\ell}$.

Thus height is $y = \ell\cdot\dfrac{k}{\ell} - \dfrac{\ell^2}{2k}\left(1+\dfrac{k^2}{\ell^2}\right) = k - \dfrac{1}{2k}(\ell^2+k^2) = \dfrac{k^2-\ell^2}{2k}$

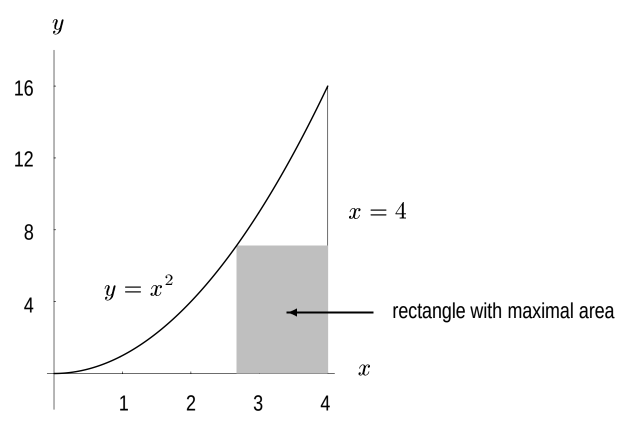
6. A rectangle with its base on the x-axis is inscribed in the region bounded by the curve $f(x) = x^2$, the x-axis and the line $x = 4$. Find the dimensions of the rectangle with maximal area.

ANSWER:

Choose a rectangle that has one corner at $(4,0)$. If the other vertical side of the rectangle intersects the x-axis at $(x,0)$, then the length of the base of the rectangle is $4-x$, and the height is x^2. The area will thus be $(4-x)x^2$. To maximize this, set the derivative equal to 0. This gives

$$\frac{d}{dx}(4x^2 - x^3) = 8x - 3x^2 = x(8-3x) = 0.$$

There are two roots to this equation, $x = 0$ and $x = 8/3$. The root $x = 0$ corresponds to the rectangle of height 0 lying along the x-axis, and it certainly does not maximize the area. Since the second derivative of the area, $8-6x$ is negative when $x = 8/3$, area is indeed maximized when $x = 8/3$. Thus the base has length ≈ 1.33 and the height is ≈ 7.11.



7. A submarine can travel 30 mi/hr submerged and 60 mi/hr on the surface. The submarine must stay submerged if within 200 miles of shore. Suppose that this submarine wants to meet a surface ship 200 miles off shore. The submarine leaves from a port 300 miles along the coast from the surface ship. What route of the type sketched below should the sub take to minimize its time to rendezvous?

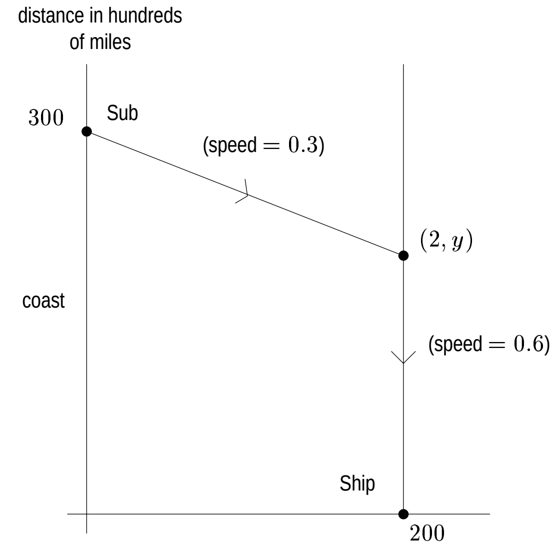
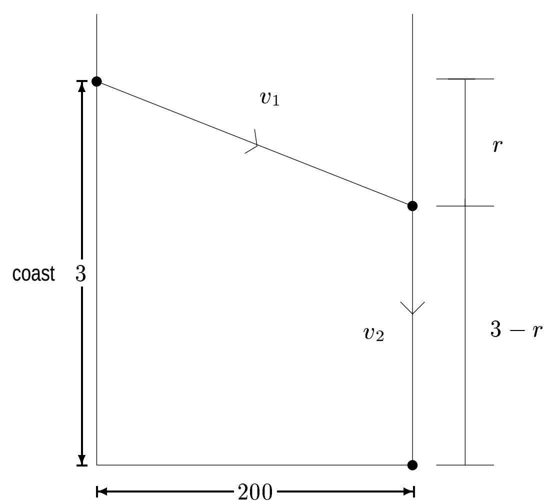


Figure 4.5.61

ANSWER:



We wish to minimize $t = t_1 + t_2$, where t_1 is spent traveling with speed 0.3, and t_2 is spent traveling with speed 0.6. Let $r = 3 - y$. We know (by the Pythagorean Theorem) that the distance traveled at the slow speed is $\sqrt{4+r^2}$, so we have

$$\begin{aligned} t &= t_1 + t_2 \\ &= \frac{\sqrt{4+r^2}}{0.3} + \frac{3-r}{0.6} \\ &= \frac{1}{0.6}(2\sqrt{4+r^2} + 3 - r) \end{aligned}$$

So $\dfrac{dt}{dr} = \dfrac{1}{0.6}(2r(4+r^2)^{-\frac{1}{2}} - 1)$, which is zero when $r = \frac{2}{\sqrt{3}}$.

$$\text{Since } \left.\frac{d^2t}{dr^2}\right|_{r=\frac{2}{\sqrt{3}}} = \frac{1}{0.3}((4+r^2)^{-\frac{1}{2}} + r\cdot(-\frac{1}{2})(4+r^2)^{-\frac{3}{2}}\cdot 2r\Big|_{r=\frac{2}{\sqrt{3}}}$$

$$= \frac{1}{0.3}\left((4+r^2)^{-\frac{1}{2}} - r^2(4+r^2)^{-\frac{3}{2}}\right)\Big|_{r=\frac{2}{\sqrt{3}}}$$
$$= \frac{1}{0.3}\left(\frac{4}{(4+r^2)^{\frac{1}{2}}}\right)\Big|_{r=\frac{2}{\sqrt{3}}}$$
$$= \frac{1}{0.3}\left(\frac{4}{\left(\frac{10}{3}\right)^{\frac{3}{2}}}\right) > 0,$$

$r = \frac{2}{\sqrt{3}}$ is indeed a minimum. So y should be equal to $3 - \frac{2}{\sqrt{3}}$ for the submarine to minimize time.

8. Daily production levels in a plant can be modeled by the function $G(t) = -3t^2 + 12t - 12$ which gives units produced at t, the number of hours since the factory opened at 8am. At what time during the day is factory productivity a maximum?
   ANSWER:
   To find the maximum productivity, first find the critical points of $G(t)$:

$$G'(t) = -6t + 12 = 0$$
$$t = 2$$

   Since $G''(t) = -6$, productivity is a maximum at 10am, 2 hours after the factory opens.

9. The number of plants in a terrarium is given by the function $P(c) = -1.2c^2 + 4c + 10$, where c is the number of mg of plant food added to the terrarium. Find the amount of plant food that produces the highest number of plants.
   ANSWER:
   Find the critical points:

$$P'(c) = -2.4c + 4 = 0$$
$$c \approx 1.67$$

   Since $P''(c) = -2.4, c \approx 1.67$ is where the maximum occurs.

10. The function $y = .2(x+2)^2 - 5x + 2$ gives the population of a town (in 1000's of people) at time x where x is the number of years since 1980. When was the population a minimum?
    ANSWER:

$$y' = .4(x+2) - 5$$
$$= .4x - 4.2$$
$$x = 10.5$$

    Since $y'' = .4$, the population was a minimum halfway through 1990.

Questions and Solutions for Section 4.6

1. Find the derivative:

$$y = \cosh^2(9x) + \sinh^2(9x)$$

   ANSWER:

$$y' = 18x\cosh(9x)\sinh(9x) - 18\sinh(9x)\cosh(9x) = 36\cosh(9x)\sinh(9x).$$

2. Find the derivative:

$$y = \frac{\cosh(2x)}{\cosh^2(x)}$$

   ANSWER:

$$y' = \frac{2\sinh(2x)\cdot\cosh^2(x) - 2\sinh x\cdot\cosh x\cdot\cosh(2x)}{(\cosh x)^4}$$

3. Find the derivative:

$$y = \sinh(\sin x)$$

ANSWER:

$$y' = \cos x \cdot \cosh(\sin x)$$

4. Find the derivative:

$$y = \tanh(3x)$$

ANSWER:

$$y' = \frac{3}{\cosh^2(3x)}$$

5. An arch over a lake has the form $y = 200 - 40\cosh(x/40)$ where x is the number of feet from a point on one side of the lake. What is the highest point on the arch?

ANSWER:

Find the value of x such that $y' = 0$:

$$y' = -\sinh\frac{x}{40}$$
$$x = 0$$

We need to check the second derivative to determine the concavity:

$$y'' = \frac{-\cosh\frac{x}{40}}{40}$$

At $x = 0, y'' = -0.025$, so $x = 0$ is where the maximum occurs and the height is 160 feet.

Questions and Solutions for Section 4.7

1. Prove that if $f(x) = x^2$ then there exists a number $c,\ 2 < c < 5$, such that $f'(c) = 7$.

ANSWER:

By the Mean Value Theorem, there is a number c, with $2 < c < 5$, such that

$$f'(c) = \frac{f(5) - f(2)}{5 - 2}$$
$$= \frac{5^2 - 2^2}{5 - 2} = \frac{25 - 4}{3} = 7.$$

2. Prove that if $f(x) = x^2 + 2$, then f is increasing on $[0, 10]$.

ANSWER:

$f(x) = x^2 + 2$ is continuous on $[0, 10]$ and differentiable on $(0, 10)$. Since $f'(x) = 2x$, $f'(x) > 0$ on $(0, 10)$. By the Increasing Function Theorem, f is increasing on $[0, 10]$.

3. What can you conclude about f if f is continuous on $[0, 5]$, differentiable on $(0, 5)$, and $f'(x) > 0$ on $(0, 5)$?

ANSWER:

By the Increasing Function Theorem, we conclude that f is increasing on $[0, 5]$.

4. The speed of a car at time $t,\ 5 \le t \le 10$, is given by $f(t) = -2(t - 10)^2$. What is guaranteed by the Mean Value Theorem about

$$\frac{f(10) - f(5)}{10 - 5}?$$

ANSWER:

Since $f(t)$ is continuous on $[5, 10]$ and differentiable on $(5, 10)$, there exists a number $c,\ 5 < c < 10$, such that

$$f'(c) = \frac{f(10) - f(5)}{10 - 5} = \frac{0 - (-2(5 - 10)^2)}{5} = 10.$$

So there will be a point in the interval $[5, 10]$ when the instantaneous acceleration will be equal to 10, the average acceleration over the interval.

5. What does the Extreme Value Theorem allow us to conclude about f if f is continuous on $[0, 100]$?

ANSWER:

f has a global maximum and a global minimum on $[0, 100]$.

Review Questions and Solutions for Chapter 4

1. One fine day you take a hike up a mountain path. Using your trusty map you have determined that the path is approximately in the shape of the curve

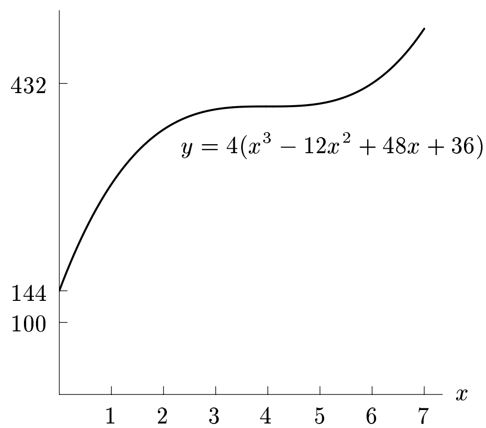
$$y = 4(x^3 - 12x^2 + 48x + 36)$$

Here y is the elevation in feet above sea level and x is the horizontal distance in miles you have traveled, but your map only shows the path for 7 miles, horizontal distance.

(a) How high above sea level do you start your hike?
(b) How high above sea level are you after 7 miles?
(c) Use your calculator to draw an informative graph of the path (i.e. One that looks like a cubic.) and sketch your answer. Show the scale you use. (Take your answers in parts (a) and (b) into account!)
(d) Where on the path is a nice flat place to stop for a picnic? Explain.
(e) Estimate the elevation after 7.5 horizontal miles. (You do not know the shape of the path explicitly after 7 miles!)
(f) If your friend, who is *not* in "good shape" followed this path for 15 miles total, in horizontal distance, the day before, does it make sense that the equation for the elevation continues to hold much beyond the 7 mile mark? Explain.

ANSWER:

(a) At the start of the hike, $x = 0$, so $y = 4(36) = 144$ feet.
(b) After 7 miles, $x = 7$, so $y = 508$ feet.
(c)



(d) A flat place occurs when the elevation is neither increasing nor decreasing, i.e., when $\dfrac{dy}{dx} = 4(3x^2 - 24x + 48) = 0$. This happens when $x = \frac{24 \pm \sqrt{24^2 - 4 \cdot 3 \cdot 48}}{6} = 4$, so stop about 4 miles along for a picnic.
(e) When $x = 7$, $\dfrac{dy}{dx} = 4(3(7)^2 - 24(7) + 48) = 108$. Local linearization yields

$$\begin{aligned} y(7.5) &\approx y(7) + 0.5(y'(7)) \\ &= 508 + (0.5)(108) \\ &= 562 \text{ feet.} \end{aligned}$$

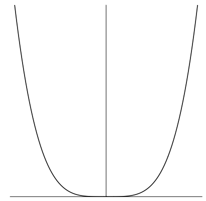
(f) No, since according to the equation $y(15) = 5724$, my friend would have to hike over a mile up, which is not easy. Beyond the seventh mile, the equation says things go uphill quickly. So if he does walk for 15 miles and is also out of shape, then in all likelihood the equation does not apply beyond the 7 mile point.

For Problems 2– 4, decide whether each statement is true or false, and provide a short explanation or a counterexample.

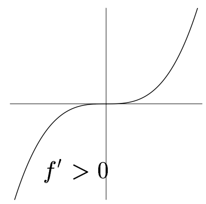
2. If $f'' = 0$ at $x = 0$, then the graph of f changes concavity at $x = 0$.

ANSWER:

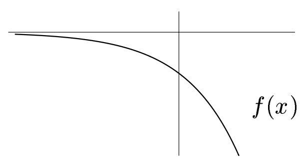
FALSE. The graph of $f(x) = x^4$ has $f''(0) = 0$, but it is concave up everywhere.



3. If $f' > 0$ on an interval, the function is concave UP on the interval.
ANSWER:
FALSE. The graph of $f(x) = x^3$ has $f' > 0$ for all $x \neq 0$, even though it is concave down for $x < 0$.



4. If f is always decreasing and concave down, then f must have at least one root.
ANSWER:
FALSE. The function $f(x) = -e^x$ is always decreasing and concave down and it has no roots.



5. Given the function $f(x) = x^2e^{-2x}$, find all the maxima, minima and points of inflection and use this information to sketch the graph.
ANSWER:

$$\begin{aligned} f(x) &= x^2e^{-2x} \\ f'(x) &= 2xe^{-2x} - 2x^2e^{-2x} = 2(x - x^2)e^{-2x} \\ f''(x) &= 2(1 - 2x)e^{-2x} - 4(x - x^2)e^{-2x} = 2(1 - 4x + 2x^2)e^{-2x}. \end{aligned}$$

The zeros of these functions are as follows:

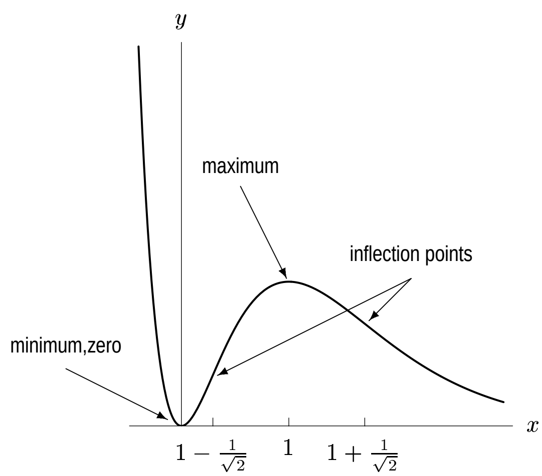
$f(x)$	is zero at	$x = 0$.
$f'(x)$	is zero at	$x = 0$ and $x = 1$.
$f''(x)$	is zero at	$x = 1 + \frac{1}{\sqrt{2}}$ and $x = 1 - \frac{1}{\sqrt{2}}$.

So the critical points of $f(x)$ are at 0 and 1, and its possible inflection points are at $1 + \frac{1}{\sqrt{2}}$ and $1 - \frac{1}{\sqrt{2}}$. We now need to classify these points, and learn what the graph looks like. We make a table of the approximate values of f, f', and f'' at these points, and their signs on the intervals between:

x	$f(x)$	$f'(x)$	$f''(x)$	
$(-\infty, 0)$	+	–	+	
0	0	0	2	zero, minimum
$(0, 1-\frac{1}{\sqrt{2}})$	+	+	+	
$1-\frac{1}{\sqrt{2}}$	0.0478	0.2306	0	inflection point
$(1-\frac{1}{\sqrt{2}}, 1)$	+	+	–	
1	0.1353	0	-0.2707	maximum
$(1, 1+\frac{1}{\sqrt{2}})$	+	–	–	
$1+\frac{1}{\sqrt{2}}$	0.0959	-0.0794	0	inflection point
$(1+\frac{1}{\sqrt{2}}, \infty)$	+	–	+	

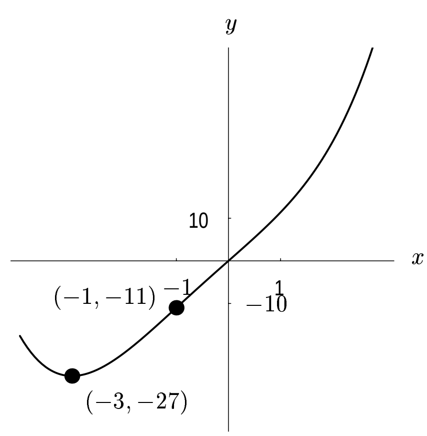
Notice that the table tells us that $f''(x)$ does indeed change sign across $x = 1 + \frac{1}{\sqrt{2}}$ and $x = 1 - \frac{1}{\sqrt{2}}$, so these are inflection points.

Using the information in the table, we now draw the graph:



6. Draw the graph of a polynomial of degree four that has a local minimum at $(-3, -27)$ and inflection points at $(-1, -11)$ and the origin.

ANSWER:



Chapter 5 Exam Questions

Questions and Solutions for Section 5.1

1. Consider a sports car which accelerates from 0 ft/sec to 88 ft/sec in 5 seconds (88 ft/sec = 60 mph). The car's velocity is given in the table below.

t	0	1	2	3	4	5
$V(t)$	0	30	52	68	80	88

(a) Find upper and lower bounds for the distance the car travels in 5 seconds.
(b) In which time interval is the average acceleration greatest? Smallest?

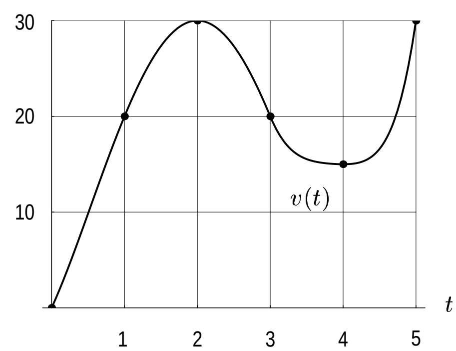
ANSWER:

(a) Since $v(t)$ is increasing, a lower bound is given by the left-hand sum, and an upper bound is given by the right-hand sum.

$$\text{lower bound } = 0 + 30 + 52 + 68 + 80 = 230 \text{ feet};$$
$$\text{upper bound } = 30 + 52 + 68 + 80 + 88 = 318 \text{ feet}.$$

(b) In the first interval it is greatest. In the last interval it is smallest.

2. The graph shown below is that of the velocity of an object (in meters/second).



(a) Find an upper and a lower estimate of the total distance traveled from $t = 0$ to $t = 5$ seconds.
(b) At what times is the acceleration zero?

ANSWER:

(a) $v(0) = 0$, $v(1) = 20$, $v(2) = 30$, $v(3) = 20$, $v(4) = 15$, $v(5) = 30$.
Since this function is increasing over $[0, 2]$, decreasing over $[2, 4]$, and increasing over $[4, 5]$, we need to break the function into three parts in order to determine an overestimate and an underestimate of the distance traveled.

Over $[0, 2]$, lower bound $= 0 + 20 = 20$,
upper bound $= 20 + 30 = 50$.
Over $[2, 4]$, lower bound $= 20 + 15 = 35$,
upper bound $= 30 + 20 = 50$.
Over $[4, 5]$, lower bound $= 15$,
upper bound $= 30$.

So, adding the upper and lower bounds for the separate intervals, we get

$$\text{lower bound on distance traveled} = 20 + 35 + 15 = 70 \text{ meters};$$
$$\text{upper bound on distance traveled} = 50 + 50 + 30 = 130 \text{ meters}.$$

(b) $v' = 0$ at $t = 2$ and $t = 4$.

3. At time t, in seconds, the velocity, v, in miles per hour, of a car is given by

$$v(t) = 5 + .5t^2 \text{ for } 0 \leq t \leq 8.$$

Use $\Delta t = 2$ to estimate the distance traveled during this time. Find the left- and right-hand sums, and the average of the two.

ANSWER:

Using $\Delta t = 2$,

$$\begin{aligned} \text{Left} - \text{hand sum} &= v(0)\cdot 2 + v(2)\cdot 2 + v(4)\cdot 2 + v(6)\cdot 2 \\ &= 2(5) + 2(5 + .5(2)^2) + 2(5 + .5(4)^2) + 2(5 + .5(6)^2) \\ &= 96 \\ \text{Right} - \text{hand sum} &= v(2)\cdot 2 + v(4)\cdot 2 + v(6)\cdot 2 + v(8)\cdot 2 \\ &= 160 \end{aligned}$$

The average is $(96 + 160)/2 = 128$.

4. A car travels exactly 20 mph faster than the car from Exercise 3. What are the left- and right-hand estimates of the distance traveled by the new car (also using $\Delta t = 2$)?

ANSWER:

Each term of the new estimates will be $2 \cdot 20$ miles greater for a total of $4 \cdot 2 \cdot 20 = 160$. The left-hand sum will be 256 and the right-hand sum will be 320.

5. If an upper estimate of the area of a region bounded by the curve in Figure 5.1.62, the horizontal axis and the vertical lines $x = 3$ and $x = -3$ is 15, what is the upper estimate if the graph is shifted up one unit?

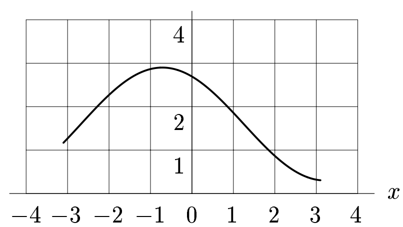


Figure 5.1.62

ANSWER:

The new area will be 6 units more, or 21 units.

6. Figure 5.1.63 shows the graph of the velocity, v, of an object (in meters/sec.). If the graph were shifted up two units, how would the total distance traveled between $t = 0$ and $t = 6$ change? What would it mean for the motion of the object?

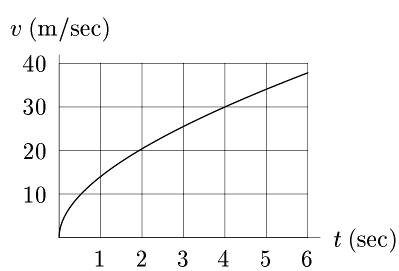


Figure 5.1.63

ANSWER:

The distance would increase by $6 \cdot 2$ or 12 meters. Shifting the graph up 2 units means the velocity at each time would be 2 mph greater.

7. Estimate the area bounded by the x-axis, y-axis, $y = e^{-x} + 1$ and $x = 2$ with an error of at most 0.8.
   ANSWER:
   We want the error to be less than 0.8, so we take Δx such that $|f(2) - f(0)|\Delta x < 0.8$, giving

$$\Delta x < \frac{0.08}{|(e^{-2}+1) - (e^0+1)|} \approx 0.93$$

   So take $\Delta x = 1$ or $n = 2$. Then the left sum $\approx 2 + 1.37 = 3.37$ and the right sum $\approx 1.37 + 1.14 = 2.51$, so a reasonable estimate for the sum is $(3.37 + 2.51)/2 = 2.94$.

8. A car is observed to have the following velocities at times $t = 0, 2, 4, 6$:

Table 5.1.10

time(sec)	0	2	4	6
velocity(ft/sec)	0	21	40	66

   Give lower and upper estimates for the distance the car traveled.
   ANSWER:
   Lower estimate $= 0(2) + 21(2) + 40(2) = 122$ feet.
   Upper estimate $= 21(2) + 40(2) + 66(2) = 254$ feet.

9. At time t, in seconds, your velocity v, in meters/sec, is given by

$$v(t) = 2 + 2t^2 \text{ for } 0 \leq \text{t} \leq 6$$

   Use $\Delta t = 2$ to estimate distance during this time.
   ANSWER:

$$\begin{aligned}\text{Left} - \text{hand sum} &= v(0)\cdot 2 + v(2)\cdot 2 + v(4)\cdot 2\\ &= 2\cdot 2 + 10\cdot 2 + 34\cdot 2 = 92 \text{ meters}\\ \text{Right} - \text{hand sum} &= v(2)\cdot 2 + v(4)\cdot 2 + v(6)\cdot 2\\ &= 10\cdot 2 + 34\cdot 2 + 74\cdot 2 = 236 \text{ meters}\end{aligned}$$

   Average $= (92 + 236)/2 = 164$ meters. Distance traveled ≈ 164 meters.

10. Repeat Exercise 9 with $\Delta t = 1$. Compare the accuracy of your two answers.
   ANSWER:

$$\begin{aligned}\text{Left} - \text{hand sum} &= v(0)\cdot 1 + v(1)\cdot 1 + v(2)\cdot 1 + v(3)\cdot 1 + v(4)\cdot 1 + v(5)\cdot 1\\ &= 2\cdot 1 + 4\cdot 1 + 10\cdot 1 + 20\cdot 1 + 34\cdot 1 + 52\cdot 1\\ &= 122 \text{ meters}\\ \text{Right} - \text{hand sum} &= v(1)\cdot 1 + v(2)\cdot 1 + v(3)\cdot 1 + v(4)\cdot 1 + v(5)\cdot 1 + v(6)\cdot 1\\ &= 4\cdot 1 + 10\cdot 1 + 20\cdot 1 + 34\cdot 1 + 52\cdot 1 + 74\cdot 1\\ &= 194 \text{ meters}\end{aligned}$$

   Average $= (122 + 194)/2 = 158$ meters. Distance traveled ≈ 158 meters.

   Since the estimate of 158 meters was obtained using more and smaller intervals of time, it is more accurate than the estimate obtained by using $\Delta t = 2$.

Questions and Solutions for Section 5.2

1. Estimate $\int_8^{10} \ln x\, dx$ with accuracy 0.1. Show why you chose the Δx that you did.
   ANSWER:

Since $\ln x$ is increasing on [8,10],

$$\text{LEFT}(n) < \int_8^{10} \ln x\, dx < \text{RIGHT}(n),$$

and

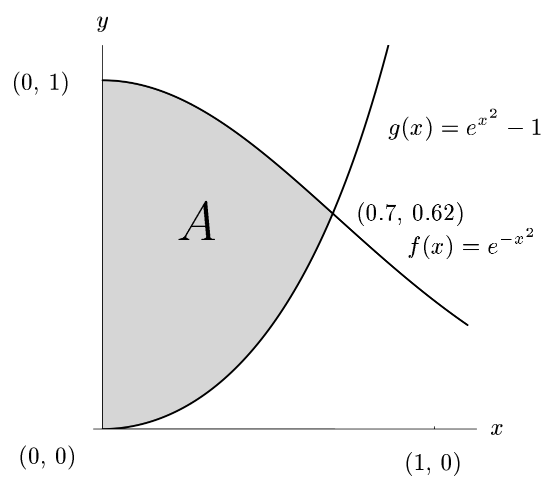
$$|\,\text{RIGHT}(n) - \text{LEFT}(n)| \leq |\ln 10 - \ln 8|\Delta x$$

We need $0.224\,\Delta x < 0.1$, so we take $\Delta x = \frac{1}{3}, n = 6$, and we find $\text{LEFT}(6) \approx 4.35 < \int_8^{10} \ln x\, dx < \text{RIGHT}(6) \approx 4.43$, so $\int_8^{10} \ln x\, dx \approx 4.39$ with an accuracy of 0.1.

2. Consider the region A bounded above by the graph of $f(x) = e^{-x^2}$, bounded below by the graph of $g(x) = e^{x^2} - 1$, and bounded on the left by the y-axis.
   (a) Sketch and label the curves $f(x)$ and $g(x)$ and shade the region A. Find (approximately if necessary) and label the coordinates of the three corner points of A.
   (b) By just looking at your sketch in Part (a), decide whether the area of A is more or less than 0.7. Is it more or less than 0.3? Give a graphical justification of your answers.
   (c) Express the area of the region A as an integral, or as a sum or difference of integrals. Approximate the value(s) of the integral(s) with an accuracy that allows you to decide whether the area of A is more or less than 0.5. Explain what you are doing.
   (d) Name the possible sources of error in your calculation of the area of A in part (b).

   ANSWER:

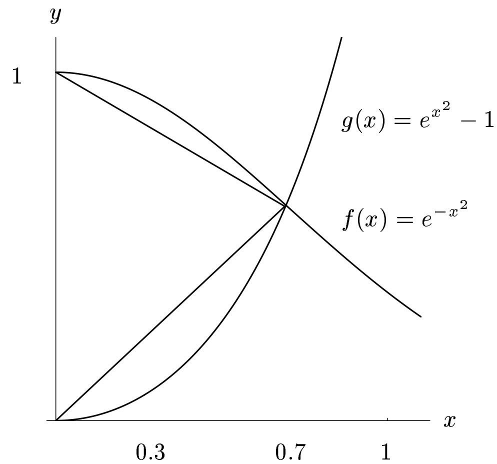
   (a)



   (b) The area of A is more than 0.3 but less than 0.7. To visualize this, draw a triangle connecting all three "corners" of the area. If we take the base of the triangle to be the left side of the area then:

$$A \approx \frac{1}{2}(1)(x) = \frac{x}{2}$$

   Since $x \approx 0.7$, $A \approx 0.35 < 0.7$. Not only is this approximation of A greater than 0.3, we also know it is less than the actual value of A. Thus $A > 0.3$.



(c)

$$\int_0^{0.7} e^{-x^2} dx - \int_0^{0.7} (e^{x^2} - 1) dx = \int_0^{0.7} \left(e^{-x^2} - e^{x^2} + 1 \right) dx$$

We can approximate the value of this integral with left and right Riemann sums, dividing the region $[0, 0.7]$ into fourteen subdivisions. Then

$$0.441235 = \text{RIGHT} < A < \text{LEFT} = 0.492219,$$

which shows that the exact value of A is less than 0.5.

(d) The sources of error from this method are the extra space in A not included in the triangle, and our approximation for the point of intersection of the functions f and g.

3. Using Figure 5.2.64, draw rectangles representing each of the following Riemann sums for the function f on the interval $0 \le t \le 12$. Calculate the value of each sum.

   (a) Left-hand sum with $\Delta t = 3$
   (b) Right-hand sum with $\Delta t = 3$
   (c) Left-hand sum with $\Delta t = 6$
   (d) Right-hand sum with $\Delta t = 6$

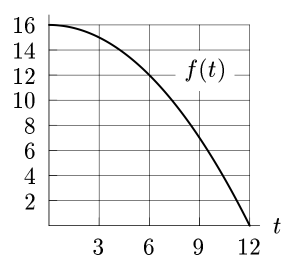


Figure 5.2.64

ANSWER:

(a)

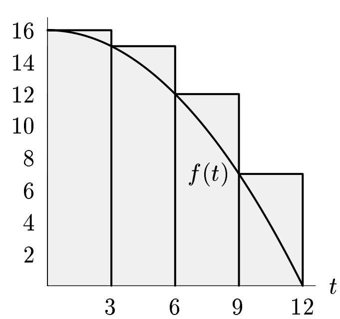


Figure 5.2.65

(b) Left-hand sum $= 16(3) + 15(3) + 12(3) + 7(3) = 150$

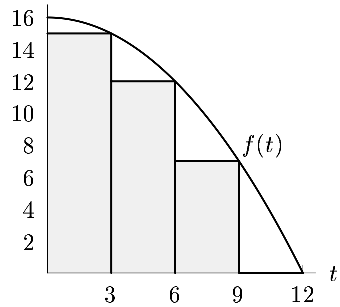


Figure 5.2.66

(c) Right-hand sum $= 15(3) + 12(3) + 7(3) = 108$

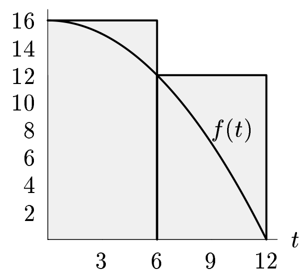


Figure 5.2.67

(d) Left-hand sum $= 16(6) + 12(6) = 168$

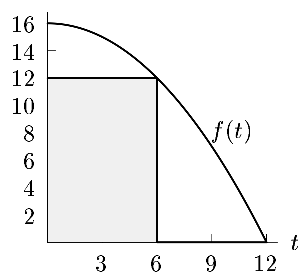


Figure 5.2.68

Right-hand sum $= 12(6) = 72$

4. Use the table to estimate $\int_0^{50} f(x)\,dx$. What values of n and Δx did you use?

Table 5.2.11

x	0	10	20	30	40	50
$f(x)$	30	35	45	50	70	85

ANSWER:

We estimate $\int_0^{50} f(x)\,dx$ using left- and right-hand sums.

Left sum $= 30(10) + 35(10) + 45(10) + 50(10) + 70(10) = 2300$
Right sum $= 35(10) + 45(10) + 50(10) + 70(10) + 85(10) = 2850$
We estimate that $\int_0^{50} f(x)\,dx \approx \dfrac{2300 + 2850}{2} = 2575$. In this estimate, we used $n = 5$ and $\Delta x = 10$.

5. Estimate the area of the region above the curve $y = \cos x$ and below $y = 1$ for $0 \leq x \leq \pi/2$.
ANSWER:

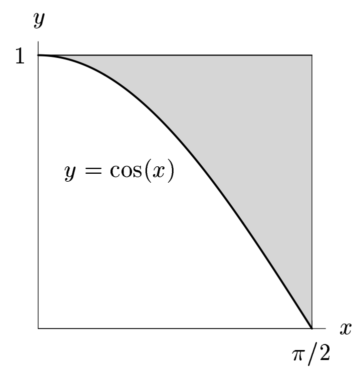


Figure 5.2.69

Area= $\int_0^{\pi/2} (1 - \cos x)\, dx \approx 0.57$

6. Estimate the area of the region between $y = \cos x$, $y = x$, $x = -\pi/2$, and $x = 0$.
ANSWER:

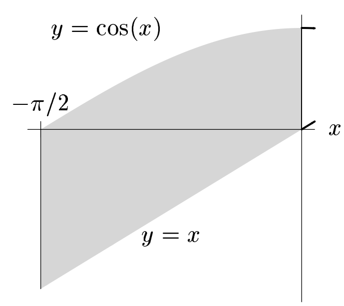


Figure 5.2.70

Area= $\int_{-\pi/2}^{0} \cos x\, dx + \int_{-\pi/2}^{0} -x\, dx \approx 2.2337$

7. Estimate the area of the region under the curve $y = -x^3 + 5$ and above the x-axis for $0 \leq x \leq \sqrt[3]{5}$.
ANSWER:

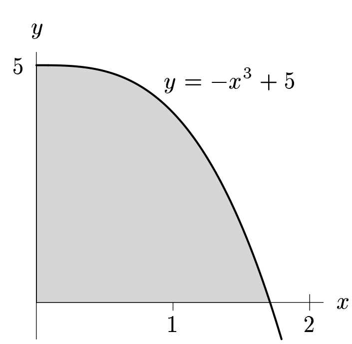


Figure 5.2.71

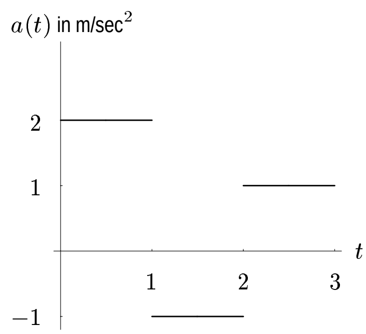
Area= $\int_0^{\sqrt[3]{5}} (-x^3 + 5)\, dx \approx 6.41241$

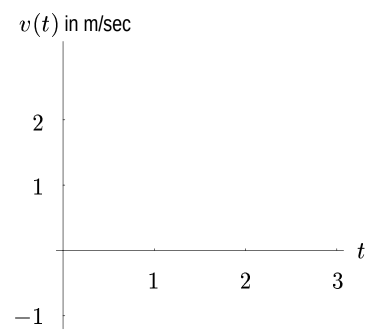
8. Estimate the area of the region under the curve $y = \sin(x/2)$ for $0 \le x \le 2$.
   ANSWER:
   Since $\sin(x/2) > 0$ for $0 \le x \le 2$, the area is given by

   $$\text{Area} = \int_0^2 \sin(x/2)\,dx \approx 0.91939$$

Questions and Solutions for Section 5.3

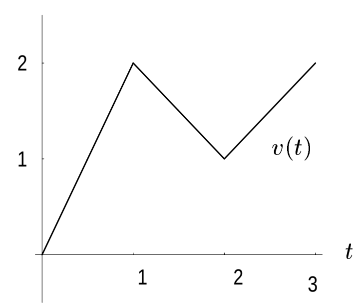
1. (a) The acceleration of an object is given by the graph shown below. Make a graph of the velocity function v, of this object if $v(0) = 0$.





   (b) What is the relationship between the total change in $v(t)$ over the interval $0 \le t \le 3$ and $a(t)$?
   ANSWER:

   (a)

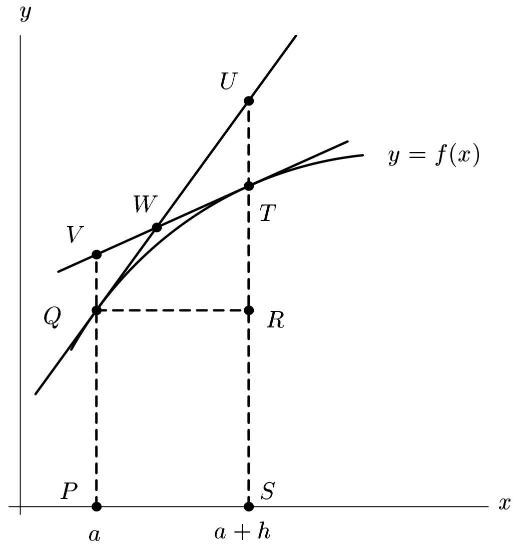


   (b) $\displaystyle\int_0^3 a(t)\,dt = v(3) - v(0)$

   Check this:

   $$\int_0^3 a(t)\,dt = 2 - 1 + 1 = 2; \qquad v(3) - v(0) = 2 - 0 = 2.$$

2. Each of the quantities below can be represented in the picture. For each quantity, state whether it is represented by a length, slope, or an area. Then using letters on the picture, make clear exactly which length, slope or area represents it. [Note: The letters P, Q, R, etc., represent points.]



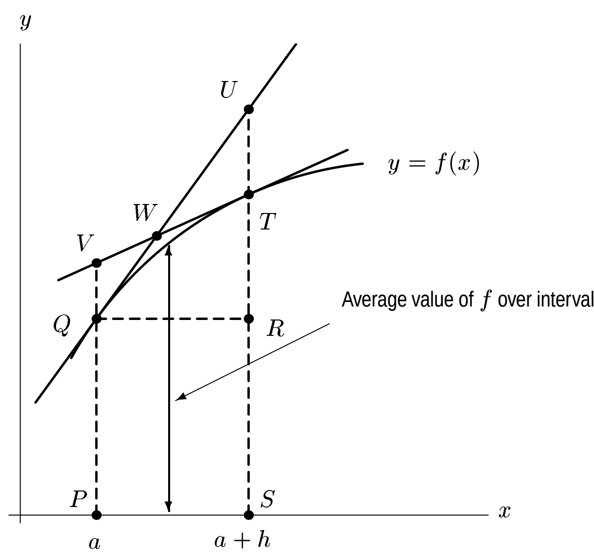
(a) $f(a+h) - f(a)$
(b) $f'(a+h)$
(c) $f'(a)h$
(d) $f(a)h$
(e) State whether the quantity

$$\frac{1}{h}\int_a^{a+h} f(x)\,dx$$

is represented by a length or area in the picture. Draw the length or shade the area in the picture above.

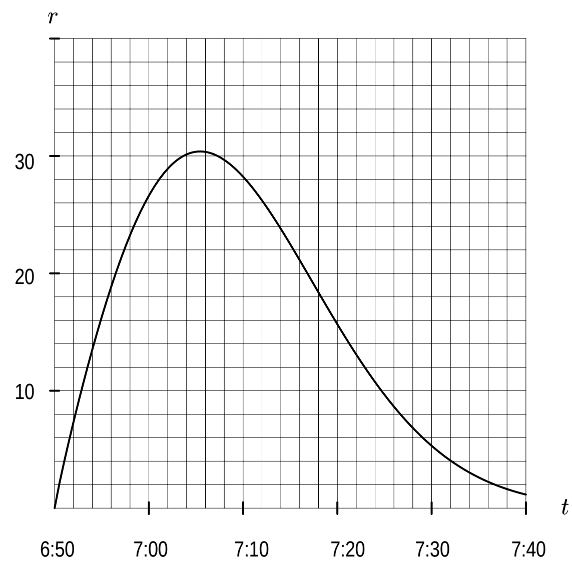
ANSWER:

(a) length TR
(b) slope TV
(c) length UR
(d) area $PQRS$
(e) The average of f over the interval $[a, a+h]$ is a length.



Breakfast at Cafeteria Charlotte

3. Below is the graph of the rate r in arrivals/minute at which students line up for breakfast. The first people arrive at 6:50 a.m. and the line opens at 7:00 a.m.



Suppose that once the line is open, checkers can check peoples' meal cards at a constant rate of 20 people per minute. Use the graph and this information to find an estimate for the following:

(a) The length of the line (i.e. the number of people) at 7:00 when the checkers begin.
(b) The length of the line at 7:10.
(c) The length of the line at 7:20.
(d) The rate at which the line is growing in length at 7:10.
(e) The length of time a person who arrives at 7:00 has to stand in line.
(f) The time at which the line disappears.

ANSWER:

(a) The length of the line at $7:00$ will simply be the number of people who arrived before $7:00$ a.m. This is just $\int_{6:50}^{7:00} r\,dt$. By counting squares, this turns out to be 150 students.

(b) This will simply be the [number of people who have arrived] - [number of people checked]

$$\begin{aligned}
&= \int_{6:50}^{7:10} r\,dt - 10(20) \\
&= \int_{6:50}^{7:00} r\,dt + \int_{7:00}^{7:10} r\,dt - 200 \\
&\approx 150 + 280 - 200 \\
&= 230.
\end{aligned}$$

(c) Similarly, at $7:29$ we have the number of people in line

$$\begin{aligned}
&= \int_{6:50}^{7:20} r\,dt - 400 \\
&= 430 + \int_{7:10}^{7:20} r\,dt - 400 \\
&\approx 430 + 220 - 400 \\
&= 250.
\end{aligned}$$

(d) At $7:10$, the rate of arrivals is about 28 people per minute. The checking rate is 20 people per minute, so the line is growing at a rate of 8 people per minute.

(e) A person who arrives at $7:00$ has about 150 people waiting in front of her. At a checking rate of 20 people per minute, she will spend approximately 7.5 minutes in line.

(f) The total number of arrivals, from the graph, $\int_{6:50}^{7:40} r\,dt$, appears to be about 800. At a checking rate of 20 people per minute, this will take 40 minutes, beginning at $7:00$ a.m. So the line will disappear at $7:40$ a.m.

4. (a) Explain, using words and pictures, how you would decide whether or not the quantity

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin t\,dt$$

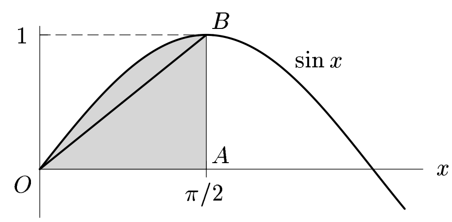
is greater than, less than, or equal to 0.5 without doing any calculations. (Please be as concise as possible.)

(b) Which of the following best approximates $\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin t\,dt$? Circle one. No explanation needed.

0.35 0.4 0.45 0.5 0.55 0.6 0.65

ANSWER:

(a)



Compare the areas:
area OAB ¡ area under sine curve
$\frac{2}{\pi}\times$ area $OAB < \frac{2}{\pi}\times$ area under sine curve.
$\frac{2}{\pi}\times\frac{1}{2}\times 1\times\frac{\pi}{2} < \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\sin t\,dt$
$.5 < \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\sin t\,dt$

(b) 0.65

5. (a) The average value of a function g on $0 \le x \le 1$ is a constant $\bar{g}$ given by

$$\bar{g} = \frac{1}{1-0}\int_0^1 g(x)\,dx = \int_0^1 g(x)\,dx.$$

Show that

$$\int_0^1 \bar{g}g(x)\,dx = \bar{g}^2.$$

(b) Since $(g(x)-\bar{g})^2 \ge 0$ (being a square), we have

$$0 \le \int_0^1 (g(x)-\bar{g})^2\,dx.$$

Use this and part (a) to show that

$$\left(\int_0^1 g(x)\,dx\right)^2 \le \int_0^1 (g(x))^2\,dx.$$

ANSWER:

(a)

$$\int_0^1 \overline{g}g(x)\,dx = \overline{g}\int_0^1 g(x)\,dx = (\overline{g})^2.$$

Notice that $\overline{g}$ is a constant and therefore can be factored out of the integral.

(b) Since $\int_0^1 (g(x)-\overline{g})^2\,dx \ge 0$ and

$$\int_0^1 (g(x)-\overline{g})^2\,dx = \int_0^1 \left((g(x))^2 - 2\overline{g}g(x) + \overline{g}^2\right)\,dx$$

$$
\begin{aligned}
&= \int_0^1 (g(x))^2\,dx - 2\overline{g}\int_0^1 g(x)\,dx + \overline{g}^2\int_0^1 dx \\
&= \int_0^1 (g(x))^2\,dx - 2\overline{g}^2 + \overline{g}^2 \\
&= \int_0^1 (g(x))^2\,dx - \overline{g}^2,
\end{aligned}
$$

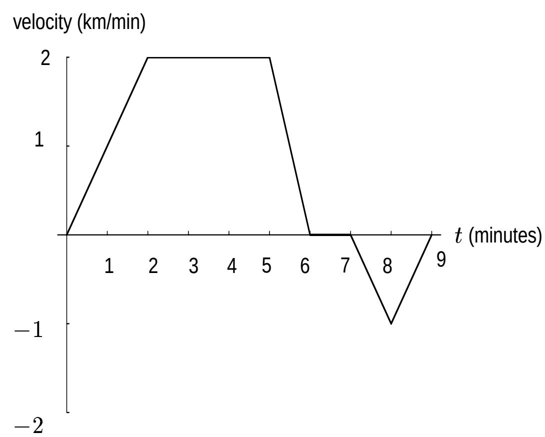
we have

$$\int_0^1 (g(x))^2\,dx - \overline{g}^2 \geq 0$$

i.e., $$\int_0^1 (g(x))^2\,dx \geq \left(\int_0^1 g(x)\,dx\right)^2.$$

6.

A car is moving along a straight road from A to B, starting from A at time $t = 0$. Below is the velocity (positive direction is from A to B) plotted against time.

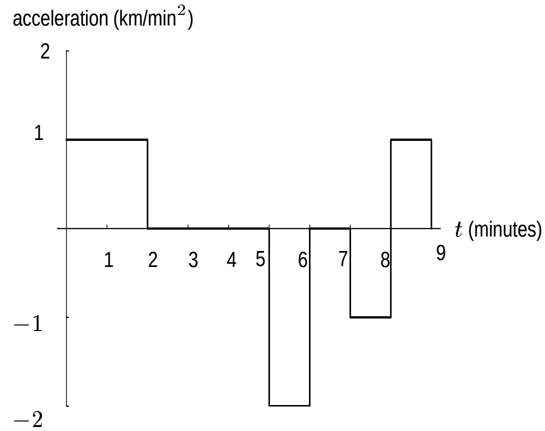


(a) How many kilometers away from A is the car at time $t = 2$, 5, 6, 7, and 9?
(b) Carefully sketch a graph of the acceleration of the car against time. Label your axes.

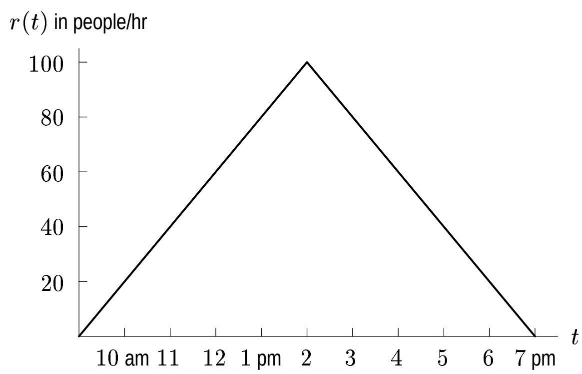
ANSWER:

(a) Since distance is found by integrating velocity, we find the area under the curve:
$t = 2$, the distance from A is $\frac{1}{2}(2)(2) = 2$
$t = 5$, the distance from A is $2 + 3(2) = 8$
$t = 6$, the distance from A is $8 + \frac{1}{2}(1)(2) = 9$
$t = 7$, the distance from A is 9
$t = 9$, the distance from A is $9 - \frac{1}{2}(2)(1) = 8$

(b)

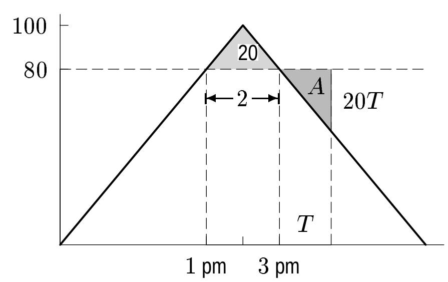


7. A shop is open from 9am–7pm. The function $r(t)$, graphed below, gives the rate at which customers arrive (in people/hour) at time t. Suppose that the salespeople can serve customers at a rate of 80 people per hour.



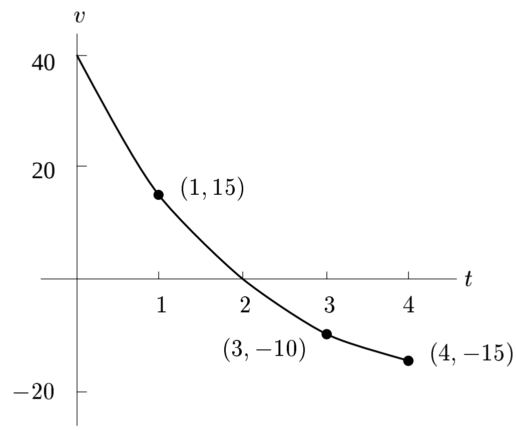
(a) When do people have to start waiting in line before getting served? Explain clearly how you get your answer.
(b) When is the line longest, and how many people are in the line then? Explain your answer.
(c) When does the line vanish? Justify your answer.

ANSWER:



(a) The line starts forming when people arrive faster than they can be served, which happens when $r(t) > 80$. This happens first when $t \geq 1$ pm.
(b) The line builds up from 1pm to 3pm. After 3pm, the rate of arrivals falls below 80 and so the line starts to shrink again. The line is longest at 3pm, and the number in line is the shaded area above line at 80, i.e., length of line $= \frac{1}{2} \cdot 2 \cdot 20 = 20$ people.
(c) The line vanishes when an extra number served (over and above new arrivals) equals the 20 people in line before. This occurs when area is marked $A = 20$, slope of $r(t)$ (for $2 \leq t \leq 7$pm) is $\frac{-100}{5} = -20$. Thus, if T is the time beyond 3pm when the line vanishes, Area $= \frac{1}{2} \cdot T \cdot 20T = 10T^2 = 20$ so $T = \sqrt{2}$ or 1.41 hours $\approx$ 1 hour and 25 minutes, so around 4:25pm.

8. To the right is the graph of the *velocity*, in feet per second, of a hat that is thrown up in the air from ground level. Positive velocity means upward motion.
[Note that this is the graph of *velocity*, not distance.]



(a) When does the hat reach the top of its flight and about how high is it then?
(b) About how high is the hat at time $t = 4$?
(c) About what is the average velocity, $0 \leq t \leq 4$?
(d) About how big is the average *speed*, $0 \leq t \leq 4$?

ANSWER:

(a) At the top of its flight, the velocity of the hat (the derivative of its height) is zero and this occurs when $t = 2$. Since the hat starts at ground level, the height it reaches is the area under the velocity graph between $t = 0$ and $t = 2$, which we approximate by the trapezoid rule with $n = 2$ as $\frac{40+15}{2} + \frac{15+0}{2} = 35$ feet. Since the graph is concave up, this is an overestimate.
(b) We now subtract the area from $t = 2$ to $t = 4$ to the value obtained above for the area between $t = 0$ and $t = 2$, giving a total distance of $\approx 35 + \frac{0-10}{2} + \frac{-10-15}{2} = 17.5$ feet.
(c) The average velocity is given by $\frac{\text{total displacement}}{\text{total time}} \approx \frac{17.5}{4} \approx 4.4$ ft/sec.
(d) Speed is defined as $|v|$, so the average speed is given by

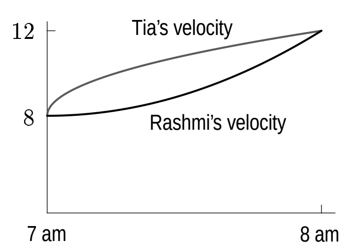
$$\frac{\int_0^4 |v|\, dt}{4} = \frac{\int_0^2 v\, dt + \int_2^4 (-v)\, dt}{4} = \frac{35 + 17.5}{4} \approx 13.1 \text{ ft/sec.}$$

9. Rashmi and Tia both go running from 7:00 am to 8:00 am. Both women increase their velocity throughout the hour, both beginning at a rate of 8 mi/hr. at 7:00 am and running at a rate of 12 mi/hr by 8:00 am. Rashmi's velocity increases at an increasing rate and Tia's velocity increases at a decreasing rate.

(a) Who has run the greater distance in the hour? Explain your reasoning clearly and convincingly.
(b) Who has the greatest average velocity, Rashmi or Tia, or do they have the same average velocity?

ANSWER:

(a) Consider the graphs of the two velocities:



$$\text{Then, total distance covered} = \int_7^8 (\text{velocity})dt = (\text{area under curve between 7 and 8}).$$

Area under Tia's curve is larger so she has covered more distance.
(b) average velocity = (total distance covered)/(time it took to cover it)
Since both spend one hour running and Tia covered more ground, her average velocity must be greater!
N.B. Average velocity is <u>not</u> defined to be $\frac{\text{change in velocity}}{\text{change in time}}$!! That would be average acceleration.

10. Find the average value of the function over the given interval.
    (a) $h(x) = 2x + 2$ over $[1, 3]$
    (b) $f(x) = e^{2x}$ over $[0, 10]$

    ANSWER:

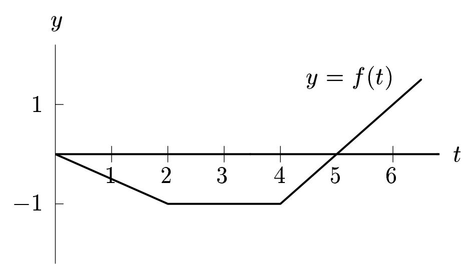
    (a) Average value $= \dfrac{1}{3-1}\displaystyle\int_1^3 (2x+2)\,dx = \frac{1}{2}(12) = 6$

    (b) Average value $= \dfrac{1}{10-0}\displaystyle\int_0^{10} e^{2x}\,dx \approx \frac{1}{10}(2.4\times 10^8) = 2.4\times 10^7$

Questions and Solutions for Section 5.4

1. Suppose $f(t)$ is given by the graph to the right. Complete the table of values of the function $F(x) = \int_0^x f(t)\,dt$.

x	$F(x)$
0	
1	
2	
3	
4	
5	
6	



ANSWER:

x	$F(x)$
0	0
1	$-\frac{1}{4}$
2	-1
3	-2
4	-3
5	$-\frac{7}{2}$
6	-3

2. Suppose $\displaystyle\int_a^b g(x)\,dx = 5$, $\displaystyle\int_a^b (g(x))^2\,dx = 8$, $\displaystyle\int_a^b h(x)\,dx = 1$, and $\displaystyle\int_a^b (h(x))^2\,dx = 4$.
   Find the integrals:
   (a) $\displaystyle\int_a^b \left((g(x))^2 + (h(x))^2\right)\,dx$
   (b) $\displaystyle\int_a^b g(x)\,dx - \left(\int_a^b (2h(x))\,dx\right)^2$

   ANSWER:

   (a) $\displaystyle\int_a^b \left((g(x))^2 + (h(x))^2\right)\,dx = \int_a^b (g(x))^2\,dx + \int_a^b (h(x))^2\,dx = 8 + 4 = 12$

   (b) $\displaystyle\int_a^b g(x)\,dx - \left(\int_a^b (2h(x))\,dx\right)^2 = \int_a^b g(x)\,dx - \left(2\int_a^b h(x)\,dx\right)^2 = 5 - (2(1))^2 = 5 - 4 = 1.$

3. Let $\displaystyle\int_0^4 f(x)\,dx = C_1$
   (a) What is the average value of $f(x)$ on the interval $x = 0$ to $x = 4$?

(b) If $f(x)$ is even, what is $\int_{-4}^{4} f(x)\,dx$?

ANSWER:

(a) For $0 \le x \le 4$, we have

$$\begin{aligned}\text{Average value } &= \frac{1}{4-0}\int_0^4 f(x)\,dx \\ &= \frac{1}{4}(C_1) = \frac{C_1}{4}\end{aligned}$$

(b) If $f(x)$ is even, the graph is symmetric about the y-axis. By symmetry, the area between $x = 4$ and $x = -4$ is twice the area between $x = 0$ and $x = 4$, so

$$\int_{-4}^{4} f(x)\,dx = 2C_1$$

4. Evaluate the definite integral $\int_0^4 (2 + 5x)\,dx$ exactly.

ANSWER:

We can break the integral up as follows:

$$\begin{aligned}\int_0^4 (2+5x)\,dx &= \int_0^4 2\,dx + \int_0^4 5x\,dx \\ &= \int_0^4 2\,dx + 5\int_0^4 x\,dx\end{aligned}$$

From the area interpretation of the integral, we see that

$$\int_0^4 2\,dx = 8$$

and

$$5\int_0^4 x\,dx = 5(8) = 40$$

Therefore,

$$\int_0^4 f(x)\,dx = 8 + 40 = 48.$$

5. Compute $\int_1^4 3x^2\,dx$ using two different methods.

ANSWER:

We can use left- or right-hand sums to approximate the integral. For example, using left-hand sums, we obtain

$$3(1) + 12(1) + 27(1) = 42$$

for $n = 3$ and $\Delta x = 1$.

We can also compute the integral exactly. We take $f(x) = 3x^2$. We know that if $F(x) = x^3$, then $F'(x) = 3x^2$. So, by the Fundamental Theorem, we use $f(x) = 3x^2$ and $F(x) = x^3$ and obtain

$$\int_1^4 3x^2\,dx = F(4) - F(1) = 63.$$

6. If $f(x) = \frac{x}{2} + 4$ and $g(x) = x + 1$, how do $\int_0^4 f(x)\,dx$ and $\int_0^4 g(x)\,dx$ compare? Interpret your answer in terms of areas.

ANSWER:

Since $g(x) \le f(x)$ for $0 \le x \le 4$, we know $\int_0^4 g(x)\,dx \le \int_0^4 f(x)\,dx$. We conclude that the area under $g(x)$ is less than the area under $f(x)$ on $0 \le x \le 4$.

7. If $\int_1^3 f(x)\,dx = 5$ and $\int_3^7 f(x)\,dx = 9$, evaluate

(a) $\int_1^7 f(x)\,dx$

(b) $\int_7^3 f(x)\,dx$

ANSWER:

(a) We know that $\int_1^7 f(x)\,dx = \int_1^3 f(x)\,dx + \int_3^7 f(x)\,dx.$

Therefore, $\int_1^7 f(x)\,dx = 5 + 9 = 14.$

(b) $\int_7^3 f(x)\,dx = -\int_3^7 f(x)\,dx = -9.$

8. How can you know the value of
$\int_{-a}^{a} x^{15}\,dx$ and $\int_{-b}^{b} \sin x\,dx$?

ANSWER:

$f(x) = x^{15}$ is odd and the limits of integration are symmetric around 0, so $\int_{-a}^{a} x^{15}\,dx = 0.$

Same is true for $f(x) = \sin x$ so $\int_{-b}^{b} \sin x\,dx = 0.$

9. If f is even and $\int_0^3 f(x)\,dx = 16$, what does $\int_{-3}^{3} f(x)\,dx = ?$

ANSWER:

Since f is even, $\int_{-3}^{3} f(x)\,dx = 2 \cdot \int_0^3 f(x)\,dx = 2(16) = 32.$

10. The average value of $y = h(x)$ equals a for $0 \le x \le 3$, and equals b for $3 \le x \le 9$. What is the average value of $h(x)$ for $0 \le x \le 9$?

ANSWER:

We know that the average value of $h(x) = a$ for $0 \le x \le 3$, so $\frac{1}{3-0}\int_0^3 h(x)\,dx = a$, and thus $\int_0^3 h(x) = 3a.$

Similarly, we know
$\frac{1}{9-3}\int_3^9 h(x)\,dx = b$, so $\int_3^9 h(x) = 6b.$

The average value for $0 \le x \le 9$ is given by

$$\begin{aligned}\text{Average value } &= \frac{1}{9-0}\int_0^9 h(x)\,dx \\ &= \frac{1}{9}\left(\int_0^3 h(x)\,dx + \int_3^9 h(x)\,dx\right) \\ &= \frac{1}{9}(3a + 6b) = \frac{a+2b}{3}\end{aligned}$$

Review Questions and Solutions for Chapter 5

For Problems 1– 3, circle the correct answer(s) or fill in the blanks. No reasons need be given.

1. Suppose $A =$ the area under the curve $y = e^{-x^2}$ over the interval $-1 \le x \le 1$. Which of the following is true?

(a) $A = F(1) - F(-1)$, where $F(x) = \dfrac{e^{-x^2}}{-2x}.$

(b) $1.495 < A$.
(c) $1.492 < A < 1.495$.
(d) $1.487 < A < 1.492$.

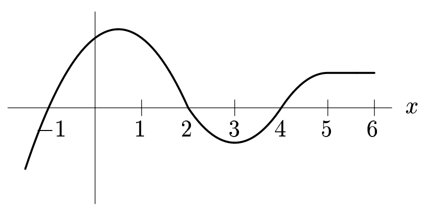
ANSWER:
(c) Use calculator; $A = 1.49365$.

2. If $r(t)$ represents the rate at which a country's debt is growing, then the increase in its debt between 1980 and 1990 is given by

(a) $\dfrac{r(1990) - r(1980)}{1990 - 1980}$
(b) $r(1990) - r(1980)$
(c) $\dfrac{1}{10}\displaystyle\int_{1980}^{1990} r(t)dt$
(d) $\displaystyle\int_{1980}^{1990} r(t)dt$
(e) $\dfrac{1}{10}\displaystyle\int_{1980}^{1990} r'(t)dt$

ANSWER:
(d) Fundamental Theorem

3. The graph of f'' is shown below.



If f is increasing at $x = -1$, which of the following <u>must</u> be true? (Circle all that apply.)

(a) $f'(2) = f'(4)$
(b) $f'(4) > f'(-1)$
(c) $f'(4) > 0$
(d) $f(5) = f(6)$

ANSWER:
f increasing at $x = -1$ means $f'(-1) > 0$.

(a) false: $f'(4) - f'(2) = \displaystyle\int_2^4 f''(t)dt < 0$
(b) true: $f'(4) - f'(-1) = \displaystyle\int_{-1}^4 f''(t)dt > 0$
(c) true: $f'(4) > f'(-1) > 0$
(d) false: $f'(t) > 0$ for all $t > -1$ so $f(6) > f(5)$.

For Problems 4– 6, decide whether each statement is true or false, and provide a short explanation or a counterexample.

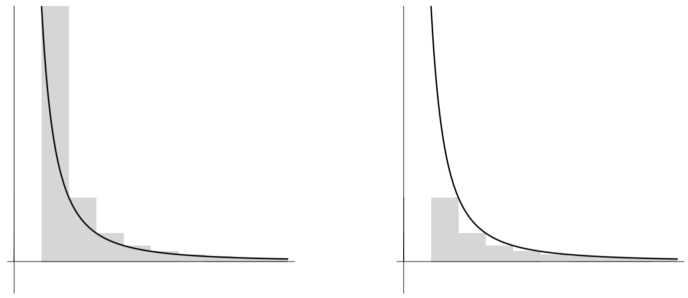
4. $\displaystyle\int_{-1}^{1} \sqrt{1 - x^2}\, dx = \frac{\pi}{2}$.

ANSWER:
TRUE. The integral represents the area under the upper half of the circle radius 1 centered at the origin. The area is thus $\frac{\pi}{2}$.

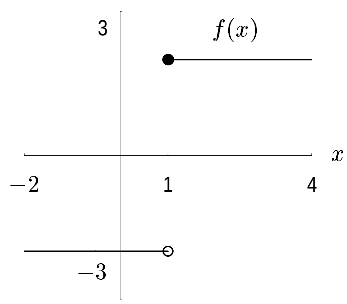
5. If a function is concave UP, then the left-hand Riemann sums are always less than the right-hand Riemann sums with the same subdivisions, over the same interval.

ANSWER:

FALSE. On the example below, the function is concave up, yet the left-hand sum is clearly larger than the right-hand sum.



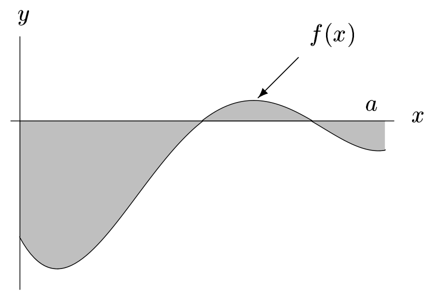
6. If $\int_a^b f(x)\,dx = 0$, then f must have at least one zero between a and b (assume $a \neq b$).
ANSWER:
TRUE, if we allow only continuous functions. FALSE, if we allow discontinuous functions. The function shown below has no roots, yet $\int_{-2}^{4} f(x)\,dx = 0$.



Chapter 6 Exam Questions

Questions and Solutions for Section 6.1

1. Sketch a function that represents the total area between $f(x)$ and the x-axis from 0 to a.

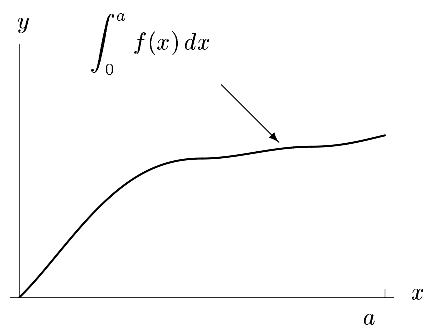


[Hint: This is not the same as $\int_0^a f(x)\,dx$.]

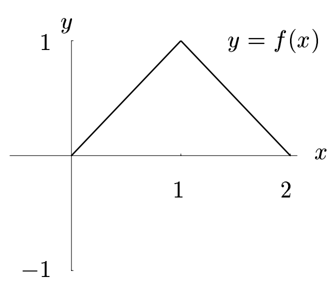
ANSWER:

We recall that since the area is always non-negative, the area between $f(x)$ and the x-axis from 0 to a is $\int_0^a |f(x)|\,dx$.

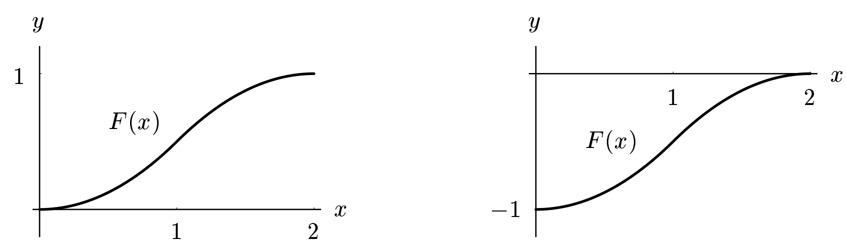
We therefore obtain the graph below:



2. If f is given as below, sketch two functions F, such that $F' = f$. In one case, have $F(0) = 0$ and in the other, have $F(0) = -1$.

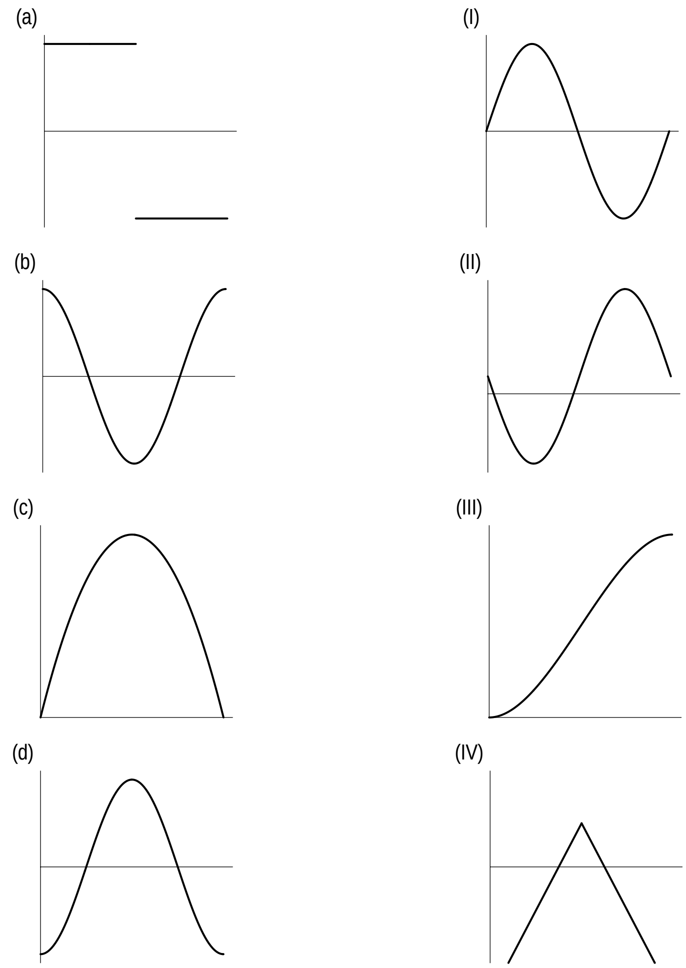


ANSWER:



3. Match the following functions with their antiderivatives:

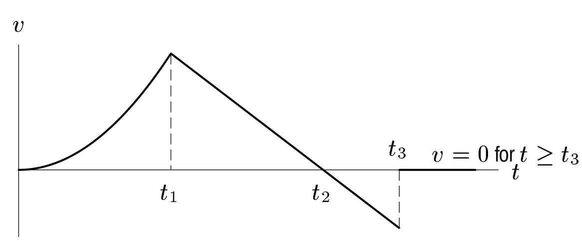
Function | Antiderivative



ANSWER:

(a) $\longrightarrow$ IV
(b) $\longrightarrow$ I
(c) $\longrightarrow$ III
(d) $\longrightarrow$ II

4. A young girl who aspires to be a rocket scientist launches a model rocket from the ground at time $t = 0$. The rocket travels straight up in the air, and the following graph shows the upward velocity of the rocket as a function of time:



Let h be the height (or vertical displacement) of the rocket, let a be the acceleration of the rocket, and let $h = 0$ be the ground level from which the rocket was launched. Recall that the first derivative of displacement is velocity and that the first derivative of velocity is acceleration.

(a) Sketch a graph of the acceleration of the rocket as a function of time.

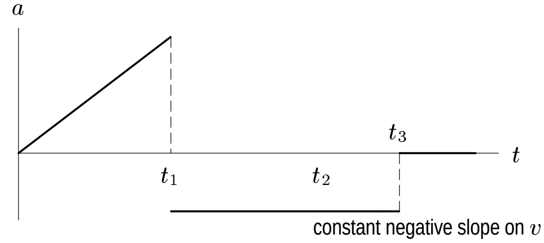
(b) Sketch a graph of the height of the rocket as a function of time.

(c) Let $v(t)$ be the function that gives the velocity of the rocket at time t. From the graph of the rocket's velocity, which is larger, $\left|\int_0^{t_2} v(t)dt\right|$ or $\left|\int_{t_2}^{t_3} v(t)dt\right|$? What do you know about the sign of $\int_0^{t_3} v(t)dt$? What does this mean physically?

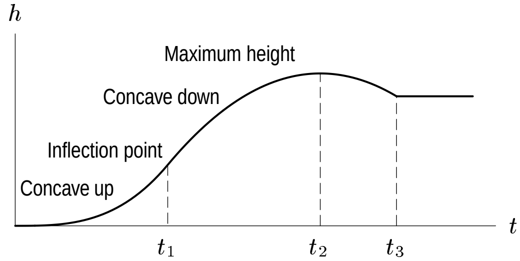
(d) Write a story about what happened to the rocket. In particular, describe what could explain the features of the graphs of the height, velocity, and acceleration at times t_1 , t_2, and t_3.

ANSWER:

(a)



(b)



(c) The area under $v(t)$ from $t = 0$ to $t = t_2$ is larger than the area above $v(t)$ from $t = t_2$ to $t = t_3$. Therefore,

$$\left|\int_0^{t_2} v(t)dt\right| > \left|\int_{t_2}^{t_3} v(t)dt\right|$$

Since $\int_0^{t_2} v(t)dt > 0$ and $\int_{t_2}^{t_3} v(t)dt < 0$, we can conclude that

$$\int_0^{t_3} v(t)dt = \int_0^{t_2} v(t)dt + \int_{t_2}^{t_3} v(t)dt > 0$$

Because $\int_0^{t_3} v(t)dt$ is the height of the rocket after t_3 time units (and $h =$ (constant after time t_3), the rocket came to rest somewhere above the ground. (It landed on a roof, got caught in a tree, etc.)

(d) A possible explanation:

The engine on the rocket fired from $t = 0$ to $t = t_1$ yielding increasing upward acceleration (increasing probably because of less mass/less drag/etc.) At $t = t_1$, the engine stopped (ran out of fuel, perhaps), and there was constant negative acceleration (due to gravity) from $t = t_1$ to $t = t_3$. Since the velocity was still positive (upward) until time t_2, the rocket continued to ascend. At time $t = t_2$,it was at its maximum height ($v(t_2) = h'(t_2) = 0$). At time t_3, the velocity suddenly went to zero, implying the rocket hit something and stayed put thereafter. From part (c), whatever the rocket hit was above the level from which it was launched.

5. You decide to take a trip down a stretch of road that runs straight east and west. The following table gives your eastward velocity (in miles per minute) measured at one-minute intervals for the first ten minutes of your trip.

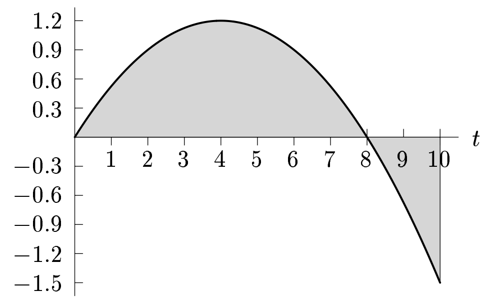
Time (min)	0	1	2	3	4	5	6	7	8	9	10
Velocity (mi/min)	0.00	0.53	0.90	1.13	1.20	1.13	0.90	0.53	0.00	-0.68	-1.50

(a) Sketch a graph of your velocity as a function of time, and give a description of what happened in words.

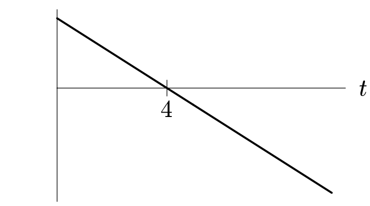
(b) Sketch a graph of your acceleration as a function of time. (This only needs to be a sketch. You don't have to calculate any values of acceleration.)
(c) Sketch a graph of your total eastward distance from your starting place as a function of time. (Again, this only needs to be a sketch.)
(d) What is your best estimate of the total eastward distance of your car from your starting position after ten minutes? You may do this by any means of your choosing, but explain your method.

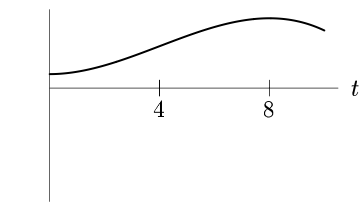
ANSWER:

(a)



You begin driving east, increasing your speed for four minutes, when you reached a peak speed of 1.2 mi/min. During the next four minutes, you continued east, but began slowing down, and at eight minutes, you reversed direction and began going west.





(d) The total distance can be represented as the area under the velocity function (geometric representation). Since the velocity graph looks like an upside down parabola, approximate it by $v(t) = kt(8-t)$. When $t = 4$, $v = 1.2$, so $1.2 = 16k$. Thus $k = 0.075$. To find distance traveled, integrate from 0 to 10, as follows.'

$$\int_0^{10} 0.075t(8-t)\,dt = 0.075 \cdot \left(4t^2 - \frac{t^3}{3}\right)\Big|_0^{10}$$
$$= 0.075 \cdot (400 - 333.3) \approx 5 \text{ miles}$$

6.

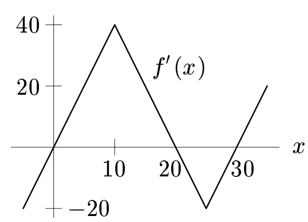


Figure 6.1.72

If Figure 6.1.72 shows the graph of $f'(x)$, sketch $f(x)$ if $f(0) = 100$.
ANSWER:

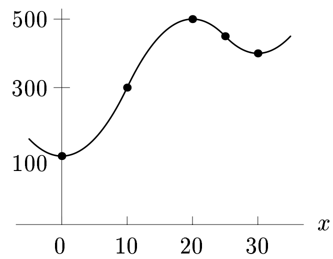


Figure 6.1.73

7. Given the values of the derivative $f'(x)$ in the table and that $f(0) = 40$, estimate $f(x)$ for $x = 2, 4, 6$.

Table 6.1.12

x	0	2	4	6
$f'(x)$	3	15	27	39

ANSWER:
By the Fundamental Theorem of Calculus, we know that

$$f(2) - f(0) = \int_0^2 f'(x)\,dx.$$

Using a left-hand sum, we estimate $\int_0^2 f'(x)\,dx \approx (3)(2) = 6$.

Using a right-hand sum, we estimate $\int_0^2 f'(x)\,dx \approx (15)(2) = 30$.

Averaging, we have $\int_0^2 f'(x)\,dx \approx \frac{6+30}{2} = 18$.

We know $f(0) = 40$, so

$$f(2) = f(0) + \int_0^2 f'(x)\,dx \approx 40 + 18 = 58.$$

Similarly, we estimate $\int_2^4 f'(x)\,dx \approx \frac{(15)(2) + (27)(2)}{2} = 42$

So $f(4) = f(2) + \int_2^4 f'(x)\,dx \approx 58 + 42 = 100$

Similarly, $\int_4^6 f'(x)\,dx \approx \frac{(27)(2) + (39)(2)}{2} = 66$

So $f(6) = f(4) + \int_4^6 f'(x)\,dx \approx 100 + 66 = 166$.

The values are shown in the table:

Table 6.1.13

x	0	2	4	6
$f(x)$	40	58	100	166

8.

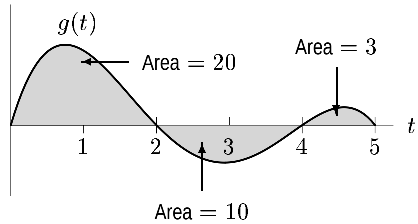


Figure 6.1.74

Using Figure 6.1.74, sketch 2 graphs of antiderivatives $G_1(t)$ and $G_2(t)$ of $g(t)$ satisfying $G_1(0) = 10$ and $G_2(0) = -5$. Label each critical point with its coordinates.
ANSWER:

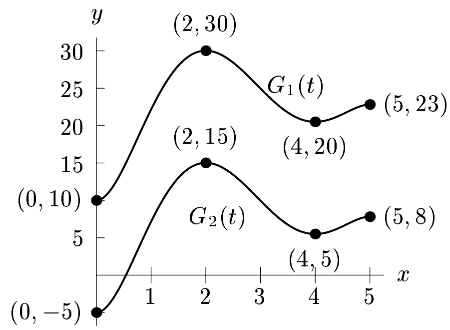


Figure 6.1.75

9. The rate of change in concentration of a certain medication in a person's body, $H'(t)$, in micrograms per milliliter per minute, is -1 for the first 2 seconds. Then it increases at a constant rate for 2 seconds, reaching 1 at $t = 4$. Then it remains constant for 1 second. Sketch $H'(t)$ and $H(t)$, assuming $H(0) = 6$.
ANSWER:

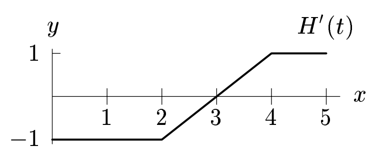


Figure 6.1.76

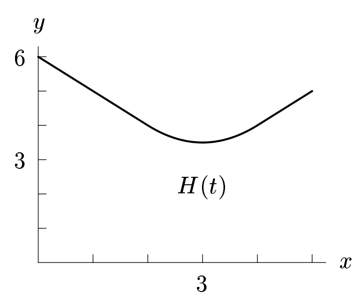


Figure 6.1.77

10. Figure 6.1.78 shows the graph of f. If $F' = f$ and $f(0) = 0$, sketch $f(t)$.

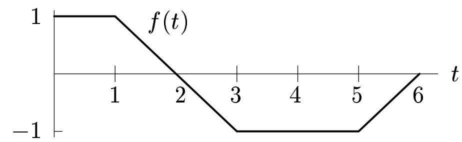


Figure 6.1.78

ANSWER:

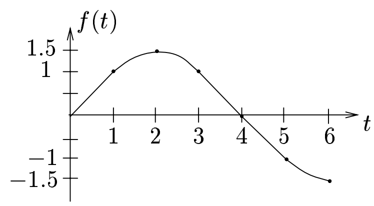


Figure 6.1.79

Questions and Solutions for Section 6.2

1. (a) Find the derivative of $\dfrac{x}{\sqrt{a^2+x^2}}$. Be sure to simplify your answer as much as possible.

   (b) Use part (a) to find an antiderivative of $\dfrac{1}{(a^2+x^2)^{\frac{3}{2}}}$.

   ANSWER:

   (a) Use the product rule (or the quotient rule):

$$\begin{aligned}\frac{d}{dx}x(a^2+x^2)^{-1/2} &= (a^2+x^2)^{-1/2} - \frac{1}{2}x(a^2+x^2)^{-3/2}(2x)\\ &= (a^2+x^2)^{-3/2}\left(a^2+x^2-x^2\right)\\ &= \frac{a^2}{(a^2+x^2)^{3/2}}.\end{aligned}$$

   (b) By part (a),

$$\begin{aligned}\int \frac{a^2}{(a^2+x^2)^{\frac{3}{2}}}\,dx &= \frac{x}{\sqrt{a^2+x^2}} + C\\ \int \frac{1}{(a^2+x^2)^{\frac{3}{2}}}\,dx &= \frac{1}{a^2}\cdot\frac{x}{\sqrt{a^2+x^2}} + C'.\end{aligned}$$

2. Find a function F such that:

   (a) $F'(x) = x^3 - \dfrac{1}{\sqrt{x}}$

   (b) $F'(x) = \sin x + \dfrac{1}{x}$

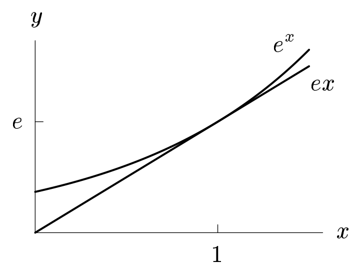
   ANSWER:

   (a) $F(x) = \frac{1}{4}x^4 - 2\sqrt{x} + C$
   (b) $F(x) = -\cos x + \ln|x| + C$

3. (a) Sketch the graphs of $y = e^x$ and $y = ex$.
   (b) For which values of x is $e^x > ex$? Explain how you can be certain of your answer.
   (c) Find the average value of the difference between e^x and ex on the interval between $x = 0$ and $x = 2$.

   ANSWER:

   (a)



   (b) The equation of the tangent line of $y = e^x$ at $x = 1$ is $y - e^1 = e^1(x-1)$, i.e. $y = ex$. Since $y'' = (e^x)'' = e^x > 0$ for all x, the graph is concave up for all x, and lies above its tangent line $y = ex$. Hence $e^x \geq ex$ for all x.
   (c) The average difference is given by

$$\frac{1}{2-0}\int_0^2 (e^x - ex)\,dx = \frac{1}{2}\left(e^x\Big|_0^2 - \frac{e}{2}x^2\Big|_0^2\right) = \frac{1}{2}(e^2 - 1 - 2e).$$

4. Decide whether the following statement is true or false and provide a short explanation or counterexample.

   An antiderivative of $2x\cos x$ is $x^2\sin x$.

   ANSWER:
   FALSE. The derivative of $x^2\sin x$ is $2x\sin x + x^2\cos x$, not $2x\cos x$.

5. Find the total area bounded between the curve $f(x) = x^3 - 5x^2 + 4x$ and the x-axis.

   ANSWER:

   Since $x^3 - 5x^2 + 4x = x(x-4)(x-1)$, the roots of $f(x)$ are 0, 1, and 4. As the magnitude of x gets large, for $x > 4$ and $x < 0$, $f(x)$ goes to ∞ and $-\infty$ respectively, and thus the area between $f(x)$ and the x-axis is unbounded in these regions.

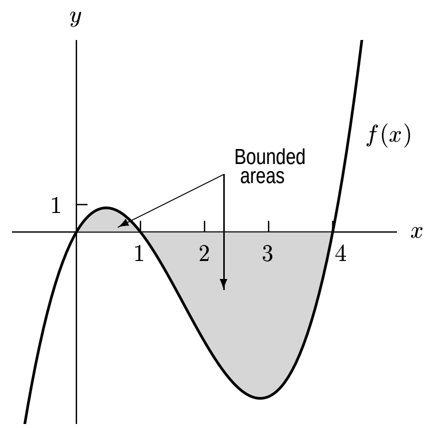
   Between $x = 0$ and $x = 1$, $f(x)$ is positive, and the area it bounds over this interval will be

$$\int_0^1 f(x)\,dx = \int_0^1 (x^3 - 5x^2 + 4x)\,dx$$
$$= \frac{x^4}{4} - \frac{5}{3}x^3 + 2x^2\Big|_0^1 \approx 0.5833.$$

   Between $x = 1$ and $x = 4$, $f(x)$ drops below the x-axis, and the area it bounds over the interval will be:

$$-\int_1^4 f(x)\,dx = -\frac{x^4}{4} + \frac{5}{3}x^3 - 2x^2\Big|_1^4 \approx 11.2500.$$

   Thus the total bounded area is $\approx 11.25 + 0.5833 = 11.8333$



[Note that simply taking the integral $\int_0^4 f(x)\,dx$ does not give the area bounded by the curve $f(x)$, because area, as we have defined it, is a positive quantity. What we have done above is to calculate the area using $\int_0^4 |f(x)|\,dx$]

6. A factory is dumping pollutants into a lake continuously at the rate of $\frac{t^{\frac{2}{3}}}{60}$ tons per week, where t is the time in weeks since the factory commenced operations.
   (a) After one year of operation, how much pollutant has the factory dumped into the lake?
   (b) Assume that natural processes can remove up to 0.15 ton of pollutant per week from the lake and that there was no pollution in the lake when the factory commenced operations one year ago. How many tons of pollutant have now accumulated in the lake? (Note: The amount of pollutant being dumped into the lake is never negative.)

   ANSWER:

   (a)

$$\text{amount dumped} = \int_0^{52\text{weeks}} \overbrace{\frac{1}{60}(t^{\frac{2}{3}})}^{\text{rate}} \overbrace{dt}^{\text{time}}$$
$$= \frac{1}{60}\cdot\frac{3}{5}t^{\frac{5}{3}}|_0^{52}$$
$$= .01t^{\frac{5}{3}}\Big|_0^{52} = .01(52)^{\frac{5}{3}} = 7.24tons$$

   (b) Natural processes will remove all pollutants until the dumping rate exceeds 0.15 tons/week. This will occur when $\frac{t^{\frac{2}{3}}}{60} = 0.15 \Rightarrow t = (9)^{\frac{3}{2}} = 27$ weeks. So we need to compute the difference between amount dumped and amount removed for the last 25 weeks of the year:

$$\int_{27}^{52} \Big(\underbrace{\frac{1}{60}t^{\frac{2}{3}}}_{\text{dumping rate}} - \underbrace{0.15}_{\text{removal rate}}\Big)\, dt = \Big(.01t^{\frac{5}{3}} - .15t\Big)\Big|_{27}^{52}$$

$$= 1.06tons$$

7. Find an antiderivative of each of the following functions.
   (a) $\pi + x^2 + \frac{1}{\pi x^2}$
   (b) $\sqrt{x} - \frac{1}{x\sqrt{x}}$
   (c) $\cos(2\theta)$
   (d) $e^t + e^3$
   (e) $\frac{2}{x} + \frac{x}{2}$

ANSWER:

(a) $\pi x + \dfrac{x^3}{3} - \dfrac{x^{-1}}{\pi} + C$

(b) $\dfrac{2}{3}x^{\frac{3}{2}} + 2x^{-\frac{1}{2}} + C$

(c) $\dfrac{1}{2}\sin(2\theta) + C$

(d) $e^t + e^3 t + C$

(e) $2\ln|x| + \dfrac{x^2}{4} + C$

8. Suppose the rate at which ice in a skating pond is melting is given by $dV/dt = 4t + 2$, where V is the volume of the ice in cubic feet, and t is the time in minutes.

(a) Write a definite integral which represents the amount of ice that has melted in the first 4 minutes.

(b) Evaluate the definite integral in part (a).

ANSWER:

(a) Amount of ice melted in the first four minutes $= \displaystyle\int_0^4 \frac{dV}{dt}\,dt = \int_0^4 (4t+2)\,dt.$

(b) $\displaystyle\int_0^4 (4t+2)dt = (2t^2 + 2t)\Big|_0^4 = 40 \text{ ft}^3.$

9. Suppose $F(x) = 3\sin x + x + 5$. Find the total area bounded by $F(x), x = 0, x = \pi$ and $y = 0$.

ANSWER:

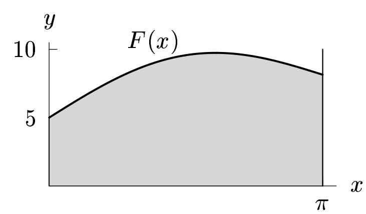


Figure 6.2.80

$$
\begin{aligned}
\text{Area} &= \int_0^\pi (3\sin x + x + 5)\,dx \\
&= \int_0^\pi 3\sin x\,dx + \int_0^\pi x\,dx + \int_0^\pi 5\,dx \\
&= -3\cos x\Big|_0^\pi + \frac{x^2}{2}\Big|_0^\pi + 5x\Big|_0^\pi \\
&= (-3(-1) + 3(1)) + \left(\frac{\pi^2}{2} - 0\right) + (5\pi - 0) \\
&= \frac{\pi^2}{2} + 5\pi
\end{aligned}
$$

10. Find the exact area between the graphs of $y = x^3 + 5$ and $y = -x + 5$ for $0 \le x \le 2$.

ANSWER:

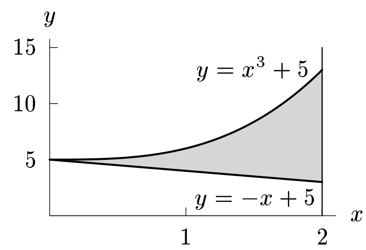


Figure 6.2.81

$$\begin{aligned}\text{Area} &= \int_0^2 \big((x^3+5) - (-x+5)\big)\ dx \\ &= \int_0^2 x^3\,dx + \int_0^2 x\,dx \\ &= \frac{x^4}{4}\bigg|_0^2 + \frac{x^2}{2}\bigg|_0^2 = 4 + 2 = 6\end{aligned}$$

Questions and Solutions for Section 6.3

1. A car is going 80 feet per second and the driver puts on the brakes, bringing the car to a stop in 5 seconds. Assume the deceleration of the car is constant while the brakes are on.
   (a) What is the acceleration (really deceleration) of the car?
   (b) How far does the car travel from the time the brakes are applied until it stops?
   (c) Suppose the car is traveling twice as fast and the brakes are applied with the same force as before. How far does the car travel before it stops?
   (d) Suppose the brakes are twice as strong (can stop the car twice as fast). How far does the car travel if its speed is 80 feet per second? How far if its speed is 160 feet per second?

   ANSWER:

   (a) Using $v = v_0 + at$, where $v_0 = 80$ ft/sec, we can find a by using the condition that $v = 0$ when $t = 5$. So since $0 = 80 + 5a$, $a = -16\text{ft}/\text{sec}^2$.
   (b) Remember that distance traveled is the integral of velocity with respect to time. So

$$\begin{aligned}d &= \int_0^5 (80 - 16t)\,dt = 80t - 8t^2\bigg|_0^5 \\ &= 400 - 200 = 200 \text{ ft}.\end{aligned}$$

   (c) Now, $v_0 = 2\cdot 80 = 160$, so $v = 160 - 16t$. The car will come to a stop when $t = 10$. So

$$\begin{aligned}d &= \int_0^{10} (160 - 16t)\,dt = 160t - 8t^2\bigg|_0^{10} \\ &= 1600 - 800 = 800 \text{ ft}.\end{aligned}$$

   (This is four times as great as the prior stopping distance.)
   (d) If the brakes are twice as strong, then the deceleration will be twice as great, so $v = 80 - 32t$. The car will come to a stop when $v = 0$, that is, when $t = 80/32 = 2.5$ sec. So

$$\begin{aligned}d &= \int_0^{2.5} (80 - 32t)\,dt = 80t - 16t^2\bigg|_0^{2.5} \\ &= 200 - 100 = 100 \text{ ft}.\end{aligned}$$

   Finally, if $v_0 = 160$, then $v = 160 - 32t$; the car will come to a stop when $t = 5$. So

$$\begin{aligned}d &= \int_0^{5} (160 - 32t)\,dt = 160t - 16t^2\bigg|_0^{5} \\ &= 800 - 400 = 400 \text{ ft}.\end{aligned}$$

2. A ball is dropped from a window 100 feet above the ground. Assume that its acceleration is $a(t) = -32\ \text{ft/sec}^2$ for $t \geq 0$.
   (a) Find the velocity of the ball as a function of time t.
   (b) Find its height above the ground as a function of time t.
   (c) After how many seconds does it hit the ground?

   ANSWER:

   (a) Let $v(t)$ be the velocity of the ball as a function of time t. $\dfrac{d(v(t))}{dt} = a(t) = -32$. Integrating $a(t)$ gives $v(t) = -32t + C$. When $t = 0$, we know that $v(0) = 0$, so $C = 0$. Hence $v(t) = -32t$ ft/sec.
   (b) Let $s(t)$ be the height above the ground as a function of time t. $\dfrac{d(s(t))}{dt} = v(t) = -32t$. Integrating $v(t)$ yields $s(t) = -16t^2 + C$. Since $s(0) = 100$, we get $C = 100$. Thus, $s(t) = -16t^2 + 100$ feet.
   (c) The ball lands when $s(t) = 0$. Set $s(t) = -16t^2 + 100 = 0$ to obtain:

$$16t^2 = 100$$
$$t = \frac{5}{2} \text{ sec.}$$

   [Note that we discard the root $t = -\frac{5}{2}$ sec as not physically meaningful.]
   Thus, the ball hits the ground after 2.5 seconds.

3. On planet Janet the gravitational constant g is -15 feet per second per second: that is, for every second an object falls it picks up an extra 15 feet per second of velocity downward. A ball is thrown upward at time $t = 0$ at 60 feet per second.
   (a) When does the ball reach the peak of its flight?
   (b) Find the peak height of the ball by giving equations for the ball's acceleration a, then its velocity v (including computation of the antidifferentiation constant C), then its position s.
   (c) Find the peak height instead by left and right sums for $v(t)$. (First give a table of values for v.)
   (d) Find the peak height instead by graphing v versus t and recalling how distance traveled is related to the graph of velocity.
   (e) On planet Nanette, g is $\frac{1}{3}$ as great as on Janet. Use the method in Part (d) to find the peak height (same initial velocity of 60 feet per second.)

   ANSWER:

   (a) The ball reaches its peak when the velocity is 0. $\frac{dv}{dt} = -15$, so $v = -15t + C$, but the initial velocity $v(0) = 60$ so $C = 60$. Since $v(t) = -15t + 60$ is 0 when $t = 4$, the ball reaches its peak after 4 seconds.
   (b) Since $a = -15$ and $v = -15t + 60$, we have

$$s(t) = \int_0^t v(x)\,dx = -\frac{15}{2}t^2 + 60t + C.$$

   Since the ball is thrown from the ground level, $C = 0$. The peak height, occurring at $t = 4$, is therefore $s(4) = -15\frac{t^2}{2} + 60t\Big|_0^4 = 120$ ft.
   (c)

t	0	1	2	3	4
$v(t)$	60	45	30	15	0

   A left-hand sum is $\text{LEFT}(4) = 60 + 45 + 30 + 15 = 150$ ft., and the corresponding right-hand sum is $\text{RIGHT}(4) = 45 + 30 + 15 + 0 = 90$ ft., with $\Delta t = 1$ sec. These sums approximate the integral of the velocity of the ball, which is the the distance it travels.
   (d) The distance traveled equals the area under the graph of velocity between $t = 0$ and $t = 4$. Since the area is just that of a triangle, it is equal to $bh/2 = 4(60)/2 = 120$ ft. See Figure 6.3.82.

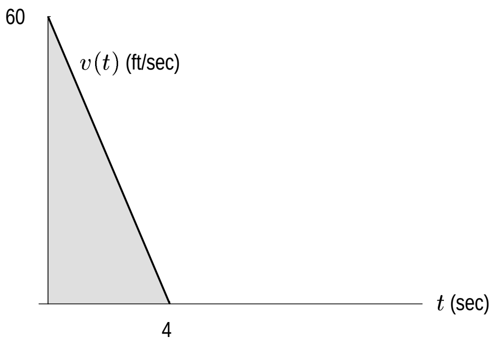


Figure 6.3.82: Ball on Planet Janet

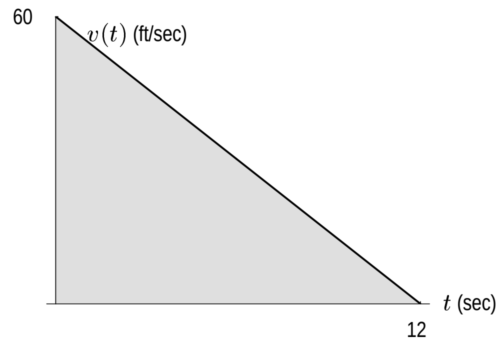


Figure 6.3.83: Ball on Planet Nanette

(e) On Nanette, $v = 60 - 5t$, and so the ball will reach its peak at $t = 12$. The area under the curve is thus $12(60)/2 = 360$ ft., so the peak height is 360 ft. See Figure 6.3.83

4. A ball is thrown vertically upwards from the top of a 320-foot cliff with initial velocity of 128 feet per second. Find:
   (a) how long it takes to reach its peak;
   (b) its maximum height;
   (c) how long it takes until impact;
   (d) the velocity upon impact.

ANSWER:

Taking upwards as the positive direction, the velocity of the ball is $v(t) = v_0 + at = 128 - 32t$, so the displacement (which is the integral of velocity) is $s(t) = \int v(t)\,dt = 128t - 16t^2 + C$. Since $s(0) = 320$, $C = 320$ and $s(t) = 128t - 16t^2 + 320$.

(a) The peak is reached when the velocity is zero:

$$128 - 32t = 0$$
$$t = 4 \text{ s.}$$

(b) $s(4) = 576$ ft.
(c) The ball hits the ground at the base of the cliff when $s(t) = 0$:

$$128t - 16t^2 + 320 = 0$$
$$t^2 - 8t - 20 = 0$$
$$t = -2, 10.$$

We discard the meaningless negative root and conclude that the ball lands after 10 seconds.
(d) $v(10) = -192$ ft/sec. (That is, 192 ft/sec downwards.)

5. Assuming the 440 feet is accurate and you neglect air resistance, determine the accuracy of the following paragraph:

MY JOURNEY BENEATH THE EARTH

Condensed from "A Wolverine is Eating My Leg"

Tim Cahill:
I am in Ellison's Cave, about to rappel down Incredible Pit, the second-deepest cave pit in the continental United States. The drop is 440 feet, about what you'd experience from the top of a 40-story building. If you took the shaft in a free fall, you'd accelerate to more than 100 miles an hour and then—about five seconds into the experience—you'd decelerate to zero. And die.

ANSWER:

Assuming that depth is the positive number that gives the distance below the Earth's surface, and that $g = 32$ ft/sec^2, we have $v(t) = 32t$ since $v(0) = 0$, and $d(t) = 16t^2$ since $d(0) = 0$.

When $d = 440$, we have $t^2 = \frac{440}{16} = 27.5$, so $t \approx 5.24$ sec. The maximum speed reached will occur at the bottom of the cave, where $t \approx 5.24$, so $v_{\max} \approx 32 \cdot 5.24 \approx 168$ ft/sec. Converting this to mph, we get 168 ft/sec $\cdot$ 3600 sec/hour $\cdot$ $\frac{1}{5280}$ miles/foot ≈ 115 mph. The paragraph is accurate!

6. (a) A function g is known to be linear on the interval from $-\infty$ to 2 (inclusive) and also linear on the interval from 2 to ∞ (again inclusive.) Furthermore, $g(1) = 2$, $g(2) = 0$, $g(4) = 8$. What are $g(0)$, $g(3)$?
   (b) Another function f satisfies $f(0) = 0$ and $f' = g$. What are $f(2)$, $f(3)$?
   (c) Give formulas that express $f(t)$ directly in terms of t.
   ANSWER:
   (a) For $x \le 2$, g is of the form $g(x) = mx + b$, because g is linear. The equations $g(1) = m + b = 2$ and $g(2) = 2m + b = 0$ give $m = -2$, $b = 4$ so $g(x) = -2x + 4$ when $x \le 2$. Using a similar method for $x \ge 2$ with the points $(2, 0)$ and $(4, 8)$, we get $g(x) = 4x - 8$ for $x \ge 2$. Thus,
$$g(0) = -2(0) + 4 = 4$$
$$g(3) = 4(3) - 8 = 4.$$
   (b) For $x \le 2$, $f(x) = \displaystyle\int g(x)\,dx = -x^2 + 4x + C$. Since $f(0) = 0$, $f(x) = -x^2 + 4x$ for $x \le 2$. For $x \ge 2$, $f(x) = \displaystyle\int g(x)\,dx = 2x^2 - 8x + C$. From the work we've just done, we know that $f(2) = -(2)^2 + 4(2) = 4$, so since $f(2) = 4 = 2(2)^2 - 8(2) + C$, we know that $C = 12$. Thus, $f(3) = 2(3)^2 - 8(3) + 12 = 6$.
   (c) $f(t) = -t^2 + 4t$ for $t < 2$; $f(t) = 2t^2 - 8t + 12$ for $t \ge 2$.

7. The police observe that the skidmarks made by a stopping car are 200 ft long. Assuming the car decelerated at a constant rate of 20 ft/sec^2, skidding all the way, how fast was the car going when the brakes were applied?
   ANSWER:
   We are given that $\frac{dv}{dt} = -20$ so, integrating, $v = v_0 - 20t$, where v_0 is the speed when the brakes were first applied. Integrating this to get an expression for distance, we have $s = s_0 + v_0 t - 10t^2$. $s_0 = 0$ since we start measuring distance from the place the brakes were first applied. When $v = 0$, $t = \frac{v_0}{20}$, and $s = 200$ so $200 = \frac{v_0^2}{20} - 10\frac{v_0^2}{400}$ so $v_0 = \sqrt{200 \cdot 40} = 89.44$ ft/sec.

8. (a) Find the general solution of the differential equation $\dfrac{dy}{dx} = -4x + 3$.
   (b) Find the solutions satisfying $y(1) = 5$ and $y(1) = 0$.
   ANSWER:
   (a) $y = \displaystyle\int (-4x + 3)\,dx$, so the solution is $y = -2x^2 + 3x + C$.
   (b) If $y(1) = 5$, we have $-2(1)^2 + 3(1) + C = 5$ and so $C = 4$. Thus we have the solution $y = -2x^2 + 3x + 4$.

   If $y(1) = 0$, we have $-2(1)^2 + 3(1) + C = 0$ and so $C = -1$. Thus we have the solution $y = -2x^2 + 3x - 1$.

9. Find the solutions of the initial value problems:
   (a) $\dfrac{dK}{dt} = 3 - \cos 2t$ when $K(0) = -10$.
   (b) $\dfrac{dP}{dt} = -10e^{-t}$ when $P(0) = 20$.
   ANSWER:
   (a) $K = \displaystyle\int (3 - \cos 2t)\,dt = 3t - \frac{\sin 2t}{2} + C$. If $K(0) = -10$, then $3(0) - \dfrac{\sin(2\cdot 0)}{2} + C = -10$ and $C = 10$. Thus $K = 3t - \dfrac{\sin 2t}{2} - 10$.
   (b) $P = \displaystyle\int -10e^{-t}\,dt = 10e^{-t} + C$. If $P(0) = 20$, then $10e^0 + C = 20$ and $C = 10$. Thus $P = 10e^{-t} + 10$.

10. Show that $y = (1 - e^{-2x})\sin 2x$ is the solution to the initial value problem
$$\frac{dy}{dx} = 2(1 - e^{-2x})(\cos 2x) + e^{-2x}\sin 2x,\ y(0) = 0$$
   ANSWER:
   Since $y = (1 - e^{-2x})\sin 2x$, we differentiate to see that $\dfrac{dy}{dx} = 2(1 - e^{-2x})(\cos 2x) + e^{-2x}\sin 2x$.
   To show that it also satisfies the initial condition, we check that $y(0) = 0$:
$$y(0) = (1 - e^0)\sin 0 = 0$$

Questions and Solutions for Section 6.4

1. For $-1 \leq x \leq 1$, define

$$F(x) = \int_{-1}^{x} \sqrt{1-t^2}\, dt.$$

(a) What does $F(1)$ represent geometrically?
(b) What is the value of $F(-1)$? $F(0)$?
(c) Find $F'(x)$.

ANSWER:

(a) $F(1)$ represents the area bounded by $y = \sqrt{1-x^2}$ and $y = 0$ between $x = -1$ and $x = 1$, which is really the area of a semicircle of radius 1.

(b)

$$\begin{aligned}
F(-1) &= \int_{-1}^{-1} \sqrt{1-t^2}\, dt = 0. \\
F(0) &= \int_{-1}^{0} \sqrt{1-t^2}\, dt \\
&= \int_{0}^{1} \sqrt{1-u^2}\, du \quad \text{(substitute } u = -t\text{)} \\
&= \frac{1}{2}\left(u\sqrt{1-u^2}\Big|_0^1 + \int_0^1 \frac{1}{\sqrt{1-u^2}}\, du \right) \quad \text{(using integral table)} \\
&= \frac{1}{2}\left(u\sqrt{1-u^2}\Big|_0^1 + \arcsin u\Big|_0^1 \right) \quad \text{(again using integral table)} \\
&= \frac{1}{2}\left(0 + \frac{\pi}{2}\right) \\
&= \frac{\pi}{4}.
\end{aligned}$$

Note that this is $\frac{1}{4}$ the area of a circle, radius 1.

(c) Using the Fundamental Theorem, $F'(x)$ will simply be $\sqrt{1-x^2}$.

2. Evaluate each of the following:

(a) $\int (x^2 + e^{3x})(x^3 + e^{3x})^{\frac{4}{5}}\, dx$
(b) $\int \frac{\cos(\ln x) dx}{x}$
(c) $\int \frac{8e^{-2w}}{6 - 5e^{-2w}}\, dw$
(d) $\frac{d}{dx} \int_e^x \log_5 \left(t^{21}\right) \sin(\sqrt{t})\, dt.$

ANSWER:

(a) Substitute $w = x^3 + e^{3x}$, and $dw = (3x^2 + 3e^{3x})\, dx$; then the integral becomes

$$\frac{1}{3}\int w^{4/5}\, dw = \frac{5}{27} w^{9/5} + C = \frac{5}{27}(x^3 + e^{3x})^{9/5} + C.$$

(b) Substitute $w = \ln x$ and $dw = \frac{1}{x}dx$. Then the integral becomes

$$\int \cos w\, dw = \sin w + C = \sin(\ln x) + C.$$

(c) Set $u = 6 - 5e^{-2w}$ and $du = 10e^{-2w}\, dw$. The integral becomes

$$\frac{4}{5}\int \frac{du}{u} = \frac{4}{5}\ln|u| + C = \frac{4}{5}\ln|6-5e^{-2w}| + C.$$

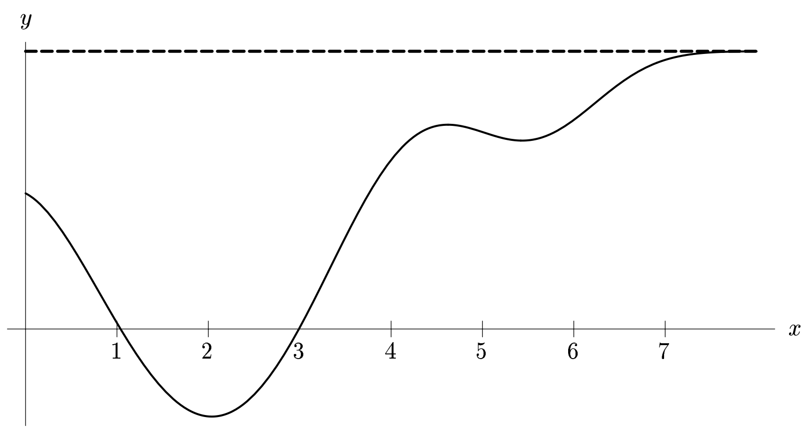
(d) Let $\log_5(t^{21})\sin\sqrt{t}$ be the derivative of some function $f(t)$. Then we have:

$$\begin{aligned}\frac{d}{dx}\int_e^x \log_5(t^{21})\sin\sqrt{t}\,dt &= \frac{d}{dx}\int_e^x f'(t)\,dt\\ &= \frac{d}{dx}\left(f(t)\Big|_e^x\right)\\ &= \frac{d}{dx}\left(f(x)-f(e)\right)\end{aligned}$$

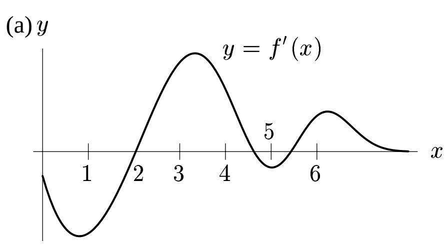
But $f(e)$ is just a constant, so we have

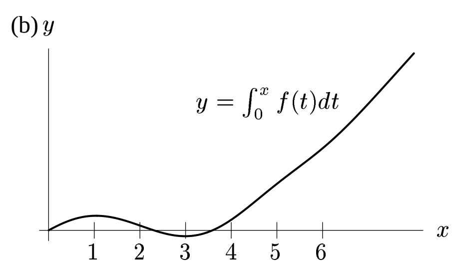
$$\frac{d}{dx}\left(f(x)-f(e)\right) = \frac{d}{dx}f(x) = f'(x) = \log_5(x^{21})\sin\sqrt{x}$$

3. Given below is the graph of a function f.
   (a) Draw a graph of f', the derivative of f.
   (b) Draw a graph of $\displaystyle\int_0^x f(t)\,dt$.

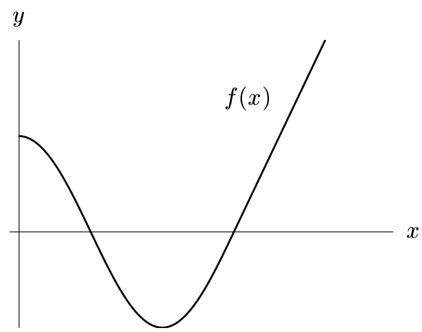


ANSWER:

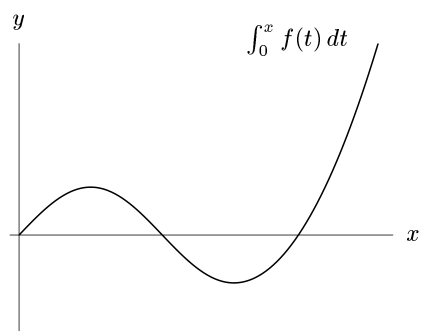




4. Given the graph of f below, draw a graph of the function $\int_0^x f(t)\,dt$.



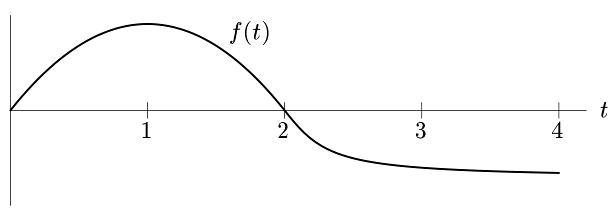
ANSWER:



5. The function $f(t)$ is graphed below and we define

$$F(x) = \int_0^x f(t)dt.$$

Are the following statements true or false? Give a brief justification of your answer.



(a) $F(x)$ is positive for all x between 2 and 3.
(b) $F(x)$ is decreasing for all x between 1 and 3.
(c) $F(x)$ is concave down for $x = \frac{1}{2}$.

ANSWER:

(a) **True**. Although $f(t)$ is negative there, the area below the curve (between 2 and 3) is so much smaller than the area between 0 and 2 that the integral $\int_0^x f(t)dt = \underbrace{\int_0^2 f(t)dt}_{\text{large } +} + \underbrace{\int_2^x f(t)dt}_{\text{small } -}$ is positive for $2 \le x \le 3$.
(b) **False**. F is increasing $1 \le x \le 2$, because f is positive there.
F is decreasing $2 \le x \le 3$ because f is negative there.
(c) **False**. At $x = \frac{1}{2}$, $F' = f$ is increasing so F is concave up.

6. Let $F(x) = \int_0^x \sin t\, dt$, and $G(x) = \int_0^x \sin^2 t\, dt$ and consider the quantities

$$G\left(\frac{\pi}{2}\right), \qquad F(\pi), \qquad G(\pi), \qquad F(2\pi).$$

(a) Show how each quantity can be represented on a graph.
(b) Use the graph to rank these quantities in ascending (i.e., increasing) order.

ANSWER:

(a)

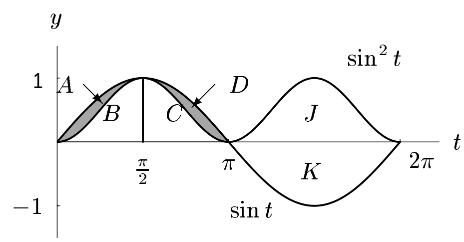


Figure 6.4.84

In the above graph, divide the area under the curves into regions $A, B, C, D, J,$ and K, with positive area. We can now see that

$$G(\frac{\pi}{2}) = B$$
$$F(\pi) = A + B + C + D$$
$$G(\pi) = B + C$$
$$F(2\pi) = A + B + C + D - K$$

(b) $A + B + C + D = K$, so $F(2\pi) = 0$. Therefore,

$$F(2\pi) < G(\frac{\pi}{2}) < G(\pi) < F(\pi).$$

7. Find the derivatives:

(a) $\frac{d}{dx}\int_0^x (\cos(t^2) + \sin(t^2))\, dt$
(b) $\frac{d}{dx}\int_x^1 (1+t)^5\, dt$

ANSWER:

(a) $\cos x^2 + \sin x^2$
(b) $\frac{d}{dx}\int_x^1 (1+t)^5\, dt = \frac{d}{dx}\left(-\int_1^x (1+t)^5\, dt\right) = -(1+x)^5$

8. Write an expression for the function, $f(x)$, with $f'(x) = \sin^2 x + \sin x$ and $f(\pi/2) = 5$.

ANSWER:

If $f'(x) = \sin^2 x + \sin x$, then $f(x)$ is of the form

$$f(x) = C + \int_a^x (\sin^2 t + \sin t)\, dt.$$

Since $f(\pi/2) = 5$, we take $a = \pi/2$ and $C = 5$, giving

$$f(x) = 5 + \int_{\pi/2}^x (\sin^2 t + \sin t)\, dt.$$

9. Find the value of $G(\pi/2)$ where $G'(x) = 2\sin x \cos x$ and $G(0) = 1$.

ANSWER:

Since $G'(x) = 2\sin x\cos x$ and $G(0) = 1$, we have

$$G(x) = G(0) + \int_0^x 2\sin t\cos t\,dt$$
$$= 1 + \int_0^x 2\sin t\cos t\,dt$$

Substituting $x = \pi/2$ and evaluating the integral gives us

$$G(\pi/2) = 1 + \int_0^{\pi/2} 2\sin t\cos t\,dt$$
$$= 1 + \sin^2 t\Big|_0^{\pi/2} = 1 + 1^2 = 2$$

10. For $x = 0, 1, 2,$ and 3, make a table of values for $H(x) = \int_0^x 9t^2\,dt$.

ANSWER:

Table 6.4.14

x	0	1	2	3
$H(x)$	0	3	24	81

Questions and Solutions for Section 6.5

1. At time $t = 0$, a bowling ball rolls off a 375-foot ledge with velocity 30 meters/sec downward. Express its height, $h(t)$, in meters above the ground as a function of time, t, in seconds.

ANSWER:

Since height is measured upward, the initial position of the bowling ball is $h(0) = 375$ meters and the initial velocity is $v = -30$ m/sec. The acceleration due to gravity is $g = -9.8$ m/sec^2. Thus, the height at time t is given by $h(t) = -4.9t^2 - 30t + 375$ meters.

2. A boulder is dropped from a 200-foot cliff. When does it hit the ground and how fast is it going at the time of impact?

ANSWER:

The velocity as a function of time is given by $v = v_0 + at$. Since the object starts from rest, $v_0 = 0$, and the velocity is just the acceleration times time: $v = -32t$. Integrating this, we get position as a function of time: $y = -16t^2 + y_0$, where the last term, y_0, is the initial position at the top of the cliff, so $y_0 = 200$ feet. Thus we have a function giving position as a function of time: $y = -16t^2 + 200$.

To find at what time the object hit the ground, we find t when $y = 0$. We solve $0 = -16t^2 + 200$ for t, getting $t^2 = 200/16 = 12.5$, so $t \approx 3.54$. Therefore, the object hits the ground after about 3.5 seconds. At that time it is moving with an approximate velocity $v = -32(3.54) = -113.28$ feet/second.

3. How does the answer change if the boulder from Exercise 2 is dropped from a cliff that has twice the height instead?

ANSWER:

Since $y = -16t^2 + y_0$, $y = -16t^2 + 400$. To find the time when the boulder hits the ground, solve $0 = -16t^2 + 400$, getting $t^2 = 400/16 = 25$, so $t = 5$. At this time the boulder is moving with a velocity $v = -32(5) = -160$ feet/second.

4. On the moon the acceleration due to gravity is 5 feet/sec^2. A brick is dropped from the top of a tower on the moon and hits the ground in 20 seconds. How high is the tower? (Give your answer in feet.)

ANSWER:

$a(t) = -5$. Since $v(t)$ is the antiderivative of $a(t)$, $v(t) = -5t + v_0$. But $v_0 = 0$, so $v(t) = -5t$. Since $s(t)$ is the antiderivative of $v(t)$, $s(t) = \dfrac{-5t^2}{2} + s_0$, where s_0 is the height of the tower. Since the brick hits the ground in 20 seconds, $s(20) = 0 = \dfrac{-5(20)^2}{2} + s_0$. Hence $s_0 = 1000$ feet, so the tower is 1000 feet high.

5. If acceleration on another planet is twice what it is on the moon, how would the answer to Problem 4 change?

ANSWER:

$a(t) = -10$. $v(t) = -10t + v_0$. Since $v_0 = 0$, $v(t) = -10t$. $s(t) = -5t^2 + s_0$, where s_0 is the height of the tower. $s(20) = 0 = -5(20)^2 + s_0$. Hence $s_0 = 2000$. So the tower is 2000 feet high, or twice the height of the tower on the moon.

Review Questions and Solutions for Chapter 6

1. Find an antiderivative $F(x)$ with $F'(x) = f(x)$ and $F(0) = 6$ when $f(x) = \sin x - \cos x$.

ANSWER:

$F(x) = \int (\sin x - \cos x)\, dx = -\cos x - \sin x + C.$

If $F(0) = 6$, then $F(0) = -\cos(0) - \sin(0) + C = -1 - 0 + C = 6.$
So $C = 7$ and $F(x) = -\cos x - \sin x + 7.$

2. Use the Fundamental Theorem of Calculus to evaluate $\int_{-2}^{2} (3x^2 - 4x + 5)\, dx$.

ANSWER:

$$\begin{aligned}\text{We have } \int_{-2}^{2} (3x^2 - 4x + 5)\, dx &= x^3 - 2x^2 + 5x\Big|_{-2}^{2} \\ &= \big((2^3) - 2(2)^2 + 5(2)\big) - \big((-2)^3 - 2(-2)^2 + 5(-2)\big) \\ &= 8 - 8 + 10 + 8 + 8 + 10 = 36\end{aligned}$$

3. Find the exact area of the region between $y_1 = -2(x-3)^2 + 18$ and $y_2 = x$.

ANSWER:

We need to find where y_1 and y_2 cross

$$\begin{aligned}-2(x-3)^2 + 18 &= x \\ -2x^2 + 12x - 18 + 18 &= x \\ x &= 11/2\end{aligned}$$

The area is $\int_0^{11/2} (-2(x-3)^2 + 18 - x)\, dx$

$$\begin{aligned}&= \int_0^{11/2} (-2x^2 + 11x)\, dx \\ &= -\frac{2}{3}x^3 + \frac{11}{2}x^2\Big|_0^{11/2} \\ &= -\frac{2}{3}\left(\frac{11}{2}\right)^3 + \frac{11}{2}\left(\frac{11}{2}\right)^2 \\ &= \frac{11^3}{3 \cdot 2^3} \approx 55.5.\end{aligned}$$

4. The area between $y = x^{3/2}$, the x-axis, and $x = b$ is $64/5$. Find the value of b using the Fundamental Theorem.

ANSWER:

Since the area between the curves is $64/5$, we have

$$\int_0^b x^{3/2}\, dx = \frac{2}{5}x^{5/2}\Big|_0^b = \frac{2}{5}b^{5/2} = \frac{64}{5}$$

Thus $b^{5/2} = 32$ and $b = 4$.

5. A boat has constant deceleration. It was initially moving at 90 mph and stopped in a distance of 350 feet. What is its rate of deceleration? (Give answer in ft/sec^2. Note: 1 mph=22/15 ft/sec.)

ANSWER:

The velocity of the boat decreases at a constant rate, so we can write $\frac{dv}{dt} = -a$. Integrating this gives $v = -at + C_1$. The constant of integration C_1 is the velocity when $t = 0$, so $C_1 = 90$ mph $= 132$ ft/sec., and $v = -at + 132$. From this equation we can see that the car comes to rest at time $t = 132/a$. Integrating the expression for velocity we get $s = -\frac{a}{2}t^2 + 132t + C_2$, where C_2 is the initial position, so $C_2 = 0$. We can use the fact that the car comes to rest at time $t = 132/a$ after traveling 350 feet. Start with

$$s = -\frac{a}{2}t^2 + 132t,$$

and substitute $t = 132/a$ and $s = 350$.

$$350 = -\frac{a}{2}\left(\frac{132}{a}\right)^2 + 132\left(\frac{132}{a}\right) = \frac{(132)^2}{2a}$$

$$a = \frac{(132)^2}{2(350)} \approx 24.89 \text{ ft/sec}^2$$

6. Find the general solution of the differential equation $\frac{dy}{dx} = \frac{1}{x} + \sin 3x$.

ANSWER:

$$y = \int \left(\frac{1}{x} + \sin 3x\right) dx = \ln|x| - \frac{\cos 3x}{3} + C$$

7. An object falls from the top of a 500-foot building. When does it hit the ground and how fast is it going at the time of impact?

ANSWER:

Velocity is given by $v = v_0 + at$. The object starts at rest so $v_0 = 0$, and velocity is $v = -32t$. Integrating this, we get position as a function of time: $y = -16t^2 + y_0$ where y_0 is the initial position, so $y_0 = 500$ feet and we get $y = -16t^2 + 500$.

To find when the object hits the ground, we find t when $y = 0$.

$$0 = -16t^2 + 500$$
$$16t^2 = 500$$
$$t^2 = 31.25$$
$$t \approx 5.59$$

The object hits the ground at approximately 5.59 seconds and at that time is moving with a velocity $v = -32(5.59) = -178.88$ feet/second.

8. Find antiderivatives for the following functions. Check by differentiating.

(a) $g(x) = 3\sin 9x - 4\cos 2x$

(b) $h(x) = \frac{a}{x} + \sin ax, a$ is constant

ANSWER:

(a)

$$G(x) = \int (3\sin 9x - 4\cos 2x)\, dx = -\frac{1}{3}\cos 9x - 2\sin 2x + C$$

(b)

$$H(x) = \int \left(\frac{a}{x} + \sin ax\right) dx = a\ln|x| - \frac{1}{a}\cos ax + C$$

9. Find antiderivatives for the following functions. Check by differentiating.

(a) $j(x) = ae^x + a\cos x + ax^2, a$ is constant

(b) $k(x) = \frac{5}{x} + 5e^{5x}$

ANSWER:

(a)

$$J(x) = \int (ae^x + a\cos x + ax^2)\, dx$$
$$= ae^x + a\sin x + \frac{ax^3}{3} + C$$

(b)

$$K(x) = \int \left(\frac{5}{x} + 5e^{5x}\right) dx = 5\ln|x| + e^{5x} + C$$

Chapter 7 Exam Questions

Questions and Solutions for Section 7.1

1. Evaluate the following indefinite integrals:

(a) $\int 4x^5\,dx$ (b) $\int \frac{x^2-x+1}{x}\,dx$ (c) $\int e^x \cos(e^x)\,dx$

ANSWER:

(a) $\int 4x^5\,dx = \frac{2}{3}x^6 + C.$

(b) $\int \frac{x^2-x+1}{x}\,dx = \int (x-1+\frac{1}{x})\,dx = \frac{x^2}{2} - x + \ln|x| + C.$

(c) Let $u = e^x$, $du = e^x\,dx$. Substituting, we get

$$\int \cos(e^x)e^x\,dx = \int \cos u\,du = \sin u + C = \sin(e^x) + C.$$

2. Find antiderivatives of the following functions:

(a) $x^2 - \frac{3}{x} + \frac{2}{x^3}$

(b) $\left(x + \sqrt{\sin(2x)+3}\right)\left(x - \sqrt{\sin(2x)+3}\right)$

ANSWER:

(a) $\int (x^2 - \frac{3}{x} + \frac{2}{x^3})\,dx = \frac{x^3}{3} - 3\ln|x| - \frac{1}{x^2} + C.$

(b) $\int (x + \sqrt{\sin(2x+3)})(x - \sqrt{\sin(2x+3)})\,dx = \int (x^2 - \sin(2x+3))\,dx = \frac{x^3}{3} + \frac{1}{2}\cos(2x+3) + C.$

3. Compute:

(a) $\int_{-R}^{R} \frac{\sin x}{1+x^4}\,dx$

(b) $\int_0^R \frac{1}{(4+x)^2}\,dx$

ANSWER:

(a) Since $\frac{\sin x}{1+x^4}$ is an odd function, the integral from $-R$ to R is zero.

(b)

$$\begin{aligned}\int_0^R \frac{1}{(4+x)^2}\,dx \\ &= -\frac{1}{4+x}\bigg|_0^R = -\frac{1}{4+R} - \left(-\frac{1}{4}\right) \\ &= \frac{1}{4} - \frac{1}{4+R}\end{aligned}$$

4. Find the integrals. Check by differentiating.

(a)

$$\int 6y(y^2+6)^3\,dy$$

(b)

$$\int \frac{4x}{\sqrt{9-x^2}}\,dx$$

ANSWER:

(a) Let $u = y^2 + 6$
$du = 2y\, dy$

$$\int 6y(y^2+6)^3\, dy = 3\int (y^2+6)^3(2y\, dy) = 3\int u^3\, du$$
$$= \frac{3u^4}{4} + C = \frac{3}{4}(y^2+6)^4 + C$$

(b) Let $u = 9 - x^2$
$du = -2x\, dx$

$$\int \frac{4x}{\sqrt{9-x^2}}\, dx = -2\int \frac{-2x\, dx}{\sqrt{9-x^2}} = -2\int \frac{du}{\sqrt{u}} = -2\int u^{-1/2}\, du$$
$$= -4u^{1/2} + C = -4(9-x^2)^{1/2} + C$$

5. Integrate

(a)

$$\int \cos^4 2x \sin 2x\, dx$$

(b)

$$\int (\tan 2x + \cos 2x)\, dx$$

ANSWER:

(a) Let $u = \cos 2x$
$du = -2\sin 2x\, dx$

$$\int \cos^4 2x \sin 2x\, dx = -\frac{1}{2}\int u^4\, du = -\frac{u^5}{10} + C$$
$$= \frac{\cos^5 2x}{10} + C$$

(b) Recall that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$. So,

$$\int (\tan 2x + \cos 2x)\, dx = \int \frac{\sin 2x}{\cos 2x}\, dx + \int \cos 2x\, dx$$
$$= -\frac{\ln|\cos 2x|}{2} + \frac{\sin 2x}{2} + C$$

6. Use the Fundamental Theorem to calculate the definite integral

$$\int_0^\pi \cos^2 x \sin x\, dx$$

ANSWER:
Let $u = \cos x$. Then $du = -\sin x\, dx$ and

$$\int_0^\pi \cos^2 x \sin x\, dx = -\int_{x=0}^{x=\pi} u^2\, du = -\frac{u^3}{3}\bigg|_{x=0}^{x=\pi} = -\frac{\cos^3 x}{3}\bigg|_0^\pi$$
$$= -\frac{\cos^3(\pi)}{3} + \frac{\cos^3(0)}{3} = \frac{-1(-1)^3}{3} + \frac{1}{3} = \frac{2}{3}$$

7. Calculate the exact area between the curve $y = \sin^2\theta\cos\theta$ and the x-axis between $\theta = 0$ and $\theta = \pi/2$.
ANSWER:

$$\text{Area} = \int_0^{\pi/2} \sin^2\theta\cos\theta\, d\theta = \frac{\sin^3\theta}{3}\bigg|_0^{\pi/2} = \frac{1}{3}.$$

8. Find the area between $f(x) = 6xe^{-x^2}$ and $g(x) = x$ for $x \geq 0$.

ANSWER:

We need to find where $f(x)$ and $g(x)$ intersect for $x \geq 0$. They are both 0 at $x = 0$. To find other points, we solve

$$\begin{aligned} 6xe^{-x^2} &= x \\ 6e^{-x^2} &= 1 \\ e^{-x^2} &= \frac{1}{6} \\ e^{x^2} &= 6 \\ x^2 &= \ln 6 \\ x &= \sqrt{\ln 6} \approx 1.34 \end{aligned}$$

The area is $\displaystyle\int_0^{1.34} (6xe^{-x^2} - x)\, dx$.

Use the substitution $u = -x^2,\ du = -2x\, dx$

$$\begin{aligned} \int_0^{1.34} (6xe^{-x^2} - x)\, dx &= -3\int_{x=0}^{x=1.34} e^u\, du - \int_0^{1.34} x\, dx \\ &= \left(-3e^{-x^2} - \frac{x^2}{2}\right)\Bigg|_0^{1.34} \\ &= -3e^{-(1.34)^2} - \frac{(1.34)^2}{2} + 3e^0 + 0 = 1.60412 \end{aligned}$$

9. Suppose $\displaystyle\int_0^2 f(t)\, dt = a$ where a is a constant. Calculate the following:

(a)

$$\int_0^1 f(2t)\, dt$$

(b)

$$\int_0^2 3f(2-t)\, dt$$

ANSWER:

(a) Let $w = 2t,\ dw = 2\, dt$. When $t = 0, w = 0$. When $t = 1, w = 2$.

$$\int_0^1 f(2t)\, dt = \int_0^2 f(w)\, \frac{dw}{2} = \frac{1}{2}\int_0^2 f(w)\, dw = \frac{a}{2}$$

(b) Let $w = 2 - t,\ dw = -dt$. When $t = 0, w = 2$ and when $t = 2, w = 0$.

$$\int_0^2 3f(2-t)\, dt = \int_2^0 -3f(w)\, dw = 3\int_0^2 f(w)\, dw = 3a$$

10. Propellant is leaking from the pressurized fuel tanks of the space shuttle at a rate of $r(t) = 15e^{-0.1t}$ psi per second at time t in seconds.

(a) At what rate, in psi per second, is pressure reducing at 15 seconds?

(b) How many psi have leaked during the first minute?

ANSWER:

(a) At $t = 15,\ r(t) = 15e^{-0.1(15)} \approx 3.35$ psi/sec.

(b) To find the amount of propellant leaked during the first minute, we integrate the rate from $t = 0$ to $t = 60$.

$$\int_0^{60} 15e^{-0.1t}\, dt = -\frac{15}{0.1}e^{-0.1t}\Big|_0^{60} = 149.628\text{psi}.$$

Questions and Solutions for Section 7.2

1. Find the following indefinite integrals:

   (a) $\int e^{-0.5t} \sin 2t \, dt$

   (b) $\int (\ln x)^2 dx$ (Hint: Integrate by parts.)

   ANSWER:

   (a) $\int e^{-\frac{t}{2}} \sin 2t \, dt$; use formula #8 from the table with $a = -\frac{1}{2}$, $b = 2$:

$$\begin{aligned}\int e^{-\frac{t}{2}} \sin 2t \, dt &= \frac{1}{\frac{1}{4}+4} e^{-\frac{t}{2}} \left(-\frac{1}{2}\sin 2t - 2\cos 2t\right) + C \\ &= -\frac{4}{17} e^{-\frac{t}{2}} \left(\frac{1}{2}\sin 2t + 2\cos 2t\right) + C\end{aligned}$$

   (b) $\int (\ln x)^2 dx$.

   Let $u = (\ln x)^2$. Then $du = \dfrac{2\ln x}{x} dx$.
   Let $dv = dx$. Then $v = x$.

$$\begin{aligned}\int udv = uv - \int vdu \Rightarrow \int (\ln x)^2 dx &= x(\ln x)^2 - \int x \left(\frac{2\ln x}{x}\right) dx \\ &= x(\ln x)^2 - \int 2\ln x \, dx, \text{ use \#4 from the table for} \int \ln x \, dx : \\ &= x(\ln x)^2 - 2[x\ln x - x] + C = x(\ln x)^2 - 2x\ln x + 2x + C\end{aligned}$$

2. Calculate the following integrals:

   (a) $\int \sec^2\theta \, d\theta \quad \left[\text{Note: } \sec\theta = \dfrac{1}{\cos\theta}\right]$

   (b) $\int z e^{3z^2+1} \, dz$

   (c) $\int y \sec^2 y \, dy$

   (d) $\int \dfrac{dt}{1+\sqrt{t}}$

   (e) $\int \dfrac{dx}{(b+ax)^2} \quad a, b$ constants

   ANSWER:

   (a) $\int \sec^2\theta \, d\theta = \tan\theta + C$ (since $\dfrac{d}{d\theta}\tan\theta = \sec^2\theta$).

   (b) Let $w = 3z^2 + 1$. Then $dw = 6z \, dz$ so $z \, dz = dw/6$. Then

$$\begin{aligned}\int z e^{3z^2+1} \, dz &= \frac{1}{6}\int e^w \, dw \\ &= \frac{1}{6} e^w + C \\ &= \frac{1}{6} e^{3z^2+1} + C.\end{aligned}$$

   (c) Let $u = y$, $v' = \sec^2 y$, $u' = 1$, $v = \tan y$. Then, through integration by parts,

$$\begin{aligned}\int y\sec^2 y \, dy = y\tan y - \int \tan y \, dy &= y\tan y - \int \frac{\sin y}{\cos y} \, dy \\ &= y\tan y + \ln|\cos y| + C. \quad (\text{Set } z = \cos y, dz = \sin y \, dy.)\end{aligned}$$

(d) Let $u = 1 + \sqrt{t}$. Then $(u-1)^2 = t$ so $2(u-1)\,du = dt$. Then

$$\begin{aligned}\int \frac{dt}{1+\sqrt{t}} &= 2\int \left(\frac{u-1}{u}\right)\,du \\ &= 2\int \left(1 - \frac{1}{u}\right)\,du \\ &= 2u - 2\ln|u| + C \\ &= 2(1+\sqrt{t}) - 2\ln(1+\sqrt{t}) + C.\end{aligned}$$

(e) Let $u = b + ax$. Then $du = a\,dx$ so $dx = \dfrac{du}{a}$. Then

$$\begin{aligned}\int \frac{dx}{(b+ax)^2} &= \frac{1}{a}\int \frac{du}{u^2} \\ &= -\frac{1}{a}u^{-1} + C \\ &= -\frac{1}{a\,(b+ax)} + C.\end{aligned}$$

3. For each of the functions $f(x)$ below, find a function $F(x)$ with the property that $F'(x) = f(x)$ and also $F(0) = 0$.
   (a) $f(x) = x \cdot \sin(2x)$
   (b) $f(x) = x^2 \cdot e^{2x}$
   (c) $f(x) = x^2 \cdot (4 + x^3)^{10}$

   ANSWER:

   (a) By Table #15, ($p(x) = x, a = 2$)

   $$\int x\sin(2x)dx = -\frac{1}{2}x\cos(2x) + \frac{1}{4}\sin(2x) + C$$

   (or by parts with $u = x$, $dv = \sin(2x)dx$)), we get $du = dx$, $v = -\frac{\cos 2x}{2}$

   $F(x)$ is of the form $-\frac{1}{2}x\cos(2x) + \frac{1}{4}\sin(2x) + C$

   $F(0) = 0$ implies $0 = F(0) = -\frac{1}{2}0\cos(2\cdot 0) + \frac{1}{4}\sin(2\cdot 0) + C = C$. Therefore

   $F(x) = -\frac{1}{2}x\cos(2x) + \frac{1}{4}\sin(2x)$

   (b) By Table #15, ($p(x) = x^2, a = 2$)

   $$\int x^2 e^{2x}dx = \frac{1}{2}e^{2x}\cdot x^2 - \frac{1}{4}e^{2x}\cdot 2x + \frac{1}{8}e^{2x}\cdot 2 + C = F(x)$$

   $0 = F(0) = \frac{1}{4} + C,$ so $C = -\frac{1}{4}$

   $F(x) = e^{2x}\left(\frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4}\right) - \frac{1}{4}$

   (or use by parts two times, starting with $u = x^2$, $v' = e^{2x}$).

   (c) $\int x^2(4+x^3)^{10}dx$; use substitution with $w = x^3 + 4$, $\dfrac{dw}{dx} = 3x^2$

   $$\int x^2(4+x^3)^{10}\,dx = \int w^{10}\frac{dw}{3} = \frac{1}{33}w^{11} + C = \frac{1}{33}(4+x^2)^{11} + C = F(x)$$

   $0 = F(0) = \frac{1}{33}(4-0)^{11} + C = \frac{4^{11}}{33} + C \Rightarrow C = -\frac{4^{11}}{33}$

   Therefore $F(x) = \frac{1}{33}\left[(4+x^2)^{11} - 4^{11}\right]$

4. Integrate:

   (a) $\displaystyle\int \frac{x^3+1}{x^2}\,dx$

(b) $\displaystyle\int_0^2 \frac{x^2}{x^3+1}\,dx$

(c) $\displaystyle\int \frac{\sqrt{\ln x}}{x}\,dx$

(d) $\displaystyle\int \sin(3x)e^{\cos 3x}\,dx$

ANSWER:

(a)

$$\int \frac{x^3+1}{x^2}\,dx = \int \left(x + \frac{1}{x^2}\right)\,dx = \frac{x^2}{2} - \frac{1}{x} + C$$

(b) Set $u = x^3+1$, $du = 3x^2\,dx$, to get:

$$\int_0^2 \frac{x^2}{x^3+1}\,dx = \frac{1}{3}\int_1^9 \frac{du}{u} = \frac{1}{3}\ln|u|\Big|_1^9 = \frac{1}{3}\ln 9 \approx 0.7324.$$

(c) Set $u = \ln x$, $du = 1/x\,dx$, to get:

$$\int \frac{\sqrt{\ln x}}{x}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(\ln x)^{\frac{3}{2}} + C.$$

(d) Set $u = \cos 3x$, $du = -3\sin 3x dx$, to get:

$$\int \sin(3x)e^{\cos(3x)}\,dx = -\frac{1}{3}\int e^u\,du = -\frac{1}{3}e^u + C = -\frac{1}{3}e^{\cos(3x)} + C.$$

5. Find the following integrals. Show your work.

(a) $\displaystyle\int \left(\frac{1}{x+1} - \frac{1}{(x+1)^2}\right)\,dx$ (b) $\displaystyle\int x\cos 2x\,dx$

ANSWER:

(a) We substitute $y = x+1$, $dy = dx$, giving

$$\int \frac{1}{y}\,dy - \int \frac{1}{y^2}\,dy = \ln|y| + \frac{1}{y} + C = \ln|x+1| + \frac{1}{x+1} + C.$$

(b) Integrating by parts with $u = x$, $v' = \cos 2x$,

$$\int x\cos 2x\,dx = \frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x\,dx = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C.$$

6. Find $\displaystyle\int te^{at}\,dt$ where a is a constant.

ANSWER:

Let $u = t$, $\quad v' = e^{at}$

$u' = 1 \quad v = \frac{1}{a}e^{at}$

$$\int xe^{ax}\,dx = \frac{1}{a}te^{at} - \int \frac{1}{a}e^{at}\,dt = \frac{1}{a}te^{at} - \frac{1}{a^2}e^{at} + C.$$

7. Find $\displaystyle\int \theta^2\cos(a\theta)\,d\theta$ where a is a constant.

ANSWER:

Let $u = \theta^2$, $\quad v' = \cos(a\theta)$

$u' = 2\theta \quad v = \frac{1}{a}\sin(a\theta)$

$$\text{Then } \int \theta^2\cos(a\theta)\,d\theta = \frac{1}{a}\theta^2\sin(a\theta) - \int \frac{1}{a}2\theta\sin(a\theta)\,d\theta$$

Use integration by parts again on $\int \frac{2\theta}{a} \sin(a\theta)\, d\theta$

Let $u = \theta$, $\quad v' = \sin(a\theta)$

$u' = 1 \quad v = -\frac{1}{a}\cos(a\theta)$

$$\text{That gives } \int \frac{2\theta}{a}\sin(a\theta)\, d\theta = \frac{2\theta}{a}\left(-\frac{\theta}{a}\cos(a\theta) + \int \frac{1}{a}\cos(a\theta)\, d\theta\right)$$

$$\text{Thus } \int \theta^2 \cos(a\theta)\, d\theta = \frac{\theta^2}{a}\sin(a\theta) - \frac{2\theta^2}{a^2}\cos(a\theta) + \frac{2\theta}{a^3}\sin(a\theta)$$

8. Concentration in mg/ml of a drug in the blood is a function of time, t, in hours since the drug was administered, and is given by

$$C = 10te^{-0.8t}$$

The area under the concentration curve is a measure of the overall effect of the drug on the body, called the bioavailability. Find the bioavailability of the drug between $t = 2$ and $t = 4$.

ANSWER:

Bioavailability $= \int_0^4 10te^{-0.8t}\, dt$

Using integration by parts, $u = 10t$ and $v' = e^{-0.8t}$, so $u' = 10\, dt$ and $v = -\frac{5}{4}e^{-0.8t}$. Then,

$$\int_0^4 10te^{-0.8t}\, dt = (10t)\left(-\frac{5}{4}e^{-0.8t}\right) - \int \left(-\frac{5}{4}e^{-0.8t}\right) 10\, dt$$

$$= -\frac{25}{2}te^{-0.8t} + \frac{25}{2}\int e^{-0.8t}\, dt = -\frac{25}{2}te^{-0.8t} - 10e^{-0.8t} + C$$

$$\text{Thus } \int_0^4 10te^{-0.8t}\, dt = \left(-\frac{25}{2}te^{-0.8t} - 10e^{-0.8t}\right)\Big|_0^4$$

$$= -2.44573 + 10 = 7.55427 \text{ (mg/ml)} - \text{hours}$$

9. Show that $\int \ln x\, dx = x\ln x - x + C$.

ANSWER:

Let $u = \ln x$, $\quad v' = 1$

$u' = \frac{1}{x}$, $\quad v = x$

$$\int \ln x\, dx = x\ln x - \int \left(x \cdot \frac{1}{x}\right) dx$$

$$= x\ln x - x + C$$

Questions and Solutions for Section 7.3

1. Use the table of antiderivatives to evaluate each of the following:

(a) $\int \frac{dx}{x^2 - 6x + 10}$

(b) $\int \frac{dx}{\sqrt{x^2 - 6x + 10}}$

(c) $\int \frac{x^2}{x^2 + 5}\, dx$

(d) $\int \frac{dt}{\sqrt{9 - 5t^2}}$

(e) $\int e^{4x}\cos 3x\, dx$

ANSWER:

(a)

$$\int \frac{dx}{x^2 - 6x + 10} = \int \frac{dx}{(x-3)^2 + 1}$$

Let $u = x - 3$, $du = dx$, to obtain:

$$\begin{aligned}\int \frac{dx}{(x-3)^2+1} &= \int \frac{du}{u^2+1} \\ &= \arctan u + C \\ &= \arctan(x-3) + C.\end{aligned}$$

(b)

$$\int \frac{dx}{\sqrt{x^2 - 6x + 10}} = \int \frac{dx}{\sqrt{(x-3)^2 + 1}}$$

Let $u = x - 3$, $du = dx$, to obtain:

$$\begin{aligned}\int \frac{dx}{\sqrt{(x-3)^2+1}} &= \int \frac{du}{\sqrt{u^2+1}} \\ &= \int \ln|u + \sqrt{u^2+1}| + C \\ &= \int \ln|x - 3 + \sqrt{(x-3)^2+1}| + C.\end{aligned}$$

(c)

$$\begin{aligned}\int \frac{x^2}{x^2+5}\,dx &= \int \frac{x^2+5-5}{x^2+5}\,dx \\ &= \int \left(1 - \frac{5}{x^2+5}\right)\,dx \\ &= x - \sqrt{5}\arctan\frac{x}{\sqrt{5}} + C.\end{aligned}$$

(d)

$$\begin{aligned}\int \frac{dt}{\sqrt{9-5t^2}} &= \int \frac{dt}{\sqrt{5(\frac{9}{5} - t^2)}} \\ &= \frac{1}{\sqrt{5}}\int \frac{dt}{\sqrt{\frac{9}{5} - t^2}} \\ &= \frac{1}{\sqrt{5}}\arcsin\left(\frac{t}{\frac{3}{\sqrt{5}}}\right) + C \\ &= \frac{1}{\sqrt{5}}\arcsin\frac{t\sqrt{5}}{3} + C.\end{aligned}$$

(e)

$$\begin{aligned}\int e^{4x}\cos 3x\,dx &= \frac{1}{4^2+3^2}e^{4x}(4\cos 3x + 3\sin 3x) + C \\ &= \frac{1}{25}e^{4x}(4\cos 3x + 3\sin 3x) + C.\end{aligned}$$

2. Calculate the following integrals.
   (a) $\int \cos(7\theta)d\theta$
   (b) $\displaystyle\int \frac{e^x}{1+e^{2x}}dx$
   (c) $\displaystyle\int \frac{t+5}{t^2+10t+77}dt$
   (d) $\displaystyle\int \frac{2dx}{x^2+4x+3}$
   (e) $\int \sin\left(\sqrt{y}\right)dy$

   ANSWER:

   (a) Set $u = 7\theta$ so $\dfrac{du}{7} = d\theta$ i.e. $\displaystyle\int \cos(7\theta)d\theta = \frac{1}{7}\int \cos u du = \frac{1}{7}\sin(7\theta) + C$.

   (b) Set $u = e^x$ so $du = e^x dx$ i.e. $\displaystyle\int \frac{e^x}{1+e^{2x}} = \int \frac{1}{1+u^2} = \tan^{-1}(e^x) + C$.

   (c) Set $u = t^2 + 10t + 77$ so $du = 2(t+5)dt$ i.e. $\dfrac{du}{2} = (t+5)dt$. Hence

$$\int \frac{t+5}{t^2+10t+77}dt = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|t^2+10t+77| + C.$$

   (d) Notice that $x^2+4x+3 = x^2+4x+4-1 = (x+2)^2-1$. Letting $u = x+2$, $du = dx$ we get

$$\int \frac{2dx}{x^2+4x+3} = \int \frac{2dx}{(x+2)^2-1} = 2\int \frac{1}{u^2-1}du = 2\int \frac{1}{(u-1)(u+1)}du$$
$$= 2\int \frac{du}{(u-1)(u-(-1))}.$$

   By the tables (or by decomposing into partial fractions!), the last integral equals ($a = 1, b = -1$):

$$\int \frac{du}{(u-1)(u-(-1))} = 2\frac{1}{1-(-1)}\left(\ln|u-1| - \ln|u+1|\right) + C,$$

   or, going back to x,

$$\int \frac{2dx}{x^2+4x+3} = \ln|x+1| - \ln|x+3| + C.$$

   (e) Let $x = \sqrt{y}$ so $dx = \dfrac{1}{2\sqrt{y}}dy$ or $2\sqrt{y}dx = dy$. But since $\sqrt{y} = x$, we have $dy = 2xdx$ so $\int \sin\left(\sqrt{y}\right)dy = 2\int x\sin(x)dx$. Integrating by parts with $u = x$, $u' = 1$, $v' = \sin x$, $v = -\cos x$ gives

$$2\int x\sin x dx = 2\left[-x\cos x + \int \cos x dx\right] = 2\left(\sin x - x\cos x\right) + C,$$

   or

$$\int \sin\sqrt{y}dy = 2\left(\sin\sqrt{y} - \sqrt{y}\cos\sqrt{y}\right) + C.$$

3. As a storm dies down, rainfall in inches is given by $y = \dfrac{1}{.25+t^2}$ for $0 \le t \le 2$ where t is hours since the point of heaviest rainfall. What was the average rainfall over the two hours?

   ANSWER:

$$\text{Average rainfall } = \frac{1}{2-0}\int_0^2 \frac{1}{.25+t^2}dt$$
$$= \frac{1}{2}\left(\frac{1}{.5}\right)\arctan\frac{t}{.5}\Big|_0^2 = \arctan 4 - \arctan 0 \approx 1.33 \text{ inches/hour}$$

4. Is $\displaystyle\int \sin^{-2}x\,dx = \frac{-1}{-2}\sin^{-3}x\cos x + \frac{-2-1}{-2}\int \sin^{-4}x\,dx$?

   ANSWER:

   No. To use the reduction formula, the power of sin must be positive.

5. As a storm dies down, rainfall in inches is given by $y = \dfrac{1}{.25 + t^2}$ for $0 \leq t \leq 2$ where t is hours since the point of heaviest rainfall. What was the average rainfall over the two hours?

ANSWER:

$$\text{Average rainfall} = \frac{1}{2-0}\int_0^2 \frac{1}{.25+t^2}\,dt$$

$$= \frac{1}{2}\left(\frac{1}{.5}\right)\arctan\frac{t}{.5}\bigg|_0^2 = \arctan 4 - \arctan 0 \approx 1.33 \text{ inches/hour}$$

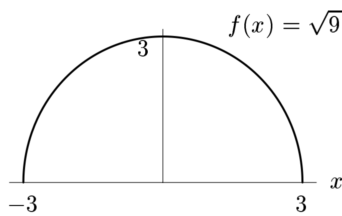
6. Is $\displaystyle\int \sin^{-2} x\,dx = \frac{-1}{-2}\sin^{-3} x\cos x + \frac{-2-1}{-2}\int \sin^{-4} x\,dx$?

ANSWER:

No. To use the reduction formula, the power of $\sin$ must be positive.

Questions and Solutions for Section 7.4

1. Consider the semicircle of radius 3 pictured below.



(a) Write a definite integral in terms of $f(x)$ that gives the area of the semicircle.
(b) Based solely on your knowledge of *geometry*, what is the value of this definite integral?
(c) Evaluate the indefinite integral.
(d) Find the area of the semicircle by substituting the limits, then compare this result with your answer in Part (b).

ANSWER:

(a) $\displaystyle\int_{-3}^{3} \sqrt{(9-x^2)}dx$

(b) $\dfrac{1}{2}\cdot\pi r^2 = \dfrac{9}{2}\cdot\pi$

(c) Use #30 with $a = 3$:

$$\int \sqrt{(9-x^2)}dx = \frac{1}{2}\cdot\left[x\sqrt{(9-x^2)} + 9\cdot\int \frac{1}{\sqrt{(9x^2+x-6)}}dx\right] + \text{constant}$$

To integrate this integral, use #28 with $a = 3$:

$$= \frac{1}{2}\cdot\left[x\sqrt{(9-x^2)} + 9\arcsin\left(\frac{x}{3}\right)\right] + C$$

(d) $\displaystyle\frac{1}{2}\cdot\left[x\sqrt{(9-x^2)} + 9\arcsin\left(\frac{x}{3}\right)\right]\bigg|_{-3}^{3} = \frac{9}{2}\cdot[\arcsin(1) - \arcsin(-1)]$

$$= \frac{9}{2}\cdot\frac{\pi}{2} - \frac{9}{2}\cdot\left(-\frac{\pi}{2}\right) = \frac{9}{2}\cdot\pi$$

2. Find the following indefinite integrals:

(a) $\displaystyle\int \frac{1}{x^2+x-6}dx$

(b) $\displaystyle\int \cos^3\beta\,d\beta$

ANSWER:

(a) $\displaystyle\int \frac{1}{(x^2+x-6)}dx = \int \frac{1}{(x+3)(x-2)}dx$

Using #26 from the table in the back of the book (or by decomposing into partial fractions!), the last integral equals ($a = -3$,$b = 2$):
$-\frac{1}{5}(\ln|x+3| - \ln|x-2|) + C$

(b)

$$\int \cos^3\beta\, d\beta = \int \cos\beta(1 - sin^2\beta)\, d\beta = \sin\beta - \frac{1}{3}\int 3\cos\beta\sin^2\beta\, d\beta$$
$$= \sin\beta - \frac{1}{3}\int \frac{d}{d\beta}(\sin^3\beta)\, d\beta = \sin\beta - \frac{1}{3}\sin^3\beta + C.$$

One may also use #18 from the table in the back of the book.

3. Find $\int \frac{2x+1}{2x^2+4x}\, dx$
ANSWER:
Factor the denominator and split the integral into partial fractions:

$$\frac{2x+1}{2x^2+4x} = \frac{2x+1}{2x(x+2)} = \frac{A}{2x} + \frac{B}{x+2}$$

Multiplying by $2x(x+2)$ gives

$$\begin{aligned} 2x + 1 &= A(x+2) + B(2x) \\ &= Ax + 2A + 2Bx \\ &= x(A + 2B) + 2A \end{aligned}$$

$$A + 2B = 2 \qquad 2A = 1,\ A = \frac{1}{2}$$
$$\frac{1}{2} + 2B = 2 \qquad 2B = \frac{3}{2},\ B = \frac{3}{4}$$
$$\int \left(\frac{A}{2x} + \frac{B}{x+2}\right) dx = \int \frac{1}{4x}\, dx + \int \frac{3}{4(x+2)}\, dx$$
$$= \frac{1}{4}\ln|x| + \frac{3}{4}\ln|x+2| + C$$

4. Find the area of the circle $y^2 + x^2 = 9$ using trigonometric substitution and integration.
ANSWER:
$y^2 = 9 - x^2$
$y = \sqrt{9 - x^2}$ gives the top half of the circle.
$\int_0^3 \sqrt{9 - x^2}\, dx$ will give area of a quarter of the circle.
Area of circle $= 4\int_0^3 \sqrt{9 - x^2}\, dx$
Choose $x = 3\sin\theta$.

$$dx = 3\cos\theta\, d\theta \qquad d\theta = \frac{dx}{3\cos\theta}$$
$$\sqrt{9 - x^2} = \sqrt{9 - 9\sin^2\theta} = 3\sqrt{1 - \sin^2\theta} = 3\sqrt{\cos^2\theta} = 3\cos\theta$$

For $x = 0$, use $\theta = 0$. For $x = 3$, use $\theta = \pi/2$.

$$\begin{aligned} 4\int_0^3 \sqrt{9 - x^2}\, dx &= 4\int_0^{\pi/2} 3\cos\theta \cdot 3\cos\theta\, d\theta \\ &= 36\int_0^{\pi/2} \cos^2\theta\, d\theta \\ &= 36\left(\frac{1}{2}\cos\theta\sin\theta + \frac{1}{2}\theta\right)\Big|_0^{\pi/2} \end{aligned}$$

$$= 18\left(\cos\frac{\pi}{2}\sin\frac{\pi}{2}+\frac{\pi}{2}\right) - 18(\cos 0 \sin 0 + 0)$$
$$= 18\left(0+\frac{\pi}{2}\right) = 9\pi$$

5. Derive the formula for the area of a circle with radius r using trigonometric substitution.

ANSWER:

$x^2 + y^2 = r^2$ describes a circle with radius r

$y = \sqrt{r^2 - x^2}$ describes the upper half of the circle.

Use $x = r\sin\theta$.

$dx = r\cos\theta\, d\theta$

When $x = 0$, use $\theta = 0$. When $x = r$, use $\theta = \pi/2$.

Area of circle = $4\int_0^r \sqrt{r^2 - x^2}\, dx = 4\int_0^{\pi/2} \sqrt{r^2 - r^2\sin^2\theta}\cdot r\cos\theta\, d\theta$

$$= 4\int_0^{\pi/2} r\cos\theta \cdot r\cos\theta\, d\theta.$$
$$= 4r^2\int_0^{\pi/2}\cos^2\theta\, d\theta = 4r^2\left(\frac{1}{2}\cos\theta\sin\theta + \frac{1}{2}\theta\right)\Big|_0^{\pi/2}$$
$$= 2r^2\left(\cos\frac{\pi}{2}\sin\frac{\pi}{2}+\frac{\pi}{2} - \cos 0\sin 0 + 0\right)$$
$$= 2r^2\left(0+\frac{\pi}{2}\right) = \pi r^2$$

6. Show that $\int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\arctan\frac{x}{a} + C, a \neq 0.$

ANSWER:

Use $x = a\tan\theta$.

If $x = a\tan\theta$, then $dx = \left(\frac{a}{\cos^2\theta}\right)\, d\theta$, so

$$\int \frac{1}{x^2+a^2}\, dx = \int \frac{1}{a^2\tan^2\theta + a^2}\cdot\frac{a}{\cos^2\theta}\, d\theta$$
$$= \frac{1}{a}\int \frac{1}{\left(\frac{\sin^2\theta}{\cos^2\theta}+1\right)(\cos^2\theta)}\, d\theta$$
$$= \frac{1}{a}\int \frac{1}{\sin^\theta + \cos^2\theta}\, d\theta$$
$$= \frac{1}{a}\int 1\, d\theta = \frac{1}{a}\theta + C$$

$x = a\tan\theta$, so $\theta = \arctan\left(\frac{x}{a}\right)$

Thus $\int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C.$

7. Find the area of the region bounded by $y = 0$ and $y = \frac{t+2}{2t^2+3t+1}$ between $t = 0$ and $t = 2$.

ANSWER:

Since $2t^2 + 3t + 1 = (2t+1)(t+1)$, we write

$$\frac{t+2}{2t^2+3t+1} = \frac{A}{2t+1} + \frac{B}{t+1},$$

$$\text{giving } t+2 = A(t+1) + B(2t+1)$$
$$t+2 = (A+2b)t + A + B$$

So $A + 2B = 1,\ A + B = 2$, thus $A = 3,\ B = -1$, so

$$\int \frac{t+2}{2t^2+3t+1}\,dt = \int \frac{3}{2t+1}\,dt - \int \frac{1}{t+1}\,dt$$
$$= \left(\frac{3}{2}\ln|2t+1| - \ln|t+1|\right)\Big|_0^2$$
$$= \frac{3}{2}\ln 5 - \ln 3 - \frac{3}{2}\ln 1 + \ln 1 \approx 1.32$$

8. Find the area of the region bounded by $y = \dfrac{1}{x^2+4x+13}$, $x = 0$, and $x = 2$.

ANSWER:
Completing the square, we get $x^2 + 4x + 13 = (x+2)^2 + 9$.

Use the substitution $x + 2 = 3\tan t$, then
$dx = \left(\dfrac{3}{\cos^2 t}\right) dt$. Since $\tan^2 t + 1 = \dfrac{1}{\cos^2 t}$, the integral becomes

$$\int \frac{1}{(x+2)^2+9}\,dx = \int \frac{1}{9\tan^2 t + 9}\cdot\frac{3}{\cos^2 t}\,dt$$
$$= \frac{1}{3}\arctan\left(\frac{x+2}{3}\right)\Big|_0^2$$
$$= \frac{1}{3}\arctan\frac{5}{3} - \frac{1}{3}\arctan\frac{2}{3} = 0.147458$$

Questions and Solutions for Section 7.5

1. The following are some of the values for a function known as the Gudermannian function, $G(x)$.

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$G(x)$	0	0.100	0.199	0.296	0.390	0.480	0.567	0.649	0.726	0.798	0.866

Use these values to approximate the value of

$$\int_0^1 G(x)\,dx.$$

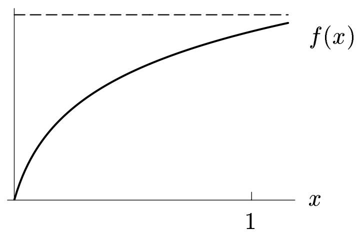
ANSWER:
Since $G(x)$ is increasing, $\text{LEFT}(10)$ is an underestimate and $\text{RIGHT}(10)$ is an overestimate. We therefore approximate $\int_0^1 G(x)\,dx$ by $\text{TRAP}(10) = \dfrac{\text{LEFT}(10) + \text{RIGHT}(10)}{2}$.

$$\text{LEFT}(10) = 0.1(0 + 0.100 + \ldots + 0.798) = 0.421$$
$$\text{RIGHT}(10) = 0.1(0.100 + 0.199\ldots + 0.866) = 0.507$$
$$\text{TRAP}(10) = \frac{\text{LEFT}(10) + \text{RIGHT}(10)}{2} = 0.464$$

2. The following numbers are the left, right, trapezoidal, and midpoint approximations to $\int_0^1 f(x)\,dx$, where $f(x)$ is as shown. (Each uses the same number of subdivisions.)
I) 0.36735
II) 0.39896
III) 0.36814
IV) 0.33575

(a) Which is which? How do you know?

(b) Write $A < \int_0^1 f(x)\,dx < B$, where $B - A$ is as small as possible.



ANSWER:

(a) f is increasing and concave down, so we expect to have

$$\mathrm{LEFT}(n) < \mathrm{TRAP}(n) < \mathrm{MID}(n) < \mathrm{RIGHT}(n).$$

Thus (IV) is left, (I) is trapezoid, (III) is midpoint and (II) is right.

(b) We know the true value is between those given by the trapezoid and midpoint rules since the curve is concave down, so

$$0.36735 < \int_0^1 f(x)\,dx < 0.36814.$$

3. Compute the following integrals. If you provide an approximation, it should be rounded to three decimal places and you must explain how you got it and how you know it has the desired accuracy.

(a) $\int_2^3 \frac{1}{\ln x}\,dx$

(b) $\int_0^3 x\sqrt{16+x^2}\,dx$

ANSWER:

(a) Using left- and right-hand sums with $n = 1200$, the left-hand sum is 1.11864 and the right-hand sum is 1.11820. Since these sums differ by only $0.00044 < 0.0005$, and since $\frac{1}{\ln x}$ is monotonically decreasing between 2 and 3,

$$\int_2^3 \frac{1}{\ln x}\,dx \approx 1.11842$$

is correct to 3 decimal places.

(b) Let $u = 16 + x^2$. So $du = 2x\,dx$. By substituting, we get

$$\begin{aligned}\frac{1}{2}\int \sqrt{u}\,du &= \frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}} + C\\ &= \frac{1}{3}(16+x^2)^{\frac{3}{2}} + C\end{aligned}$$

So

$$\begin{aligned}\int_0^3 x\sqrt{16+x^2}\,dx &= \frac{1}{3}(16+x^2)^{\frac{3}{2}}\Big|_0^3\\ &= \frac{1}{3}(25)^{\frac{3}{2}} - \frac{1}{3}(16)^{\frac{3}{2}}\\ &= \frac{1}{3}\cdot 125 - \frac{1}{3}\cdot 64\\ &= \frac{61}{3}\end{aligned}$$

4. The table to the right shows the velocity $v(t)$ of a falling object at various times (time t measured in seconds, velocity $v(t)$ measured in m/sec).

t	0	1	2	3
$v(t)$	17	23	28	32

(a) Due to air resistance, the object's acceleration is decreasing. What does this tell you about the shape of the graph of $v(t)$?

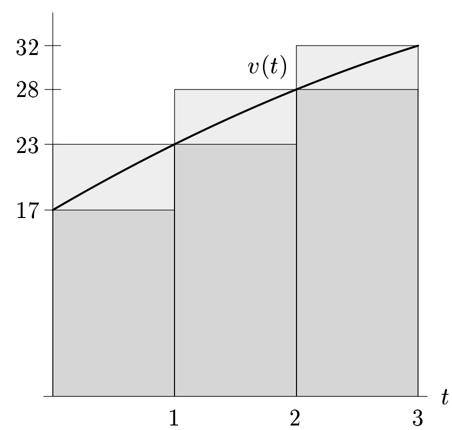
(b) Find upper and lower bounds for the distance the object fell in these three seconds. The bounds should be less than three meters apart. Illustrate your reasoning on a sketch.

ANSWER:

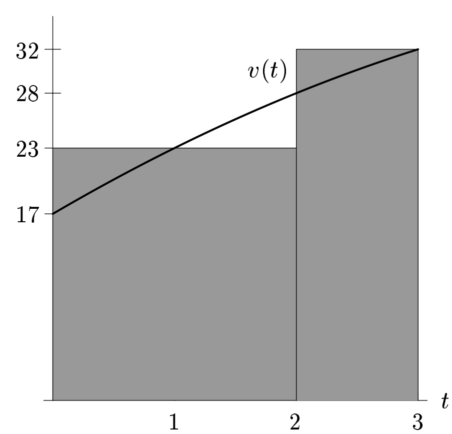
(a) $v(t)$ will be concave down.

(b) Find a left-hand sum, a right-hand sum, and a trapezoidal sum with $\Delta t = 1$.

$$\text{LEFT}(3) = 1 \cdot (17 + 23 + 28) = 68\text{m}$$
$$\text{RIGHT}(3) = 1 \cdot (23 + 28 + 32) = 83\text{m}$$
$$\text{TRAP}(3) = \frac{68 + 83}{2} = 75.5\text{m}$$



Because this curve is concave down, the trapezoid rule gives an underestimate of 75.5; the right-hand sum produces an overestimate of 83. These are not close enough! For a closer overestimate try the following. Use two intervals; the first of length 2 and the second of length 1.



Using the midpoint rule with the first interval gives an overestimate; using the right-hand rule with the second interval also gives an overestimate. Such a sum gives $2 \cdot 23 + 32 = 78$ as an overestimate. So the distance fallen is between 75.5 and 78 meters.

5. For each of the definite integrals below, confine the value of an integral to an interval of size 0.1. For each of your answers, explain carefully which method you used to obtain it, and show how you know your answer is as accurate as required.

(a) $\displaystyle\int_{\frac{\pi}{2}}^{\frac{5\pi}{2}} e^{\sin x}\,dx$

(b) $\displaystyle\int_{0}^{10} \ln\left(x^2+1\right)dx$

ANSWER:

(a) To find $\displaystyle\int_{\pi/2}^{5\pi/2} e^{\sin x}\,dx$ to within 0.1, we first note that $e^{\sin x}$ is decreasing over $[\frac{\pi}{2}, \frac{3\pi}{2}]$ and increasing over $[\frac{3\pi}{2}, \frac{5\pi}{2}]$. In fact, since the function is symmetric about $\frac{3\pi}{2}$, we can say that

$$\int_{\frac{\pi}{2}}^{\frac{5\pi}{2}} e^{\sin x}\,dx = 2\cdot\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{\sin x}\,dx$$

Since $e^{\sin x}$ is decreasing over $[\frac{\pi}{2}, \frac{3\pi}{2}]$, a left-hand sum will be an overestimate and a right-hand sum will be an underestimate. We want the error for an estimate of $\displaystyle\int_{\pi/2}^{5\pi/2} e^{\sin x}\,dx$ to be less than 0.05. Set

$$|f(b)-f(a)|\cdot\frac{b-a}{n} = \left(e^{\sin\frac{3\pi}{2}} - e^{\sin\frac{\pi}{2}}\right)\cdot\frac{\pi}{n} \le 0.05$$

So, $n \ge \dfrac{e^{\sin\frac{3\pi}{2}} - e^{\sin\frac{\pi}{2}}}{0.05}\cdot\pi \approx 147.7$. Hence use $n = 150$. Approximate $\int_{\pi/2}^{3\pi/2} e^{\sin x}\,dx$ with $n = 150$ to obtain:

$$\mathrm{LEFT}(150) \approx 4.00 \text{ (rounding up)}$$
$$\mathrm{RIGHT}(150) \approx 3.95 \text{ (rounding down).}$$

So $3.95 < \displaystyle\int_{\pi/2}^{3\pi/2} e^{\sin x}\,dx < 4.00$. Hence $7.90 < \displaystyle\int_{\pi/2}^{5\pi/2} e^{\sin x}\,dx < 8.00$.

(b) To estimate $\int_0^{10} \ln(x^2+1)\,dx$, note that $\ln(x^2+1)$ is increasing over the interval $[0, 10]$, so a left-hand sum will be an underestimate and a right-hand sum will be an overestimate. To find the number of divisions necessary, set

$$0.1 \ge |\ln 101 - \ln 1|\cdot\frac{10}{n}.$$

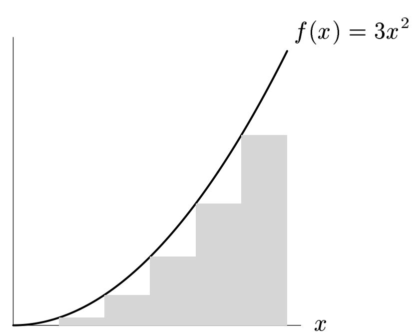
So $n \ge 461$. Thus use $n = 500$ to obtain:

$$\mathrm{LEFT}(500) \approx 29.04 \text{ (rounding down)}$$
$$\mathrm{LEFT}(500) \approx 29.14 \text{ (rounding up)}$$

So $29.04 < \displaystyle\int_0^{10} \ln(x^2+1)\,dx < 29.14.$

6. Suppose that a computer takes 10^{-6} seconds to add two numbers together, and it takes 10^{-5} seconds to multiply two numbers together. The computer is asked to integrate the function $f(x) = 3\cdot x^2$ from 0 to 1 using left hand sums with n divisions. As a function of n, introduce the time, $T(n)$, used by the computer to do the calculation. Compute $T(n)$. (The computer figures x^2 as $x\cdot x$.)

ANSWER:



$f(x) = 3x^2 = 3 \cdot x \cdot x$.

The time the computer takes to evaluate one interval (i.e., to multiply twice) $= 10^{-5} + 10^{-5} = 2 \times 10^{-5}$.

The time the computer takes to evaluate n such intervals $= (2 \times 10^{-5}) \times n$.

If there are n intervals, then the computer will have to add $(n-1)$ times. The time the computer will take to do all the additions $= (n-1) \times 10^{-6}$.

Therefore the time taken for the entire calculation is

$$T(n) = (2 \times 10^{-5}) \times n + (n-1) \times 10^{-6}.$$

7. (a) Find the exact value of $\displaystyle\int_1^{10} x\sqrt[3]{x-1}\,dx$. (An exact value is one like $\sqrt[3]{7}$, ln 10, $\tan(\pi+1)$.)

   (b) You want to estimate $\displaystyle\int_0^2 \cos(\theta^2)d\theta$ by finding values, A and B, which differ by less than 0.0001 and such that

$$A < \int_0^1 \cos(\theta^2)d\theta < B.$$

   Explain how you found A and B, and justify your assertion that they lie on either side of the integral.

   ANSWER:

   (a) Let $w = \sqrt[3]{x-1}$, then $w^3 + 1 = x$ so $3w^2dw = dx$. When $x = 1$, $w = 0$; when $x = 10$, $w = \sqrt[3]{9}$.

$$\int_1^{10} x\sqrt[3]{x-1}dx = \int_0^{\sqrt[3]{9}} (w^3+1)\cdot w \cdot 3w^2dw = 3\int_0^{\sqrt[3]{9}} w^6 + w^3dw = \frac{3w^7}{7} + \frac{3w^4}{4}\Bigg|_0^{\sqrt[3]{9}}$$
$$= \frac{3}{7}(9)^{\frac{7}{3}} + \frac{3}{4}\left(9^{\frac{4}{3}}\right)$$

   Note: To get the exact value, you need to use the Fundamental Theorem.

   (b) Graph of $y = \cos(\theta^2)$ is decreasing and concave down for $0 \le \theta \le 1$. (Draw it on the calculator and see.) Thus Trap $<$ True $<$ Midpt; we'll take Trap $= A$, Mid $= B$.

   With $n = 50$, get Trap $= .90447 = A$, Mid $= .90455 = B$, so

$$\underbrace{0.90447}_{A} < \int_0^1 \cos(\theta^2)d\theta < \underbrace{0.90455}_{B}$$

8. Last Monday we hired a typist to work from 8am to 12 noon. His typing speed decreased between 8am and his 10am cup of coffee, and increased again afterwards, between 10am and noon. His instantaneous speed (measured in characters per second) was measured each hour and the results are given below:

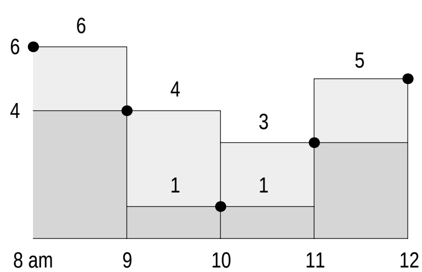
Time	8am	9	10	11	12
Speed	6	4	1	3	5

You want to estimate the total number of characters typed between 8am and 12 noon.

   (a) Make an upper and a lower estimate using Riemann sums. Represent each estimate on a sketch.
   (b) Use the trapezoidal approximation to make a better estimate. Represent this estimate on a sketch.
   (c) A good typist types at an average speed of at least four characters per second. Use your answer to (b) to decide if we got a good typist. Show your reasoning.

   ANSWER:

   (a)

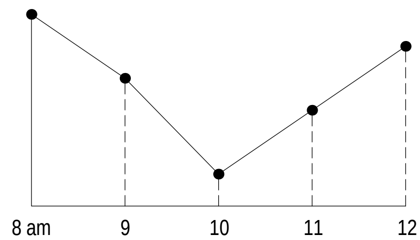


Hour $= 3600$ seconds.

Upper sum $= (6+4+3+5)3600 = 64800$ chars. (dotted rectangle).

Lower sum $= (4+1+1+3)3600 = 32400$ chars.

(b)



$\text{Trapezoid} = \dfrac{\text{Left+Right}}{2} = 48600$ chars.

(c) Average $= \dfrac{48600}{4(3600)} = 3.375$ char/sec doesn't look good. Even using the highest estimate, rate $= \dfrac{64800}{4(3600)} = 4.5$ char/sec, only a bit above 4 char/sec, so the rate is probably not above 4 char/sec. We probably didn't hire a good typist.

9. Consider the following definite integrals:

(a) $\displaystyle\int_0^{\frac{2}{5}} \frac{1}{4+25x^2}dx$

(b) $\displaystyle\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \sin^2\theta\, d\theta$

(i) Evaluate these integrals using the fundamental theorem of calculus. Write your answer both in analytical form, e.g., $\pi + \ln\frac{2}{3} - \arctan\sqrt{5}$, and in decimal form, e.g., 1.58587. . ..

(ii) Obtain a numerical approximation of both integrals correct to two decimal places, and compare the numerical results with the decimal results in part (a). Explain what kind of sum you are using (left sum, trapezoids, etc.), and how many subdivisions were necessary.

ANSWER:

(a) (i) $\displaystyle\int \frac{dx}{4+25x^2} = \frac{1}{25}\int \frac{dx}{\frac{4}{25}+x^2}$. Apply V-24 with $a^2 = \dfrac{4}{25} \Rightarrow a = \dfrac{2}{5}$

$$= \frac{1}{25}\left[\frac{5}{2}\arctan\frac{5}{2}x\right] + C = \frac{1}{10}\arctan\frac{5}{2}x + C. \text{ Thus,}$$

$$\int_C^{\frac{2}{5}} \frac{dx}{4+25x^2} = \frac{1}{10}\arctan\frac{5}{2}x\Big|_0^{\frac{2}{5}} = \frac{1}{10}(\arctan 1 - \arctan 0) = \frac{\pi}{40} \approx 0.0785$$

(ii) Apply IV-57 with $n > 2$:

$$\int \sin^2\theta\, d\theta = -\frac{1}{2}\sin\theta\,\cos\theta + \frac{1}{2}\int d\theta = -\frac{1}{2}\sin\theta\,\cos\theta + \frac{1}{2}\theta + C$$

Thus

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2\theta\, d\theta = \frac{1}{2}\Big[-\sin\theta\,\cos\theta + \theta\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left[-\left(\frac{1}{\sqrt{2}}\right)^2 + \frac{\pi}{4} - \left(-\left(-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) - \frac{\pi}{4}\right)\right]$$

$$= \frac{1}{2}\left[-\frac{1}{2} + \frac{\pi}{4} - \frac{1}{2} + \frac{\pi}{4}\right] = \frac{\pi}{4} - \frac{1}{2} \approx 0.285$$

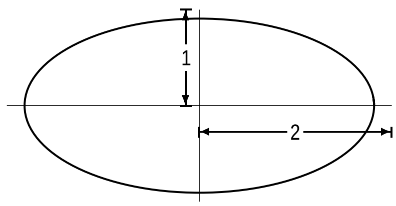
(b) (i) With $N = 20$ subdivisions, the trapezoid and midpoint are both accurate to three decimal places with

$$\overbrace{0.078545}^{\substack{\text{midpoint}\\ \text{overestimate}}} \geq \int_0^{\frac{2}{5}} \frac{dx}{4+25x^2} \geq \overbrace{0.078529}^{\substack{\text{trapezoid}\\ \text{underestimate}}}$$

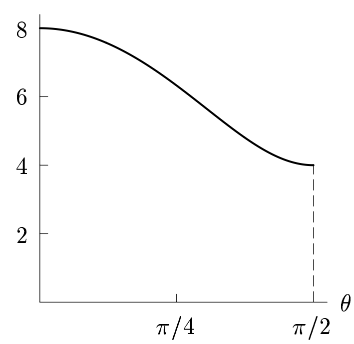
(ii) With $N = 20$ subdivisions, the trapezoid and midpoint sums are correct to two decimal places with

$$\underbrace{0.2864}_{\substack{\text{trapezoid}\\ \text{overestimate}}} \geq \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2\theta\, d\theta \geq \underbrace{0.2849}_{\substack{\text{midpoint}\\ \text{underestimate}}}$$

10. Consider the ellipse pictured below:



The perimeter of the ellipse is given by the following integral: $\int_0^{\frac{\pi}{2}} 8\sqrt{1-\frac{3}{4}\sin^2\theta}\, d\theta$. It turns out that there is no elementary antiderivative for the function $f(\theta) = 8\sqrt{1-\frac{3}{4}\sin^2\theta}$, and so the integral must be evaluated numerically. A graph of the integrand $f(\theta)$ is shown below.



(a) On the graph above, sketch a graphical representation of the *left* Riemann sum approximation of the definite integral with $N = 4$ equal divisions of the interval from $\theta = 0$ to $\theta = \frac{\pi}{2}$. Does it appear that the left sum will overestimate or underestimate the integral?

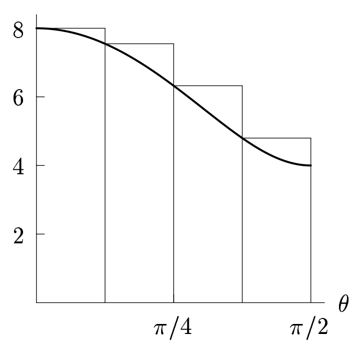
(b) Calculate the left, right, trapezoid, and midpoint sums that approximate the definite integral with $N = 4$ equal divisions of the interval.

(c) Based on part (b), what is your best estimate of the integral, and how many decimal places of accuracy do you think you have?

(d) Look at the picture of the ellipse and decide whether your answer to part (c) seems to represent a reasonable estimate of the perimeter of the ellipse.

ANSWER:

(a)



These rectangles will <u>overestimate</u> the integral.

(b) $x_0 = 0$, $x_1 = \dfrac{\pi}{8}$, $x_2 = \dfrac{\pi}{4}$, $x_3 = \dfrac{3\pi}{8}$, $x_4 = \dfrac{\pi}{2}$, $\Delta x = \dfrac{\pi}{8}$

$$\begin{aligned}\text{Left Sum} &= \sum_{i=0}^{3} f(x_i)\Delta x = (f(x_0) + f(x_1) + f(x_2) + f(x_3)) \cdot \Delta x \\ &= .3927 \cdot (8 + 7.5479 + 6.3246 + 4.7989) \\ &= .3927 \cdot (26.6714) = 10.4738\end{aligned}$$

$$\begin{aligned}\text{Right Sum} &= \sum_{i=1}^{4} f(x_i)\Delta x = (f(x_1) + f(x_2) + f(x_3) + f(x_4)) \cdot \Delta x \\ &= .3927 \cdot (7.5479 + 6.3246 + 4.7986 + 4) \\ &= .3927 \cdot (22.6711) = 8.9029\end{aligned}$$

$$\begin{aligned}\text{Trap Sum} &= \sum_{i=1}^{4} \frac{1}{2} \cdot [f(x_{i-1}) + f(x_i)]\,\Delta x = \text{the average of the left and right sums} \\ &= \frac{1}{2} \cdot (10.4738 + 8.9029) = 9.6884\end{aligned}$$

$$\begin{aligned}\text{Mid Sum} &= \sum_{i=1}^{4} f(z_i)\Delta x, \text{ where } z_i = \frac{1}{2} \cdot (x_i + x_{i=1}) \\ & z_1 = \frac{\pi}{16}, \quad z_2 = \frac{3\pi}{16}, \quad z_3 \frac{5\pi}{16}, \quad z_4 \frac{7\pi}{16} \\ &= (f(z_1) + f(z_2) + f(z_3) + f(z_4)) \cdot \Delta x \\ &= .3927 \cdot (7.88499 + 7.0132 + 5.5512 + 4.2222) \\ &= .3927 \cdot (24.67159) = 9.6885\end{aligned}$$

(c) 9.688. The midpoint and trapezoid sums agree to three decimal places, so there are three decimal places of accuracy.

(d) Yes. Consider the perimeter of circles, which we know how to calculate. A circle of radius 1, which fits inside the ellipse, has perimeter $2 \cdot \pi = 6.28$. A circle of radius 2, inside which the ellipse fits, has perimeter $4 \cdot \pi = 12.57$. So we know that the perimeter of the ellipse must be between these values.

11. It is desired to evaluate the following definite integral:

$$\int_1^3 \sin\left(\frac{1}{x}\right) dx$$

As it turns out, there is no antiderivative to the function $f(x) = \sin\left(\dfrac{1}{x}\right)$, and so it will be necessary to evaluate the integral numerically.

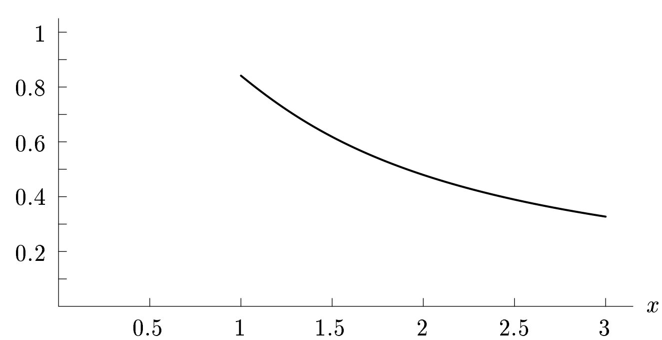
(a) Sketch a graph of $f(x)$ from $x = 1$ to $x = 3$. Be sure the slope and concavity of the curve are clear on your graph.

(b) Calculate the left, right, trapezoid, and midpoint sums that approximate the definite integral with $N = 4$ equal divisions of the interval $1 \leq x \leq 3$. State whether each sum is an overestimate or an underestimate of the integral, with an appropriate explanation.

(c) Based on part (b), what is your best estimate of the integral? How many decimal places of accuracy do you have?

ANSWER:

(a) Using the graphing calculator and plugging in points, we get a decreasing concave-up curve as such:



x	1.0	1.5	2.0	2.5	3.0
$f(x)$	.84	.62	.48	.39	.33

(b) Again, using a numerical integration program, we get
LEFT=1.1643 Overestimate, as $f(x)$ is monotone decreasing in $[1,3]$.
RIGHT=0.9072 Underestimate, as $f(x)$ is decreasing in $[1,3]$.
TRAP=1.0358 Overestimate, as $f(x)$ is concave up in $[1,3]$.
MID=1.0219 Underestimate, as $f(x)$ is concave up in $[1,3]$.

(c) The best estimate is that $1.0219 < \int_1^3 \sin\left(\frac{1}{x}\right) dx < 1.0358$. Using the weighted average, we get 1.0265. There is only 1 decimal place of accuracy (as TRAP and MID agree to only 1 decimal place).

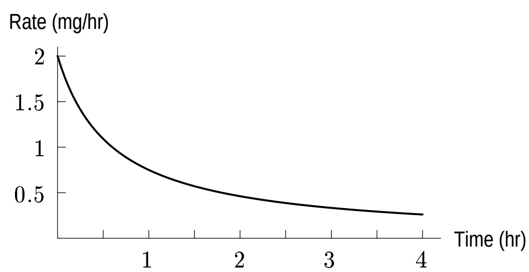
12. A drug is being administered intravenously to a patient at a constant rate of 2 mg/hr. The following table shows the rate of change of the amount of the drug in the patient's body at half-hour intervals. Initially there is none of the drug in the patient's body.

Time (hours)	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
Rate (mg/hr)	2.00	1.09	0.75	0.57	0.46	0.39	0.33	0.29	0.26

(a) Sketch a graph of the rate of change of the amount of drug in the patient's body as a function of time. Draw a smooth curve through the data points.
(b) Give a description of what is happening in words. In particular, explain why the rate of change of the amount of the drug is not simply a constant $+2$ mg/hr, which is the rate at which the drug is being administered.
(c) Sketch a graph of the total amount of the drug in the person's body as a function of time. You should be able to make this sketch just from your graph of the rate in part (a).
(d) What is your best estimate of the total amount of the drug in the patient's body after four hours? You may do this by any means of your choosing, but explain your method.

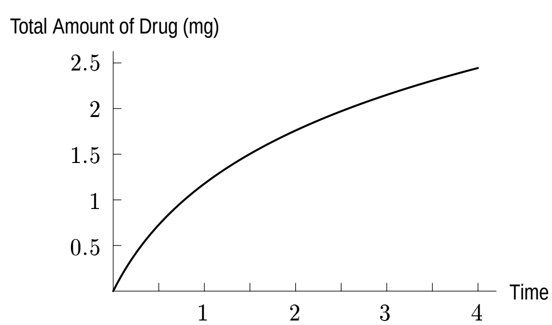
ANSWER:

(a)



(b) The rate of change of the amount of drug in the patient's body is decreasing as time increases. This could be due to the drug saturating the patient's system or changed to another form as time increases.
(c) Using trapezoids to calculate area under the rate curve, we get the graph below.

x	$f(x)$
0.0	0
0.5	0.77
1.0	1.23
1.5	1.56
2.0	1.82
2.5	2.03
3.0	2.21
3.5	2.37
4.0	2.51



(d) Calculating the total area under the rate curve, we get
LHS $\approx$ 2.94
RHS $\approx$ 2.07
TRAP $\approx$ 2.51

13. (a) Using two subdivisions, find the left, right, trapezoid and midpoint approximations to $\int_0^1 (1 - e^{-x})\, dx$.
(b) Draw sketches showing what each approximation represents.
(c) Given only the information you found in (a), what is your best estimate for the value of this integral?

ANSWER:

(a) To find $\int_0^1 (1 - e^{-x})\, dx$, try left-hand sums using two subdivisions. Let $f(x) = 1 - e^{-x}$. At $x = 0$, $f(0) = 0$ and at $x = 0.5$, $f(0.5) = 1 - e^{-0.5} \approx 0.3935$. So

$$\text{LEFT}(2) = 0.5(0 + 0.3935) = 0.1968.$$

Try right-hand sums, using two subdivisions.

At $x = 0.5$ $f(0.5) \approx 0.3935$ and at $x = 1$, $f(1) = 1 - e^{-1} \approx 0.6321$. So

$$\text{RIGHT}(2) = 0.5(0.3935 + 0.6321) = 0.5128.$$

Try midpoint sums, using two subdivisions.

At $x = 0.25$, $f(0.25) = 1 - e^{-0.25} \approx 0.2212$ and at $x = 0.75$, $f(0.75) = 1 - e^{-0.75} \approx 0.5726$. So

$$\text{MID}(2) = 0.5(0.2212 + 0.5276) = 0.3744.$$

Find the trapezoidal sum:

$$\text{TRAP}(2) = \frac{\text{LEFT}(2) + \text{RIGHT}(2)}{2} = \frac{0.1968 + 0.5128}{2} = 0.3548.$$

(b)

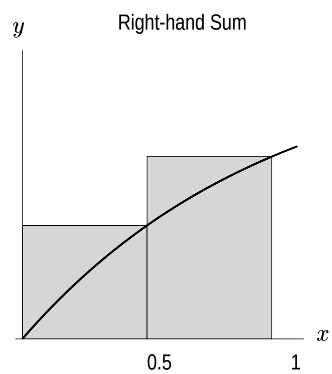


Figure 7.5.85

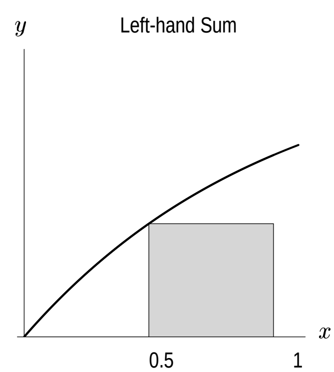


Figure 7.5.86

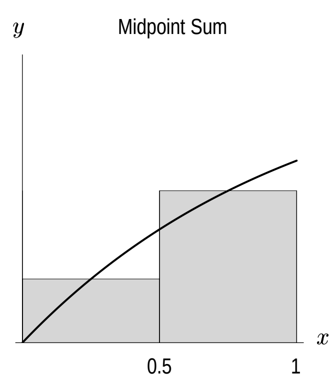


Figure 7.5.87

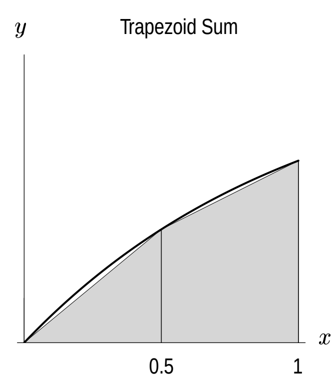


Figure 7.5.88

(c) Since the curve $1 - e^{-x}$ is concave down, the trapezoid rule must give an underestimate. By the same reasoning, the midpoint sum must provide an overestimate. A good estimate of the true area under the curve might be given by the average of these two sums, namely 0.3646.

14. Let $S(t)$ be the number of daylight hours, in Cambridge, MA, on the t^{th} day of the year. During spring (from the vernal equinox, $t = 80$, to the summer solstice, $t = 173$), the graph of $S(t)$ is concave down. In the table below we list some values of $S(t)$. What is the average length of the days in spring (in hours and minutes)? Please give an upper bound and a lower bound for the answer. These bounds should differ by less than 10 minutes.

t	$S(t)$
80	12 hours 12 minutes
111	13 hours 44 minutes
142	14 hours 50 minutes
173	15 hours 12 minutes

ANSWER:

First, we convert all values of $S(t)$ into minutes for simplicity. By inspecting the values in the table, it is clear that left- and right-hand sums will not produce the needed accuracy So, to find the understimate, we approximate $\int S(t)\,dt$ by $\text{TRAP}(3)$:

$$\begin{aligned}\int S(t)\,dt &\approx \frac{1}{2}\left(\text{LEFT}(3) + \text{RIGHT}(3)\right)\\ &= \frac{1}{2}\left(732 + 824 + 89 + 824 + 890 + 912\right)\cdot 31\\ &= 78616\end{aligned}$$

So $\overline{S(t)} = \dfrac{78616}{173-80} \approx 845.3$ minutes.

The overestimate is a bit more tricky. If we approximate $\int S(t)\,dt$ by $\text{RIGHT}(3)$, we get $\overline{S(t)} = (824 + 890 + 912)\frac{31}{93} \approx 875.5$ minutes, but then the bounds are not within 10 minutes of each other. We would try $\text{MID}(3)$, but we don't know the values for $S(\frac{80+111}{2})$, $S(\frac{111+142}{2})$, or $S(\frac{142+173}{2})$. The trick is to estimate $\int S(t)\,dt$ with $\text{MID}(1)$ between $t = 80$ and $t = 142$ $+\text{RIGHT}(1)$ between $t = 142$ and $t = 173$. We get: $\overline{S(t)} = \frac{1}{3}(824 \times 2 + 912) \approx 854$ minutes, rounding up. Therefore, our answer is 845 minutes $< \overline{S(t)} <$ 854 minutes; 14 hours 5 minutes $< \overline{S(t)} <$ 14 hours 14 minutes.

15. Give an upper bound U and a lower bound L for

$$\int_{-0.5}^{0.3} \frac{1}{1+x^4}\,dx$$

such that $U - L < 10^{-3}$. Explain your procedure.

ANSWER:

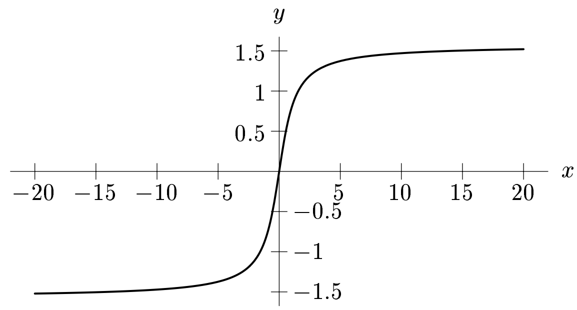
Since, by inspecting the graph of $\dfrac{1}{1+x^4}$, we can tell that the function is concave down over $[-0.5, 0.3]$, $\text{TRAP}(n)$ will be an underestimate (a lower bound), and $\text{MID}(n)$ will be an overestimate (an upper bound).

$$\begin{aligned}\text{TRAP}(50) &\approx 0.98789\\ \text{MID}(50) &\approx 0.98793\end{aligned}$$

So, $0.98789 \le \displaystyle\int_{-0.5}^{0.3} \frac{1}{1+x^4}\,dx \le 0.98793$.

The difference between upper and lower bounds is $\approx 0.00004 < 0.001$.

16. Below is the graph of $y = \arctan x$:



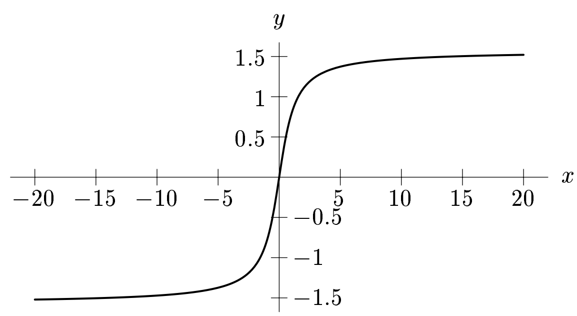
(Note that arctan is the $\boxed{\tan^{-1}}$ button on your calculator.)

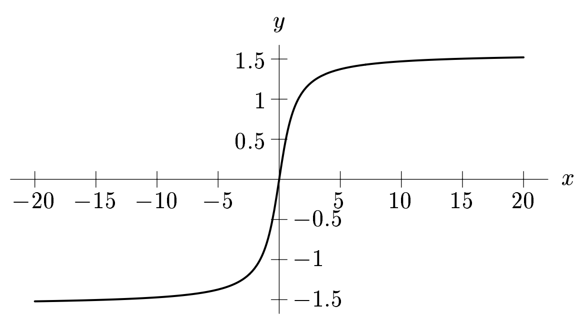
(a) For any number of subdivisions N write an inequality using $\text{RIGHT}(N)$, $\text{LEFT}(N)$ and $\int_{-10}^{16} \arctan x\, dx$. Explain.

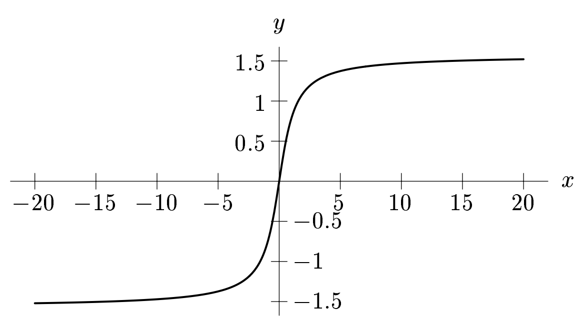
(b) In computing $\int_{-10}^{16} \arctan x\, dx$ using left- and right-hand sums we record the following table:

N	$\text{LEFT}(N)$	$\text{RIGHT}(N)$
2	-2.877	35.8465
3	-8.2473	17.474
4	-0.3163	19.0505

Why do you think there are such wide variations in this table? You might want to illustrate your reasoning using the graphs below. Be sure to distinguish between the variation for different N and the difference between left- and right-hand sums for a particular N.







(c) The point $x = 0$ is an inflection point for $y = \arctan x$. If we write

$$\int_{-10}^{16} \arctan x\, dx = \int_{-10}^{0} \arctan x\, dx + \int_{0}^{16} \arctan x\, dx,$$

then we can get over-estimates and under-estimates for the integrals on the right via the midpoint and trapezoidal rules. Explain.

(d) We have the following data:

For $\displaystyle\int_{-10}^{0} \arctan x\, dx$

N	Midpoint	Trapezoid
50	-12.40537	-12.40041

For $\displaystyle\int_{0}^{16} \arctan x\, dx$

N	Midpoint	Trapezoid
50	21.363787	21.35097

Based on your answer to part (c) find numbers A and B such that

$$A \le \int_{-10}^{16} \arctan x\, dx \le B.$$

Explain. How big is $B - A$?

(e) Evaluate $\int_{-10}^{16} \arctan x\, dx$ "symbolically" (plug in the limits but don't evaluate).

Hint: Integrate by parts using $\dfrac{d}{dx} \arctan x = \dfrac{1}{1+x^2}$.

ANSWER:

(a) Since $\arctan x$ is an increasing function, $\text{LEFT}(N) < \displaystyle\int_{-10}^{16} \arctan x\, dx < \text{RIGHT}(N)$ for any number of divisions N.

(b) The general reason for the wide variation in values in the table, for different N, is that $\arctan x$ is steep only near $x = 0$, and is relatively flat further out. A small number of rectangles (like $2, 3$, or 4) will necessarily obscure this behavior, either underestimating or overestimating it.

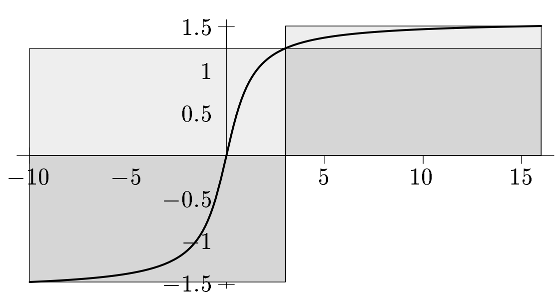


Figure 7.5.89: $N = 2$

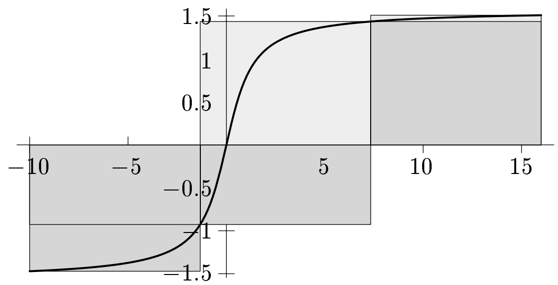


Figure 7.5.90: $N = 3$

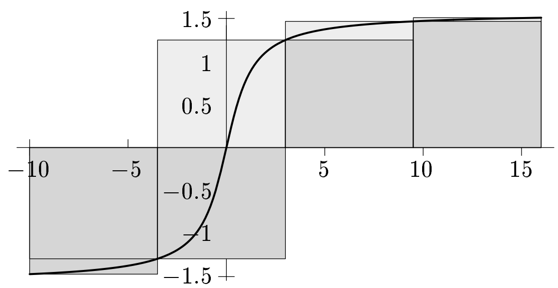


Figure 7.5.91: $N = 4$

With $N = 2$, for example, a right-hand sum completely ignores the behavior of the function for x to the left of 3. But at $x = 3$, $\arctan x$ is already beyond the steep increase at $x = 0$! Similar problems occur for $N = 3$ and $N = 4$. As for the difference between left- and right-hand sums for a particular N, consider the following:

$$\text{RIGHT}(N) - \text{LEFT}(N) = ((f(b) - f(a))\frac{b-a}{N}$$

but in this case, we have

$$\begin{aligned}\text{RIGHT}(N) - \text{LEFT}(N) &= (1.51 - (-1.47)) \cdot \frac{26}{N} \\ &\approx \frac{77.5}{N}.\end{aligned}$$

For small N, this will be very large.

(c) On $[-10, 0]$, $\arctan x$ is concave up, so

$$\text{MID}(N) < \int_{-10}^{0} \arctan x\, dx < \text{TRAP}(N) \text{ for any } N.$$

On $[0, 16]$, $\arctan x$ is concave down, so

$$\text{TRAP}(N) < \int_{-10}^{0} \arctan x\, dx < \text{MID}(N) \text{ for any } N.$$

So for an underestimate, we can use $\text{MID}(N)$ on $[-10, 0]$ and $\text{TRAP}(N)$ on $[0, 16]$. Vice versa for an overestimate.

(d) According to the previous part, we have

$$\begin{aligned}\text{A} &= \text{underestimate} = -12.40537 + 21.35097 = 8.9456 \\ \text{B} &= \text{overestimate} = -12.40041 + 21.363787 = 8.96338 \\ \text{B} - \text{A} &= 0.01778\end{aligned}$$

.

(e) Integrating by parts with $u = \arctan x$ and $v' = 1$ we get

$$\int_{-10}^{16} \arctan x\, dx = x \arctan x \Big|_{-10}^{16} - \int_{-10}^{16} \frac{x}{1+x^2}\, dx,$$

since $v = x$ and $u' = \dfrac{dx}{1+x^2}$. Notice that $\displaystyle\int \frac{x}{1+x^2}\, dx = \frac{1}{2}\ln(1+x^2) + C$. Therefore,

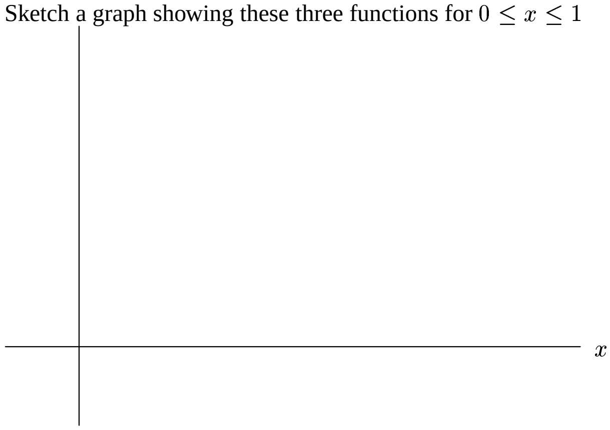
$$\begin{aligned}\int_{-10}^{16} \arctan x\, dx &= x \arctan x \Big|_{-10}^{16} - \frac{1}{2}\ln(1+x^2)\Big|_{10}^{16} \\ &= 16 \arctan 16 + 10 \arctan(-10) - \frac{1}{2}\ln(1+16^2) + \frac{1}{2}\ln(1+10^2) \\ &\approx 8.956\end{aligned}$$

.

17. Consider the functions

$$f_k(x) = (1-x)^{1/k} \quad \text{for} \quad k = 1, 2, 3.$$

(a) Sketch a graph showing these three functions for $0 \leq x \leq 1$



Consider the integrals

$$I_k = \int_0^1 (1-x^k)^{1/k} dx \quad k = 1, 2, 3.$$

(b) Suppose you use the midpoint and trapezoid rules to approximate each of these integrals. For each integral, does each rule give an overestimate, an underestimate, the exact value, or is it impossible to tell without knowing n, the number of subdivisions? (Circle one answer for each integral and give a very brief reason.)

I_1 Midpt: Over Under Exact Can't Tell

Why?

Trap: Over Under Exact Can't Tell

Why?

I_2 Midpt: Over Under Exact Can't Tell

Why?

Trap: Over Under Exact Can't Tell

Why?

I_3 Midpt: Over Under Exact Can't Tell

Why?

Trap: Over Under Exact Can't Tell

Why?

ANSWER:

(a) The functions are shown in Figure 7.5.92.

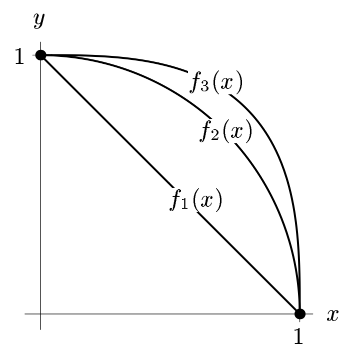


Figure 7.5.92

(b) Mid(n) and Trap(n) give exact values of I_1 since, in the case of Trap(n) it is easy to see that if n is even as low as 1 the trapezoid of the trapezoid rule is exactly the triangle whose area we want. In the case of Mid(n), if n is even as low as one, the trapezoid whose oblique side is tangent to the "curve" (and whose area equals that of the midpoint rectangle) fits the triangular area exactly since the tangents of straight lines are the the lines themselves.
For I_2 *and* I_3 we conclude that Trap(n) underestimates and Mid(n) overestimates as both regions are bounded above by curves which are concave down.

18. Suppose the points $x_0, x_1, ..., x_n$ are equally spaced and $a = x_0 < x_1 < x_2 < ... < x_n = b$. Give a formula (in terms of f and the x_i's) for each of the following approximations to $\int_a^b f(x)dx$. (No justification needed.)

(a) The left Riemann sum approximation.
(b) The right Riemann sum approximation.
(c) The trapezoid approximation.
(d) The midpoint approximation.

ANSWER:

(a) The left Riemann sum approximation uses the value of f at the left of each interval:

$$\text{Left}(n) = \sum_{i=0}^{n-1} f(x_i)\Delta x.$$

(b) The right Riemann sum uses the value of f at the right of each interval:

$$\text{Right}(n) = \sum_{i=1}^{n} f(x_i)\Delta x.$$

(c) The trapezoid approximation is the average of the left and right Riemann approximations:

$$\text{Trap}(n) = \frac{1}{2}\left(\text{Left}(n) + \text{Right}(n)\right)$$

$$= \frac{1}{2}\left(\sum_{i=0}^{n-1} f(x_i)\Delta x + \sum_{i=1}^{n} f(x_i)\Delta x\right)$$
$$= \sum_{i=0}^{n-1}\left(\frac{f(x_i)+f(x_{i+1})}{2}\right)\Delta x.$$

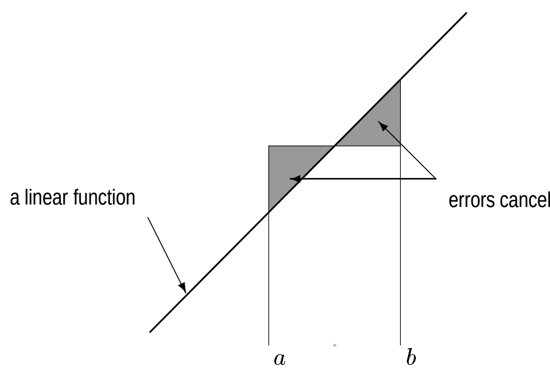
(d) The midpoint approximation uses the value of f evaluated the the midpoint of each interval:

$$\text{Mid}(n) = \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right)\Delta x.$$

19. TRUE/FALSE: For each statement, write whether it is true or false and give a short explanation.
(a) For any given function, $\text{TRAP}(n)$ is always *more* accurate than $\text{LEFT}(n)$.
(b) The midpoint rule gives *exact* answers for linear functions, no matter how many subdivisions are used.
(c) $\displaystyle\int_0^3 \cos^{36}(x)\,dx > \pi.$

ANSWER:

(a) FALSE. A simple counterexample is a constant function, say $f(x) = 5$, for which $\text{LEFT}(n)$ is just as accurate as $\text{TRAP}(n)$, since both are exact. For other functions, $\text{LEFT}(n)$ can even be more accurate than $\text{TRAP}(n)$. (See if you can come up with an example of such function.)
(b) TRUE. Using $\text{MID}(1)$ on any interval $[a,b]$ of a linear function overshoots and undershoots the exact area equally so there is exact cancellation. Consequently, for N larger than 1, $\text{MID}(N)$ gives an exact answer on the whole interval $[a,b]$ because on each subinterval it acts as $\text{MID}(1)$, which we know gives an exact answer.



(c) FALSE. Since $-1 \le \cos x \le 1$ everywhere, $\cos^{36}(x) \le 1$ everywhere, so $\int_0^3 \cos^{36}(x)\,dx \le \int_0^3 1\,dx = 3$.

20. In the field of dynamical astronomy, integrals such as the following appear in the problem of determining trajectories of planets or spacecraft:

$$\int_0^1 \frac{1}{1 + c\cos\theta}\,d\theta$$

In this definite integral, c is a constant that can go from 0 to $+\infty$ depending on the shape of the orbit. Except for some special values of c, there is no fundamental antiderivative for the integrand

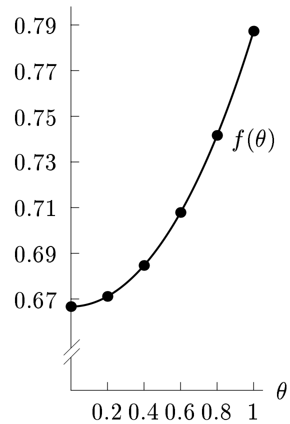
$$f(\theta) = \frac{1}{1 + c\cos\theta}$$

(a) Sketch a graph of $f(\theta)$ with $c = 0.5$ for $0 \le \theta \le 1$.
(b) Calculate the left, right, trapezoid, and midpoint sums that approximate the definite integral for $c = 0.5$ with $N = 5$ equal divisions of the interval $0 \le \theta \le 1$. State whether each sum is an overestimate or an underestimate of the integral, with an appropriate explanation.
(c) Based on part (b), what is your best estimate of the integral? How many decimal places of accuracy do you have?

ANSWER:

(a) For $c = 0.5$, $f(\theta) = \dfrac{1}{1 + (0.5)\cos\theta}$. To graph this function, we plot a few points:

θ	0	0.2	0.4	0.6	0.8	1.0
$f(\theta)$	0.6667	0.6711	0.6847	0.7079	0.7416	0.7873



From this graph we can conclude the function is increasing and concave up.

(b) For $N = 5$ equal divisions over the interval $0 \le \theta \le 1$; $x_0 = 0$, $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$, $x_4 = 0.8$, $x_5 = 1.0$ and $\Delta x = 0.2$. The values of $f(x)$ were calculated in part (a).

$$\text{LEFT} = \sum_{i=0}^{4} f(x_i)\Delta x$$
$$= [f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_n)]\,\Delta x$$
$$= [0.6667 + 0.6711 + 0.6847 + 0.7079 + 0.7416]\,(0.2)$$
$$= 0.6944$$

Because the function is an increasing function, we can conclude that the left Riemann sum is an underestimate.

$$\text{RIGHT} = \sum_{i=1}^{2} f(x_i)\Delta x$$
$$= [f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)]\Delta x$$

$$= [0.6711 + 0.6847 + 0.7079 + 0.7416 + 0.7873](0.2)$$
$$= 0.7185$$

Because the function is an increasing function, the right hand side is an overestimate.

TRAP=Average of left and right sums

$$= \frac{RHS + LHS}{2} = \frac{0.6944 + 0.7185}{2} = 0.7065$$

Because the function is concave up, the trapezoid sum will give an overestimate.

$$\text{MID}=\sum_{i=0}^{4} f\left(x_i + \frac{1}{2}\Delta x\right)\Delta x$$
$$= [f(0.1) + f(0.3) + f(0.5) + f(0.7) + f(0.9)]\,\Delta x$$
$$= [0.6678 + 0.6767 + 0.6950 + 0.7234 + 0.7629]\,(0.2)$$
$$= 0.7052$$

Because the function is concave up, a MID will give us an underestimate.

(c) Because we know that

$$\text{MID} \le \int_0^2 f(x) \le \text{TRAP}$$

$$\text{so } 0.7052 \le \int_0^2 f(x) \le 0.7065.$$

The best estimate we can make is one between these two numbers, say 0.706. This is accurate to 2 decimal places.

Questions and Solutions for Section 7.6

1. (a) Use the trapezoid rule with $n = 4$ to approximate

$$\int_0^1 \sqrt{1+e^{-x}}\,dx.$$

(b) Do the same using Simpson's rule with $n = 4$.

(c) Indicate how much more accurate you would expect the results to be if you used $n = 40$ in each case.

ANSWER:

(a)

x	$f(x)$
0.00	1.41421
0.25	1.33372
0.50	1.26749
0.75	1.21341
1.00	1.16956

$$\text{LEFT}(4) = 1.30721$$
$$\text{RIGHT}(4) = 1.24605$$
$$\text{TRAP}(4) = 1.27663$$

(b)

x	$f(x)$
0.125	1.37204
0.375	1.29896
0.625	1.23906
0.875	1.19032

Using the table to the left, we get

$$\text{MID}(4) = 1.27509$$
$$\text{SIMP}(4) = \frac{1}{3}\left(2 \times \text{MID}(4) + \text{TRAP}(4)\right) = 1.27560.$$

(c) Since the trapezoid rule has error proportional to $\frac{1}{n^2}$, using $n = 40$ will be $\frac{\frac{1}{4^2}}{\frac{1}{40^2}} = 100$ times as accurate as using $n = 4$. Since the Simpson's rule has error proportional to $\frac{1}{n^4}$, using $n = 40$ will be $\frac{\frac{1}{4^4}}{\frac{1}{40^4}} = 10{,}000$ times as accurate as using $n = 4$.

2. The table below contains numerical data for a definite integral approximated by the left-hand, midpoint, trapezoid, and Simpson's rule methods. Which column is which? Why?

N				
1	2.4737	−44.1930	0.4737	−67.5263
3	−39.6662	−45.5098	−40.3329	−48.4316
9	−44.8687	−45.5261	−45.0909	−45.8548
27	−45.45315	−45.52630	−45.62722	−45.5629
81	−45.518171	−45.526300	−45.572862	−45.530364

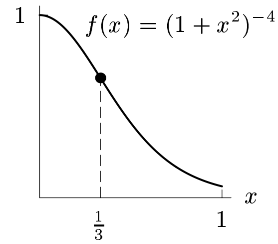
ANSWER:

The second column consistently gives answers very close to the true value (which appears to be around 45.53) so it is likely to be the Simpson's rule column. We recall that $\text{SIMP}(N) = \frac{1}{3}(2\,\text{MID}(N) + \text{TRAP}(N))$. Noting that 2^{nd} column $= \frac{1}{3}(2 \times 4^{\text{th}}$ column $+ 1^{\text{st}}$ column), we conclude that $\text{MID}(N)$ is the 4th column and $\text{TRAP}(N)$ is the 1st column. The 3rd column is therefore the $\text{LEFT}(N)$ column.

3. Compute to within 0.001 the following integral: $\int_0^1 (1+x^2)^{-4}dx$. You may use a graphing calculator or a copy of the integral tables. Please justify all of your steps.

ANSWER:

First look at the function $(1+x^2)^{-4}$:



Using Left and Right sums: Left-sum overestimates and right-sum underestimates the integral, as the function is monotonic and decreasing. The number of steps we should take, N, should verify

$|\text{Left}(N) - \text{Right}(N)| = |f(0) - f(1)|\frac{1}{N} = \left|1 - \frac{1}{1.6}\right|\frac{1}{N} < 0.001$

so $N \geq 938$ steps. Then $\begin{cases} LHS \cong 0.47413 & \text{and} \\ RHS \cong 0.47507 \end{cases}$.

The integral $\simeq$ 0.475 up to 0.001.

Using Trap and Midpoint: By concavity we must use TRAP on $[0, 1/3]$ and Mid on $[1/3, 1]$ to underestimate. Use Mid on $[0, 1/3]$ and Trap on $[1/3, 1]$ to overestimate. Use approximately 100 steps to estimate them, then sum-up to get 0.474588 $<$ integral $<$ 0.474617, well within 10^{-3}.

(C) It is possible to compute the integral precisely:

$I = \int_0^1 \frac{dx}{(1+x^2)^4}$.

Put $x = \tan\theta$; then $dx = \frac{d\tan\theta}{d\theta}d\theta = \frac{1}{\cos^2\theta}d\,\theta$. If $x = 0$, $\theta = 0$; if $x = 1$, $\theta = \frac{\pi}{4}$, so substitute and get

$I = \int_0^{\pi/4}(1+\tan^2\theta)^{-4}\cos^{-2}\theta\,d\theta = \int_0^{\pi/4}\cos^6\theta\,d\theta$

Use Table #18 to write it as (with $m = 6$)

$$
\begin{aligned}
I &= \frac{1}{6}\cos^6\theta\sin\theta\Big|_0^{\frac{\pi}{4}} + \frac{5}{6}\int_0^{\frac{\pi}{4}}\cos^4\theta\,d\theta \\
&= \frac{1}{6}\left(\frac{1}{\sqrt{2}}\right)^6 + \frac{5}{6}\left[\frac{1}{4}\cos^3\theta\sin\theta\Big|_0^{\frac{\pi}{4}} + \frac{3}{4}\int_0^{\frac{\pi}{4}}\cos^2\theta\,d\theta\right] \\
&= \frac{1}{48} + \frac{5}{24}\frac{1}{(\sqrt{2})^4} + \frac{15}{24}\left[\frac{1}{2}\cos\theta\sin\theta\Big|_0^{\frac{\pi}{4}} + \frac{1}{2}\cdot\frac{\pi}{4}\right] \\
&= \frac{1}{48} + \frac{5}{24}\frac{1}{4} + \frac{15}{96} + \frac{15}{192}\pi = \underbrace{0.4746035}_{\text{true value}}
\end{aligned}
$$

4. Compute the following integrals. If you provide an approximation, it should be rounded to three decimal places, and you must explain how you got it and how you know it has the desired accuracy.

(a) $\int_2^4 (1+\ln(x))^{-1}\,dx$

(b) $\int_0^4 x\cdot(9+x^2)^{1/2}\,dx$

ANSWER:

(a) $\int_2^4 (1+\ln x)^{-1}\,dx$.

Method 1. (With Riemann sum.)

With 100 divisions, Left= 0.97256945, Right= 0.96913832.

With 200 divisions, Left= 0.9717084, Right= 0.9699928.

With 1000 divisions, Left= 0.97102, Right= 0.97067.

Therefore the answer is 0.971. We know that this is the desired answer because it is approached from both left and right.

Method 2. (With a calculator.)

$$\int_2^4 (1+\ln x)^{-1}\,dx = 0.9708.$$

(b) $\int_0^4 x\cdot(9+x^2)^{1/2}\,dx$.

Method 1. (with a calculator) The answer is 32.667.

Method 2. (substitution)

Let $w = 9+x^2$.

Then $dw = 2x\cdot dx$.

Change the limits: when $x=0, w=9$ and when $x=4, w=25$.

$$\begin{aligned}\int_0^4 x\cdot(9+x^2)^{1/2}\,dx &= \int_9^{25} w^{1/2}\,\frac{dw}{2}\\ &= \left[\frac{1}{2}\,\frac{w^{3/2}}{3/2}\right]_9^{25}\\ &= \frac{1}{3}[25^{3/2}-9^{3/2}]\\ &= \frac{1}{3}[125-27]\\ &= 32.667.\end{aligned}$$

Method 3. (with Riemann sums)

Trap: 32.6673

Mid: 32.666

Simp: 32.66666

Therefore the answer is 32.667. We know that this has the desired accuracy because the same value is approached by different methods.

5. (a) What is the exact value of $\displaystyle\int_0^6 e^{2x}\,dx$?
(b) Find LEFT(2), RIGHT(2), TRAP(2), MID(2), and SIMP(2).
(c) Compute the error for each approximation.

ANSWER:

(a)

$$\int_0^6 e^{2x}\,dx = \left.\frac{e^{2x}}{2}\right|_0^6 = \frac{e^{12}}{2} - \frac{e^0}{2} = 81,376.9$$

(b) Computing the sums directly, since $\Delta x = 3$, we have

$$\begin{aligned}&\mathrm{LEFT}(2) = 3\cdot e^0 + 3\cdot e^6 = 1213.29\\ &\mathrm{RIGHT}(2) = 3\cdot e^6 + 3\cdot e^{12} = 489,475\\ &\mathrm{TRAP}(2) = \frac{1213.29 + 489,475}{2} = 245,344\\ &\mathrm{MID}(2) = 3\cdot e^3 + 3\cdot e^9 = 24,369.5\\ &\mathrm{SIMP}(2) = \frac{2\cdot\mathrm{MID}(2) + \mathrm{TRAP}(2)}{3} = 98027.7\end{aligned}$$

(c) Errors

$$\mathrm{LEFT}(2) : 80,163.6$$

$$\begin{aligned}&\text{RIGHT}(2): -408,098\\&\text{TRAP}(2): -163,967\\&\text{MID}(2): 57,007.4\\&\text{SIMP}(2): -16,650.8\end{aligned}$$

Questions and Solutions for Section 7.7

1. (a) Does the integral $\displaystyle\int_1^\infty \frac{x^2}{e^{-x}+x}\,dx$ converge? Why or why not?
   (b) If the following improper integral converges, find its value. Otherwise explain why it does not converge.

$$\int_0^\infty x^2e^{-x}\,dx$$

   ANSWER:

   (a) The integral does not converge, because the integrand does not approach zero as x goes to infinity. This is because $e^{-x}\to 0$ as $x\to\infty$, and the integrand behaves like $\dfrac{x^2}{x}=x$, which grows without bound as $x\to\infty$.
   (b) We need to integrate by parts twice to determine the indefinite integral:

$$\begin{aligned}\int x^2e^{-x}\,dx &= -x^2e^{-x}+2\int xe^{-x}\,dx\\&= -x^2e^{-x}+2\left(-xe^{-x}+\int e^{-x}\,dx\right)\\&= -x^2e^{-x}+2\left(-xe^{-x}-e^{-x}\right)\\&= -e^{-x}\left(x^2+2x+2\right)+C.\end{aligned}$$

   Then,

$$\begin{aligned}\int_0^\infty x^2e^{-x}\,dx &= \lim_{a\to\infty}\int_0^a x^2e^{-x}\,dx\\&= \lim_{a\to\infty} -e^{-x}(x^2+2x+2)\Big|_0^a\\&= \lim_{a\to\infty}\left(-e^{-a}(a^2+2a+2)+2\right).\end{aligned}$$

   As $a\to\infty$, e^{-a} goes to 0 faster than the polynomial a^2+2a+2 grows. So

$$\lim_{a\to\infty}\left(-e^{-a}(a^2+2a+2)+2\right)=\lim_{a\to\infty}2=2.$$

   The integral thus converges to a value of 2.

2. Does the following improper integral converge? If so, find its value. $\displaystyle\int_0^1\frac{1}{x^{\frac{1991}{1992}}}dx$

   ANSWER:

$$\begin{aligned}&\int_0^1 x^{-\frac{1991}{1992}}dx=\lim_{b\to0}\int_b^1 x^{-\frac{1991}{1992}}\,dx\\&=\lim_{b\to0}1992x^{\frac{1}{1992}}\Big|_b^1\\&=\lim_{b\to0}\left(1992-b^{\frac{1}{1992}}\right)\\&=1992\end{aligned}$$

   The integral converges to 1992.

3. If the following improper integral converges, then give its value correct to three decimal places. You may do this by any means of your choosing, but show your work clearly and explain your steps. $\int_0^2 \frac{1}{\sqrt{1-x}}dx$

   ANSWER:
   Let $w = 1 - x \Rightarrow dw = -dx$.

$$\int_0^1 \frac{1}{\sqrt{1-x}}dx$$
$$= -\int_1^0 w^{-\frac{1}{2}}dw = \int_0^1 w^{-\frac{1}{2}}dw = \lim_{b\to 0+}\int_b^1 w^{-\frac{1}{2}}dw$$
$$= \lim_{b\to 0+}\left[2\sqrt{w}\right]_b^1 = \lim_{b\to 0+}\left[2-2\sqrt{b}\right] = 2$$

   The integral converges, and it equals 2.000.

4. Find the value of the following improper integrals, or, if an integral does not converge, say so explicitly.
   (a) $\int_{-1}^0 \frac{1}{\sqrt{1+x}}dx$
   (b) $\int_0^\infty \frac{1}{\sqrt{1+x}}dx$

   ANSWER:
   (a) This integral is improper because the integrand is undefined at $x = -1$.

$$\int \frac{1}{\sqrt{1+x}}dx = 2\sqrt{1+x} + C,$$

   so $\int_{-1}^0 \frac{1}{\sqrt{1+x}}dx = \lim_{a\to -1^+}\left[2\sqrt{1+x}\right]_a^0 = \lim_{a\to -1^+}\left[2-2\sqrt{1+a}\right] = 2.$

   The integral converges to 2.

   (b) $\int_0^\infty \frac{1}{\sqrt{1+x}}dx = \lim_{b\to\infty}\left[2\sqrt{1+x}\right]_0^b = \lim_{b\to\infty}\left[2\sqrt{1+b}-2\right]$

   The integral diverges.

5. Do the following integrals converge? If so, calculate the value. If not, explain why not.
   (a) $\int_0^2 \frac{dx}{(x-1)^2}$
   (b) $\int_0^\infty xe^{-x}\,dx$

   ANSWER:
   (a) $\frac{1}{(x-1)^2}$ misbehaves at $x = 1$, as shown in Figure 7.7.93:

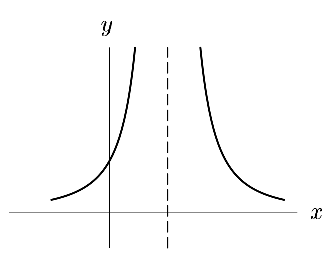


Figure 7.7.93

So

$$\int_0^2 \frac{1}{(x-1)^2}dx = \lim_{a\to 1^-}\int_0^a \frac{dx}{(x-1)^2} + \lim_{b\to 1^+}\int_b^2 \frac{dx}{(x-1)^2}$$

$$= \lim_{a\to 1^-}\left[-\left(\frac{1}{x-1}\right)\Bigg|_0^a\right] + \lim_{b\to 1^+}\left[-\left(\frac{1}{x-1}\right)\Bigg|_b^2\right]$$
$$= \lim_{a\to 1^-}\left[-\frac{1}{a-1}-1\right] + \lim_{b\to 1^+}\left[-\frac{1}{2}+\frac{1}{b-1}\right].$$

Neither limit is defined so the integral diverges.

(b)
$$\int_0^\infty xe^{-x}dx = \lim_{b\to\infty}\int_0^b xe^{-x}dx.$$

Using integration by parts with $u = x$, $u' = 1$, $v' = e^{-x}$, $v = -e^{-x}$ we have:

$$\lim_{b\to\infty}\int_0^b xe^{-x}dx = \lim_{b\to\infty}\left(-be^{-b}+0\right) + \lim_{b\to\infty}\int_0^b e^{-x}dx.$$

Since the left limit is equal to zero,

$$\int_0^\infty xe^{-x}dx = \lim_{b\to\infty}\int_0^b e^{-x}dx = \lim_{b\to\infty}\left[-e^{-x}\Bigg|_0^b\right] = \lim_{b\to\infty}\left[-\frac{1}{e^b}+1\right] = 1.$$

So the integral converges.

6. Doe the following integrals converge or diverge? If an integral converges, give its exact value. Justify your answer using antiderivatives.

(a) $\displaystyle\int_0^\infty xe^{-x}dx$

(b) $\displaystyle\int_0^2 \frac{dx}{(x-1)^{4/3}}$

ANSWER:

(a) Applying the definition of an improper integral,

$$\int_0^\infty xe^{-x}dx = \lim_{b\to\infty}\int_0^b xe^{-x}dx$$

Using integration by parts with $u = x$, $v' = e^{-x}$, $u' = 1$, $v = 1$, we have

$$\lim_{b\to\infty}\int_0^b xe^{-x}dx = \lim_{b\to\infty} -xe^{-x}\Bigg|_0^b - e^{-x}\Bigg|_0^b$$
$$= \lim_{b\to\infty}\left[-be^{-b}+0e^0\right] - \left[e^{-b}-e^0\right] = 1.$$

So $\displaystyle\int_0^\infty xe^{-x}dx$ converges to 1.

(b) Since the integrand goes to infinity at $x = 1$, we can split the integral up around this point:

$$\int_0^2 \frac{dx}{(x-1)^{4/3}} == \lim_{a\to 1^-}\int_0^a \frac{dx}{(x-1)^{4/3}} + \lim_{b\to 1^+}\int_b^2 \frac{dx}{(x-1)^{4/3}}$$
$$= \lim_{a\to 1^-} -3\cdot\frac{1}{(x-1)^{1/3}}\Bigg|_0^a + \lim_{b\to 1^+} -3\cdot\frac{1}{(x-1)^{1/3}}\Bigg|_b^2$$
$$= \lim_{a\to 1^-} -3\left[\frac{1}{(a-1)^{1/3}}-\frac{1}{(-1)^{1/3}}\right] + \lim_{b\to 1^+} -3\left[\frac{1}{1^{1/3}}-\frac{1}{(b-1)^{1/3}}\right],$$

which diverges.

7. Evaluate the following integral:

$$\int_0^\infty 3z^2 e^{-z^3}\, dz.$$

ANSWER:
Applying the formula for this improper integral,

$$\begin{aligned}\int_0^\infty 3z^2 e^{-z^3} dz &= \lim_{b\to\infty}\int_0^b 3z^2 e^{-z^3} dz \\ &= \lim_{b\to\infty} -e^{-z^3}\Big|_0^b \\ &= \lim_{b\to\infty}\left(-e^{-b^3}+e^0\right) \\ &= 1.\end{aligned}$$

8. Find $\int_{-2}^{2}\frac{1}{\sqrt{4-x^2}}\,dx$.
ANSWER:

$$\begin{aligned}\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx &= \lim_{b\to 2^-}\int_0^b \frac{1}{\sqrt{4-x^2}}\,dx \\ &= \lim_{b\to 2^-} \arcsin\frac{x}{2}\Big|_0^b \\ &= \lim_{b\to 2^-}\arcsin\frac{b}{2} = \arcsin 1 = \frac{\pi}{2} \\ \int_{-2}^0 \frac{1}{\sqrt{4-x^2}}\,dx &= \lim_{b\to -2^+}\int_b^0 \frac{1}{\sqrt{4-x^2}}\,dx \\ &= \lim_{b\to -2^+}\arcsin\frac{x}{2}\Big|_b^0 \\ &= \lim_{b\to -2^+}\left(\arcsin 0 - \arcsin\frac{b}{2}\right) \\ &= -\arcsin(-1) = \frac{\pi}{2}\end{aligned}$$

Thus the original integral converges to a value of π.

9. If the rate, r, at which people get sick during an epidemic of flu is approximately

$$r = 1000te^{-0.5t}$$

where r is people/day and t is time in days since start of epidemic, then the total number of people who get sick is 4000. How does the total number of people getting sick change if the coefficient is doubled and the rate is now $q = 2000te^{-0.5t}$? How does the total number of sick people change if the exponent is doubled to a rate of $s = 1000te^{-1t}$?
ANSWER:
Total number of sick people $= \int_0^\infty 2000te^{-0.5t}\,dt.$

Use integration by parts, with $u = t,\ v' = e^{-0.5t}$:

$$\begin{aligned}\text{Total } &= \lim_{b\to\infty} 2000\left(\frac{-t}{0.5}e^{-0.5t}\Big|_0^b - \int_0^b -\frac{1}{0.5}e^{-0.5t}\,dt\right) \\ &= \lim_{b\to\infty} 2000\left(-2be^{-0.5t} - \frac{2}{0.5}e^{-0.5t}\Big|_0^b\right) \\ &= \lim_{b\to\infty} 2000\left(-2be^{-0.5t} - 4e^{-0.5b}+4\right) \\ &= 8000 \text{ people.}\end{aligned}$$

So, if the coefficient is doubled, the number of people who become sick doubles too.

$$\begin{aligned}\text{Total } &= \int_0^\infty 1000e^{-t}\,dt. \qquad\qquad u = t,\ v' = e^{-t}\\ &= \lim_{b\to\infty} 1000\left(-te^{-t}\Big|_0^b - \int_0^b -e^{-t}\,dt\right)\\ &= \lim_{b\to\infty} 1000\left(-be^{-b} - \left(e^{-t}\right)\Big|_0^b\right)\\ &= \lim_{b\to\infty} 1000\left(-be^{-b} - e^{-b} + 1\right)\\ &= 1000 \text{ people.}\end{aligned}$$

10. Write the following improper integral as the sum of two limits of definite integrals (you don't need to evaluate the integrals).

$$\int_{-a-2}^{a+3} \frac{dx}{(a-x)}$$

ANSWER:
The integral is not defined at $x = a$, so split it there.

$$\int_{-a-2}^{a+3} \frac{dx}{(a-x)} = \lim_{b\to a+} \int_{-a-2}^{b} \frac{dx}{(a-x)} + \lim_{c\to a-} \int_{c}^{a+3} \frac{dx}{(a-x)}$$

Questions and Solutions for Section 7.8

1. Are the following integrals convergent or divergent? Give reasons for your answers and if the integral is convergent, please give an approximation rounded off to two decimal places. (Note: the function $f(x) = \dfrac{e^{-x}}{1+x}$ is concave up for $x \geq 0$.)

(a) $\displaystyle\int_0^\infty \frac{1}{e^x}\,dx$ (b) $\displaystyle\int_0^\infty \frac{e^{-x}}{1+x}\,dx$

ANSWER:

(a) This integral is convergent:

$$\int_0^\infty \frac{1}{e^x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to\infty} -e^{-x}\Big|_0^b = \lim_{b\to\infty}(e^0 - e^{-b}) = 1.$$

(b) This integral is also convergent. The integrand $\frac{e^{-x}}{1+x}$ is positive and less than e^{-x} for $x \geq 0$; since by (a), $\int_0^\infty e^{-x}\,dx = 1$, our integral is finite and, in particular, less than 1. As b tends to infinity, $\int_0^b \frac{e^{-x}}{1+x}\,dx$ tends to 0.60 (or more precisely 0.5963); that is the value of our integral.

2. Does $\displaystyle\int_1^\infty \frac{2+e^{-z}}{z}dz$ converge? Explain clearly how you know.

ANSWER:
Since $\dfrac{2+e^{-z}}{z} > \dfrac{2}{z}$ and $\displaystyle\int_1^\infty \frac{2}{z}dz$ does <u>not</u> converge, neither does $\displaystyle\int_1^\infty \frac{2+e^{-z}}{z}dz$.

3. Is $\displaystyle\int_0^\infty \frac{\sin^2 x}{(1+x)^2}\,dx$ convergent or divergent? Give reasons for your answer and if it is convergent, give an upper bound for its value.

ANSWER:

$$\begin{aligned}\int_0^\infty \frac{\sin^2 x}{(1+x)^2}\,dx &\leq \int_0^\infty \frac{1}{(1+x)^2}\,dx \quad \text{(because } \sin^2 x \leq 1)\\ &= -\frac{1}{1+x}\Big|_0^\infty\end{aligned}$$

$$= \lim_{b\to\infty}\left(-\frac{1}{1+b}+1\right) = 1.$$

It is convergent and 1 is an upper bound.

4. Does the following integral converge or diverge? Justify your answer.

$$\int_0^\infty x\cdot(x^3-3x-1)^{-1/2}dx$$

ANSWER:
For large and positive x,

$$\frac{x}{(x^3-3x-1)^{1/2}} \geq \frac{1}{x^{1/2}},$$

hence since $\int_0^\infty \frac{1}{x^{1/2}}\,dx$ diverges, so does $\int_0^\infty \frac{x}{(x^3-3x-1)^{1/2}}\,dx$.

5. The following improper integral may arise in the study of certain damped oscillatory motion, such as that of a pendulum that eventually comes to rest due to friction:

$$\int_0^\infty e^{-t}(1+\cos t)dt$$

(a) Show why this improper integral converges. (Hint: It is easier to do this with the comparison theorem than by finding the antiderivative and evaluating the improper integral exactly.)
(b) Based on your answer to part (a), give an upper bound on the value of the improper integral.

ANSWER:

(a) $e^{-t}(1+\cos t)\leq 2e^{-t}$ for all $t\geq 0$, because $-1\leq \cos t\leq 1$.

Since $\int_0^\infty 2e^{-t}dt$ converges, by comparison, $\int_0^\infty e^{-t}(1+\cos t)dt$ also converges.

(b) We can use the value of $\int_0^\infty 2e^{-t}dt$ as an upper bound on the value of $\int_0^\infty e^{-t}(1+\cos t)dt$.

$$\int_0^\infty 2e^{-t}dt = \lim_{b\to\infty}\int_0^b 2e^{-t}dt = \lim_{b\to\infty} -2e^{-t}\Big|_0^b = \lim_{b\to\infty}(-2e^{-t}+2e^0) = 2$$

Since $\int_0^\infty e^{-t}(1+\cos t)dt < \int_0^\infty 2e^{-t}$, then 2 is an upper bound for $\int_0^\infty e^{-t}(1+\cos t)dt$.

6. If we approximate $\int_1^\infty \frac{1}{x^2+4}\,dx$ with $\int_1^b \frac{1}{x^2+4}\,dx$, what value of b do we need to estimate the value of $\int_1^\infty \frac{1}{x^2+4}\,dx$ with an error of less that 0.01?

ANSWER:

$$\int_1^\infty \frac{1}{x^2+4}\,dx = \int_1^b \frac{1}{x^2+4}\,dx + \int_b^\infty \frac{1}{x^2+4}\,dx$$

We find b such that the tail of the integral satisfies the inequality $\left|\int_b^\infty \frac{1}{x^2+4}\,dx\right| < 0.01$

We get $0 < \int_b^\infty \frac{1}{x^2+4}\,dx < \int_b^\infty \frac{1}{x^2}\,dx = -b^{-1}$

We choose b such that $-b^{-1} < 0.01$, which means that $b > 100$. With an error of less that 0.01, choose $b = 120$,

We have $\int_1^\infty \frac{1}{x^2+4}\,dx \approx \int_1^{120} \frac{1}{x^2+4}\,dx \approx 0.545.$

7. Is the area between $y=\frac{1}{x^2}$ and $y=\frac{1}{x^3}$ on $(1,\infty)$ finite or infinite? Explain.

ANSWER:
The area under $y=\frac{1}{x^2}$ is 1. The area under $y=\frac{1}{x^3}$ is also finite, so the area between the two curves is also finite.

8. For what values of a does $\int_1^\infty \frac{1}{x^{a^2}}\,dx$ diverge? When does it converge?

ANSWER:

Diverges when $a^2 \leq 1,\ -1 \leq a \leq 1.$
Converges when $a^2 > 1,\ -1 > a > 1.$

9. For what values of p does $\displaystyle\int_1^\infty \frac{1}{x^{\sqrt{p}}}\,dx$ converge? When does it diverge?

ANSWER:
Converges when $\sqrt{p} < 1,\ p < 1.$
Diverges when $\sqrt{p} \geq 1,\ p \geq 1.$

Review Questions and Solutions for Chapter 7

1. Evaluate each of the following and show *all* work:

(a) $\displaystyle\int \sin^2 x \cos^3 x\,dx$

(b) $\displaystyle\int xe^{-x}\,dx$

(c) $\displaystyle\int \frac{x^2}{x^2+1}\,dx$

(d) $\displaystyle\int \frac{t^2}{\sqrt{9-t^2}}\,dt$

ANSWER:

(a) $\displaystyle\int \sin^2 x \cos^3 x\,dx = \int \sin^2 x(1-\sin^2 x)\cos x\,dx = \int (\sin^2 x\cos x - \sin^4 x\cos x)\,dx = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C.$

(b) Integrate by parts with $u = x,\, v' = e^{-x}$, to get $\displaystyle\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -e^{-x}(x+1) + C.$

(c) $\displaystyle\int \frac{x^2}{x^2+1}\,dx = \int \frac{x^2+1-1}{x^2+1}\,dx = \int (1 - \frac{1}{x^2+1})\,dx = x - \arctan x + C.$

(d)

$$\begin{aligned}\int \frac{t^2}{\sqrt{9-t^2}}\,dt &= \int \frac{-(9-t^2)+9}{\sqrt{9-t^2}}\,dt \\ &= \int \left(-\sqrt{9-t^2} + \frac{9}{\sqrt{9-t^2}}\right)\,dt \\ &= -\frac{1}{2}\left(t\sqrt{9-t^2} + 9\int \frac{dt}{\sqrt{9-t^2}}\right) + \left(9\arcsin\frac{t}{3} + C\right) \\ &= -\frac{1}{2}t\sqrt{9-t^2} + \frac{9}{2}\arcsin\frac{t}{3} + C'\end{aligned}$$

Note: The last two steps were performed using integral tables.

2. Use the results of Problem 1(a) to evaluate

$$\int_0^{\frac{\pi}{2}} \sin^2 x\cos^3 x\,dx.$$

ANSWER:

$\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 x\cos^3 x\,dx = \left(\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x\right)\Bigg|_0^{\frac{\pi}{2}} = \frac{2}{15}.$

3. Find by any method. Say briefly but clearly what you did.

(a) $\displaystyle\int_0^1 x^3 e^{x^2}\,dx$

(b) $\displaystyle\int_{-2}^2 f(x)\,dx$ where $f(x) = \begin{cases} 1 \text{ for } x \leq 1 \\ x \text{ for } x > 1 \end{cases}$

ANSWER:

(a) Substituting $t = x^2,\, dt = 2x\,dx,$

$$\int_{x=0}^{x=1} x^2 e^{x^2}(x\,dx) = \frac{1}{2}\int_{t=0}^{t=1} te^t\,dt.$$

Integrating by parts with $u = t,\, v' = e^t$, gives

$$\frac{1}{2}\int_0^1 te^t\,dt = \frac{1}{2}te^t\Big|_0^1 - \frac{1}{2}\int_0^1 e^t\,dt = \frac{1}{2}e^t(t-1)\Big|_0^1 = \frac{1}{2}.$$

(b) We do the integral in two pieces, using the additivity property. We get:

$$\int_{-2}^1 1\,dx + \int_1^2 x\,dx = x\Big|_{-2}^1 + \frac{1}{2}x^2\Big|_1^2 = 3 + \frac{3}{2} = 4\tfrac{1}{2}.$$

4. Find elementary formulas for the following:

(a) $\int (1+x)\sin(2x)\,dx$

(b) $\int \frac{1+e^{2t}}{e^t}\,dt$

ANSWER:

(a)

$$\int (1+x)\sin 2x\,dx = \int \sin 2x\,dx + \int x\sin 2x\,dx$$
$$= -\frac{\cos 2x}{2} + \int x\sin 2x\,dx + C$$

In the integral $\int x\sin 2x\,dx$, integrate by parts, with $u = x, u' = 1$;
$v' = \sin 2x, \quad v = -\frac{1}{2}\cos 2x$

$$\int x\sin 2x\,dx = -\frac{1}{2}x\cos 2x + \int \frac{1}{2}\cos 2x\,dx$$
$$= -\frac{x\cos 2x}{2} + \frac{\sin 2x}{4} + C'$$

So,

$$\int (1+x)\sin 2x\,dx = -\frac{\cos 2x}{2} - \frac{x\cos 2x}{2} + \frac{\sin 2x}{4} + C''$$

(b)

$$\int \frac{1+e^{2t}}{e^t}\,dt = \int (e^{-t} + e^t)\,dt$$
$$= -e^{-t} + e^t + C$$

5. Integrate

(a) $\int x^2(2x-1)\,dx$

(b) $\int \frac{\cos^3 x}{\sin x}\,dx$

(c) $\int \frac{x}{\cos^2 x}\,dx$

(d) $\int_2^4 \frac{e^{2x}}{\sqrt{e^x-1}}\,dx$

(e) Derive the reduction formula

$$\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx.$$

ANSWER:

(a) $\int x^2(2x-1)\,dx = \int 2x^3 - x^2\,dx = \frac{x^4}{2} - \frac{x^3}{3} + C.$

(b)

$$\int \frac{\cos^3 x}{\sin x}\,dx = \int \frac{\cos x \cos^2 x}{\sin x}\,dx = \int \frac{\cos x(1-\sin^2 x)}{\sin x}\,dx$$
$$= \int \left(\frac{\cos x}{\sin x} - \cos x \sin x\right)\,dx.$$

Let $u = \sin x$. Then $du = \cos x\,dx$ and

$$\int \left(\frac{\cos x}{\sin x} - \cos x \sin x\right)\,dx = \int \left(\frac{1}{u} - u\right)\,du = \ln|u| - \frac{u^2}{2} + C$$
$$= \ln|\sin x| - \frac{\sin^2 x}{2} + C.$$

(c) Integrate by parts with $u = x$ and $v' = \frac{1}{\cos^2 x}$. Then $u' = 1$, and $v = \tan x$. We get $\int \frac{x}{\cos^2 x}\,dx = x\tan x - \int \tan x\,dx$. But $\int \tan x\,dx = -\ln|\cos x| + C$, so we have $x\tan x - \ln|\cos x| + C$ as the final answer.

(d) Setting $u = e^x$ and $du = e^x\,dx$, we get $\int_2^4 \frac{e^{2x}\,dx}{\sqrt{e^x-1}} = \int_{e^2}^{e^4} \frac{u\,du}{\sqrt{u-1}}$. We now set $v = u - 1$, to get $\int_{e^2}^{e^4} \frac{u\,du}{\sqrt{u-1}} = \int_{e^2-1}^{e^4-1} \frac{v+1}{\sqrt{v}}\,dv = \int_{e^2-1}^{e^4-1} \left(\sqrt{v} + \frac{1}{\sqrt{v}}\right)\,dv = \frac{2}{3}v^{\frac{3}{2}} + 2v^{\frac{1}{2}}\Big|_{e^2-1}^{e^4-1} = \frac{2}{3}(e^4-1)^{\frac{3}{2}} + 2(e^4 - 1)^{\frac{1}{2}} - \frac{2}{3}(e^2-1)^{\frac{3}{2}} - 2(e^2-1)^{\frac{1}{2}} \approx 260.42$.

(e) Integrate by parts with $u = (\ln x)^n$ and $v' = 1$. Then $u' = n(\ln x)^{n-1}\cdot\frac{1}{x}$ and $v = x$. We get $\int (\ln x)^n\,dx = x(\ln x)^n - \int n(\ln x)^{n-1}\left(\frac{1}{x}\right)\cdot x\,dx$. But $\int n(\ln x)^{n-1}\left(\frac{1}{x}\right)\cdot x\,dx = n\int (\ln x)^{n-1}\,dx$, to complete the proof.

6. Evaluate each of the indefinite integrals below. Be sure to show your work.

(a) $\int xe^x\,dx$

(b) $\int \frac{e^x}{1+e^{2x}}\,dx$

(c) $\int \frac{1}{1+e^x}\,dx$

(d) $\int \frac{\sin x}{\cos x}\,dx$

(e) $\int \frac{1}{\sqrt{1-4x^2}}\,dx$

ANSWER:

(a) Using the formula for integration by parts, $\int uv'\,dx = uv - \int u'v\,dx$, with $u = x$ and $v = e^x$, we find:

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C = e^x(x-1) + C.$$

(b) Set $u = e^x$, $du = e^x\,dx$. Then

$$\int \frac{e^x\,dx}{1+e^{2x}} = \int \frac{du}{1+u^2} = \arctan u + C = \arctan(e^x) + C.$$

(c)

$$\int \frac{1}{1+e^x}\,dx = \int \frac{1+e^x-e^x}{1+e^x}\,dx$$
$$= \int \left(1 - \frac{e^x}{1+e^x}\right)\,dx$$
$$= \int 1\,dx - \int \frac{e^x}{1+e^x}\,dx.$$

In the second integral, set $u = 1 + e^x$, $du = e^x\,dx$. Substituting, we get

$$\begin{aligned} -\int \frac{e^x}{1+e^x}\,dx &= -\int \frac{du}{u} \\ &= -\ln|u| + C \\ &= -\ln(1+e^x) + C. \end{aligned}$$

So $\displaystyle\int \frac{1}{1+e^x}\,dx = x - \ln(1+e^x) + C'$.

(d) Set $u = \cos x$, $du = -\sin x\,dx$. Then

$$\begin{aligned} \int \frac{\sin x}{\cos x}\,dx = -\int \frac{du}{u} &= -\ln|u| + C \\ &= -\ln|\cos x| + C. \end{aligned}$$

(e) Set $u = 2x$, $du = 2\,dx$. Then

$$\int \frac{1}{\sqrt{1-4x^2}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\arcsin u + C = \frac{1}{2}\arcsin(2x) + C.$$

7. Calculate the following indefinite integrals. (Remember, you can always check your answers.)

(a) $\displaystyle\int x^3 \cos x\,dx$

(b) $\displaystyle\int \sin(\ln t)\,dt$

(c) $\displaystyle\int \left((\ln z)^2 + 3\ln z\right)\,dz$

ANSWER:

(a) Using the Table (#16), or parts:

$$\int x^3 \cos x\,dx = x^3 \sin x + 3x^2\cos x - 6x\sin x - 6\cos x + C$$

(b) Parts:

$$\int \sin(\ln t)dt \underbrace{=}_{\substack{u = \sin(\ln t) \;\; u' = \frac{\cos(\ln t)}{t} \\ v = t \;\; v' = 1}} t\sin(\ln t) - \int \cos(\ln t)dt$$

$$\underbrace{=}_{\substack{\text{Parts again} \\ u = \cos(\ln t) \;\; u' = \frac{-\sin(\ln t)}{t} \\ v = t \;\; v' = 1}} t\sin(\ln t) - [t\cos(\ln t) + \int \sin(\ln t)dt]$$

solving $\displaystyle\int \sin(\ln t)dt = \frac{t}{2}[\sin(\ln t) - \cos\ln t)] + C$

(Alternatively, use $w = \ln t$, $t = e$, $dt = e^2 dw$ and #8 from Table.]

(c) $\displaystyle\int \ln z\,dz = z\ln z - z + C$ (from Table #4)

$$\int (\ln z)^2 dz \underbrace{=}_{\substack{u = (\ln z)^2 \;\; u' = \frac{2\ln z}{z} \\ v = z \;\; v' = t}} z(\ln z)^2 - \int 2\ln z\,dz = z(\ln z)^2 - 2[z(\ln z) - z] + C$$

$$= z(\ln z)^2 - 2z\ \ln z + 2z + C$$

so $\displaystyle\int ((\ln z)^2 + 3\ln z)dz = z(\ln z)^2 + z\,lnz - z + C$

8. Find the following indefinite integrals:

(a) $\int e^{-2x} \cos x \, dx$

(b) $\int \frac{t}{t+1} dt$

ANSWER:

(a) Apply II-9 from the integral table; $u = -2, b = 1$.

$= \frac{1}{(-2)^2 + (1)^2} e^{-2x} [-2 \cos x + \sin x] + C$

$= \frac{1}{5} e^{-2x} (-2 \cos x + \sin x) + C$

You can integrate by parts twice to get the same answer.

(b) You have to apply long division to get the degree of the numerator to be less than the degree of the denominator.

$\frac{t}{t+1} = 1 - \frac{1}{t+1}$ Thus,

$\int \frac{t}{t+1} dt = \int \left(1 - \frac{1}{t+1}\right) dt = t - \ln|t+1| + C$

9. Find the following integrals:

(a) $\int x e^x \, dx$

(b) $\int (\sin t) e^{\cos t} dt$

(c) $\int \frac{1}{4 - y^2} dy$

ANSWER:

(a) Use parts or #14 of tables: $\int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + C$

(b) Set $w = \cos t$ and $dw = -\sin t dt$. $\int \sin t e^{\cos t} dt = -\int e^w \, dw = -e^w + C = -e^{\cos t} + C$

(c) Use #16 of tables: ($a = 2, b = -2$)

$$\int \frac{dy}{4 - y^2} = \int \frac{dy}{(2-y)(2+y)} = -\int \frac{dy}{(y-2)(y+2)} = -\frac{1}{4}(\ln|y-2| - \ln|y+2|) + C$$

10. Find the following indefinite integrals:

(a) $\int \frac{x}{x^2 - 1} dx$

(b) $\int x^2 \sin x \, dx$.

ANSWER:

(a) $\int \frac{x}{x^2 - 1} dx$

Let $w = x^2 - 1 \Rightarrow dw = 2x \, dx \Rightarrow x \, dx = \frac{1}{2} dw$

$= \frac{1}{2} \int \frac{dw}{w} = \frac{1}{2} \ln|w| + C$

$= \frac{1}{2} \ln|x^2 - 1| + C$

(b) $\int x^2 \sin x \, dx$

Apply Formula III-15 from the table of integrals, letting

$p(x) = x^2, \quad a = 1$

$\Rightarrow p'(x) = 2x$

$\Rightarrow p''(x) = 2$
$= -x^2 \cos x + (2x)\sin x + 2\cos x + C$

11. Find the following indefinite integrals:

(a) $\int \frac{x^2}{x^3+1}dx$

(b) $\int (x^2+1)\cos 2x\, dx$

ANSWER:

(a) Use substitution! Let $u = x^3+1$, then $du = 3x^2dx$, $\frac{1}{3}du = x^2dx$

$$\int \frac{x^2}{x^3+1}dx = \frac{1}{3}\int \frac{1}{u}du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|x^3+1| + C$$

(b) Using Table III #16, or integrating by parts (twice), we get

$$\int (x^2+1)\cos 2x\, dx = \frac{1}{2}(x^2+1)\sin 2x + \frac{1}{2}x\cos 2x - \frac{1}{4}\sin 2x + C$$

$$= \frac{1}{2}x^2\sin 2x + \frac{1}{2}\sin 2x + \frac{1}{2}x\cos 2x - \frac{1}{4}\sin 2x + C$$

$$= \frac{1}{2}x^2\sin 2x + \frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C$$

12. Find the following indefinite integrals:

(a) $\int \frac{1}{2x+3}dx$

(b) $\int \frac{1}{\sin^3 x}dx$

ANSWER:

(a) $w = 2x+3 \Rightarrow dw = 2x\,dx \Rightarrow x\,dx = \frac{1}{2}dw$

$$\int \frac{1}{2x+3}dx = \frac{1}{2}\int \frac{dw}{w} = \frac{1}{2}\ln|w| + C = \frac{1}{2}\ln|2x+3| + C$$

(b) $\int \frac{1}{\sin^3 x}dx$

Integral Table #19, $m = 3$

$$= -\frac{1}{2}\frac{\cos x}{\sin^2 x} + \frac{1}{2}\int \frac{1}{\sin x}dx$$

Integral Table #20

$$= -\frac{1}{2}\frac{\cos x}{\sin^2 x} + \frac{1}{4}\ln\left|\frac{\cos x - 1}{\cos x + 1}\right| + C$$

13. Integrate: (Please give exact answers.)

(a) $\int \frac{1}{\sqrt{3x+7}}\, dx$

(b) $\int_1^2 x^3 \ln x\, dx$

(c) $\int \frac{2t}{e^{3t}}\, dt$

(d) $\int_{-13}^{13} \frac{(\sin\theta)^7}{\theta^2}\, d\theta$

(e) $\displaystyle\int \frac{\sqrt{u}}{\sqrt{u}+1}du$

ANSWER:

(a) Let $w = 3x + 7$, $dw = 3dx$.

$$= \int \frac{1}{\sqrt{w}}\frac{dw}{3}$$

$$= \frac{1}{3}\int w^{-\frac{1}{2}}dw = \frac{1}{3}\frac{w^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{2}{3}\sqrt{3x+7} + C$$

Common mistake: $\displaystyle\int (3x+7)^{-\frac{1}{2}}dx = (3x+7)^{\frac{1}{2}}$

(b)

$$\int x^3 \ln x\, dx = \int \ln x\, d\left(\frac{x^4}{4}\right) = \frac{x^4}{4}\ln x - \int \frac{x^4}{4}d(\ln x)$$

$$= \frac{x^4}{4}\ln x - \int \frac{x^4}{4}\cdot\frac{1}{x}dx$$

$$= \frac{x^4}{4}\ln x - \frac{1}{4}\cdot\frac{x^4}{4} + C$$

$$\int_1^2 x^3 \ln x\, dx = 4\ln 2 - \frac{15}{16}$$

Common mistake: Miss $\frac{1}{4}$ during copying.

(c)

$$\int 2t\, d\left(\frac{e^{-3t}}{-3}\right) = 2t\frac{e^{-3t}}{-3} - \int \frac{e^{-3t}}{-3}d(2t)$$

$$= -\frac{2}{3}te^{-3t} + \frac{2}{3}\int e^{-3t}dt$$

$$= -\frac{2}{3}te^{-3t} - \frac{2}{9}e^{-3t} - C$$

Common mistake:

(1) Take $\dfrac{1}{e^{3t}}$ as e^{3t}.

(2) Miss a sign during copying.

(d) $= 0$.

Reason: $\sin(-\theta) = -\sin\theta \Rightarrow \dfrac{[\sin(-\theta)]^7}{(-\theta)^2} = -\dfrac{(\sin\theta)^7}{\theta^2}$

i.e., $\dfrac{(\sin\theta)^7}{\theta^2}$ is an <u>odd</u> function, the graph is symmetric with respect to the origin.

Note also that near zero, the function behaves like $\frac{\theta^7}{\theta^2} = \theta^5$, so it is bounded there. So $\displaystyle\int_{-13}^{0}$ cancels with $\displaystyle\int_0^{13}$.

Common mistake: Do not recognize the symmetry.

(e) Let $w = \sqrt{u} + 1$, $dw = \dfrac{1}{2}u^{-\frac{1}{2}}du$, $du = 2\sqrt{u}dw = 2(w-1)dw$.

$$\int \frac{2(w-1)^2}{w}dw = 2\int \frac{w^2 - 2w + 1}{w}dw = 2\int\left(w - 2 + \frac{1}{w}\right)dw$$

$$= 2\left[\frac{w^2}{2} - 2w + \ln|w|\right]$$

$$= w^2 - 4w + 2\ln|w| + C$$

$$= (\sqrt{u}+1)^2 - 4(\sqrt{u}+1) + 2\ln|\sqrt{u}+1| + C$$

$$= u - 2\sqrt{u} + 2\ln(\sqrt{u}+1) + C$$

Common mistake: Fail to get $\underline{du = 2(w-1)dw}$.

14. Find

(a) $\int \sqrt{1-3x}\,dx$

(b) $\int x \ln x\,dx$

ANSWER:

(a) Let $w = 1-3x$, $dw = -3dx$. By substitution,

$$\begin{aligned}\int \sqrt{1-3x}\,dx &= -\frac{1}{3}\int \sqrt{w}\,dw \\ &= -\frac{1}{3}\cdot\frac{2}{3}w^{\frac{3}{2}} + C \\ &= -\frac{2}{9}(1-3x)^{\frac{3}{2}} + C\end{aligned}$$

(b) Let $u = \ln x$, $v' = x$, so $u' = \frac{1}{x}$ and $v = \frac{x^2}{2}$. Integrating by parts gives:

$$\begin{aligned}\int x\ln x\,dx &= \frac{1}{2}\ln x \cdot x^2 - \int \frac{x}{2}\,dx + C \\ &= \frac{1}{2}x^2\ln x - \frac{x^2}{4} + C\end{aligned}$$

15. Find the following antiderivatives:

(a) $\int \frac{4+x}{\sqrt{x}}\,dx$

(b) $\int \frac{1+\sin(2x)}{2}\,dx$

(c) $\int \frac{1}{(1+x)^2}\,dx$

(d) $\int \frac{\ln(1+x)}{1+x}\,dx$

(e) $\int \ln\left(\frac{1}{x}\right)dx$

ANSWER:

(a) $\int \frac{4+x}{\sqrt{x}}\,dx = \int\left(\frac{4}{\sqrt{x}} + \sqrt{x}\right)dx = 8\sqrt{x} + \frac{2}{3}x^{\frac{3}{2}} + C$

(b) $\int \frac{1+\sin(2x)}{2}\,dx = \int\left(\frac{1}{2} + \frac{1}{2}\sin(2x)\right)dx = \frac{1}{2}x - \frac{1}{4}\cos(2x) + C$

(c) $\int (1+x)^{-2}\,dx = -\frac{1}{1+x} + C$

(d) We first set $u = 1+x$, $du = dx$ to get $\int \frac{\ln(1+x)}{1+x}\,dx = \int \frac{\ln u}{u}\,du$. Now set $v = \ln u$, $dv = \frac{1}{u}du$ to get

$$\int v\,dv = \frac{v^2}{2} + C = \frac{(\ln(1+x))^2}{2} + C$$

(e) $\int \ln\left(\frac{1}{x}\right)dx = \int -\ln x\,dx$. We integrate by parts with $u = -\ln x$, $dv = dx$, to get

$$-\int \ln x\,dx = -x\ln x + \int 1\,dx = -x\ln x + x + C$$

16. Find formulas for the following indefinite integrals (anti-derivatives):

(a) $\int (x+2)(x-2)\,dx$

(b) $\int \frac{1+e^t}{e^t}\,dt$

(c) $\int \frac{1-\ln x}{x}\,dx$

(d) $\int (x-1)e^{-x}\,dx$

ANSWER:

(a) $\int (x+2)(x-2)\,dx = \int (x^2-4)\,dx = \frac{x^3}{3} - 4x + C.$

(b) $\int \frac{1+e^t}{e^t}\,dt = \int (e^{-t}+1)\,dt = -e^{-t} + t + C.$

(c) Set $u = \ln x$, $du = \frac{1}{x}dx$. Substitute to get
$\int \frac{1-\ln x}{x}\,dx = \int (1-u)\,du = u - \frac{u^2}{2} + C = \ln|x| - \frac{(\ln|x|)^2}{2} + C.$

(d)

$$\int (x-1)e^{-x}\,dx = \int (xe^{-x} - e^{-x})\,dx.$$

We integrate $\int xe^{-x}\,dx$ by parts with $u = x$, $v' = e^{-x}$ and $u' = 1$, $v = -e^{-x}$.

$$\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C.$$

Thus

$$\begin{aligned}\int (xe^{-x} - e^{-x})\,dx &= -xe^{-x} - e^{-x} - \int e^{-x}\,dx \\ &= -xe^{-x} - e^{-x} + e^{-x} + C \\ &= -xe^{-x} + C.\end{aligned}$$

17. Evaluate the following. Give both an exact, but possibly symbolic, value (e.g., $\frac{3}{4}, \sin 3, \ln 2, \ldots$) and a decimal approximation accurate to two decimal places.

(a) $\int_0^{\frac{1}{2}} \frac{1}{1-x^2}\,dx$

(b) $\int_0^4 \frac{4x}{\sqrt{x^2+9}}\,dx$

(c) $\int_1^3 \ln x\,dx$

ANSWER:

(a) We use the table of integrals to solve the integral $\int_0^{\frac{1}{2}} \frac{1}{1-x^2}\,dx$. The table of integrals says:

$$\int \frac{1}{(x-a)(x-b)}\,dx = \frac{1}{a-b}(\ln|x-a| - \ln|x-b| + C)$$

Since $x^2 - 1 = (x-1)(x+1)$, we use $a = 1$ and $b = -1$. Substituting for a and b in the formula, we get:

$$\begin{aligned}\int_0^{\frac{1}{2}} \frac{1}{1-x^2}\,dx &= -\int_0^{\frac{1}{2}} \frac{1}{(x-1)(x+1)}\,dx \\ &= -\left(\frac{1}{2}\right)(\ln|x-1| - \ln|x+1|)\bigg|_0^{\frac{1}{2}}\end{aligned}$$

$$= -\left(\frac{1}{2}\right)\left(\ln\left|\frac{x-1}{x+1}\right|\right)\Bigg|_0^{\frac{1}{2}}$$
$$= \ln\sqrt{\left|\frac{x+1}{x-1}\right|}\,\Bigg|_0^{\frac{1}{2}}$$
$$= \ln\sqrt{3} - \ln 1 = \ln\sqrt{3} \approx 0.5943.$$

(b) Set $u = x^2 + 9$, $du = 2xdx$. So substituting, $\int_0^4 \frac{4x}{\sqrt{x^2+9}}\,dx = 2\int_{u=9}^{u=25} \frac{du}{\sqrt{u}} = 2 \cdot 2u^{1/2}\Big|_9^{25} = 4(5-3) = 8.$

(c) We integrate $\int_1^3 \ln x\,dx$ by parts with $u = \ln x$, $v' = 1$ and $u' = \frac{1}{x}$, $v = x$. $\int_1^3 \ln x\,dx = x\ln x\Big|_1^3 - \int_1^3 dx =$ $(x\ln x - x)\Big|_1^3 = 3\ln 3 - 3 - 0 + 1 = 3\ln 3 - 2 \approx 1.296.$

18. Integrate:

(a) $\int \frac{x^2+1}{\sqrt{x}}\,dx$

(b) $\int (\sin^3 2\theta + 1)\cos 2\theta\,d\theta$

(c) $\int \frac{3x^2+\cos x}{x^3+\sin x}\,dx$

(d) $\int_0^{\frac{\sqrt{3}}{2}} \frac{dx}{\sqrt{1-x^2}}$

(e) $\int_0^{\ln 2} \frac{x}{e^x}\,dx$

ANSWER:

(a) $\int \frac{x^2+1}{\sqrt{x}}\,dx = \int (x^{\frac{3}{2}} + x^{-\frac{1}{2}})\,dx = \frac{2}{5}x^{\frac{5}{2}} + 2x^{\frac{1}{2}} + C$

(b)
$$\int (\sin^3 2\theta + 1)\cos 2\theta\,dt = \int \sin^3 2\theta\cos 2\theta\,d\theta + \int \cos 2\theta\,d\theta$$

In the first integral, let $u = \sin^2 2\theta,\ du = 2\sin 2\theta\cos 2\theta \cdot 2\,d\theta = 4\sin 2\theta\cos 2\theta\,d\theta$.

So the first integral can be simplified to:

$$\frac{1}{4}\int u\,du = \frac{1}{4}\frac{u^2}{2} + C = \frac{1}{8}u^2 + C = \frac{1}{8}\sin^4 2\theta + C.$$

The second integral is simply $\frac{1}{2}\sin 2\theta + C$, so the answer is $\frac{1}{8}\sin^4 2\theta + \frac{1}{2}\sin 2\theta + C$.

(c) Substitute $u = x^3 + \sin x$ and $du = (3x^2 + \cos x)dx$. Then

$$\int \frac{3x^2+\cos x}{x^3+\sin x}\,dx = \int \frac{du}{u}$$
$$= \ln|u| + C$$
$$= \ln|x^3 + \sin x| + C.$$

(d) From the tables, $\int_0^{\frac{\sqrt{3}}{2}} \frac{dx}{\sqrt{1-x^2}} = \arcsin x\Big|_0^{\frac{\sqrt{3}}{2}} = \arcsin\frac{\sqrt{3}}{2} - \arcsin 0 = \frac{\pi}{3}$

(e)
$$\int_0^{\ln 2} \frac{x}{e^x}\,dx = \int_0^{\ln 2} xe^{-x}\,dx$$

Integrate by parts: set $u = x, v' = e^{-x}$, and $u' = 1, v = -e^{-x}$

$$\begin{aligned}\int_0^{\ln 2} xe^{-x}\,dx &= -xe^{-x}\Big|_0^{\ln 2} + \int_0^{\ln 2} e^{-x}\,dx \\ &= \left(-xe^{-x} - e^{-x}\right)\Big|_0^{\ln 2} \\ &= -\ln 2\, e^{-\ln 2} - e^{-\ln 2} - (-1) \\ &= -\ln 2 \cdot \left(\frac{1}{2}\right) - \frac{1}{2} + 1 \\ &= \frac{1}{2} - \frac{1}{2}\ln 2.\end{aligned}$$

19. Evaluate the following integrals symbolically, (i.e., don't use numerical integration programs.)

(a) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{d\theta}{\tan\theta}$.

(b) $\displaystyle\int_{-\pi}^{\pi} e^{2x}\cos 2x\,dx$.

(c) $\displaystyle\int u^{-1}\ln u\,du$.

(d) $\displaystyle\int \frac{x^2}{(1+x^2)}\,dx$.

(Hint: Use integral tables.)

ANSWER:

(a) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{d\theta}{\tan\theta} = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos\theta d\theta}{\sin\theta} = \ln|\sin\theta|\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \ln 2.$

(b) Refering to the table of integrals, we get

$$\begin{aligned}\int_{-\pi}^{\pi} e^{2x}\cos 2x\,dx &= \frac{1}{8}e^{2x}(2\cos 2x + 2\sin 2x)\Big|_{-\pi}^{\pi} \\ &= \frac{1}{4}e^{2x}(\cos 2x + \sin 2x)\Big|_{-\pi}^{\pi} = \frac{1}{4}(e^{2\pi} - e^{-2\pi}).\end{aligned}$$

(c) Substituting $x = \ln u$, $dx = \frac{1}{u}\,du$ we get

$$\int u^{-1}\ln u\,du = \int x\,dx = \frac{x^2}{2} + C = \frac{(\ln u)^2}{2} + C.$$

(d) $\displaystyle\int \frac{x^2}{1+x^2}\,dx = \int \left(1 - \frac{1}{1+x^2}\right)\,dx = x - \arctan x + C.$

20. Compute the following indefinite integrals:

(a) $\displaystyle\int \sin x(\cos x + 5)^7\,dx$

(b) $\displaystyle\int \frac{\ln x\,dx}{x}$

(c) $\displaystyle\int xe^{2x}\,dx$

ANSWER:

(a) Let $u = \cos x + 5$. Then $du = -\sin x\,dx$.

$$\int \sin x(\cos x + 5)^7\,dx = -\int u^7\,du = -\frac{u^8}{8} + C = -\frac{(\cos x + 5)^8}{8} + C.$$

(b) Let $u = \ln x$. Then $du = \dfrac{1}{x}\,dx$.

$$\int \frac{\ln x\,dx}{x} = \int u\,du = \frac{u^2}{2} + C = \frac{(\ln x)^2}{2} + C.$$

(c) Integrating by parts with $u = x$, $v' = e^{2x}$, we get

$$\int xe^{2x}\,dx = \frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2}\,dx = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C.$$

21. TRUE/FALSE questions. For each statement, write whether it is true or false and provide a short explanation or counterexample.

(a) If the left-hand sum, $\mathrm{LEFT}(n)$, for $\int_a^b f(x)\,dx$ is too large for one value of n, it will be too large for all values of n.
(b) If the average value of $f(x)$ on the interval $2 \le x \le 5$ is between 0 and 1 then f is between 0 and 1 on the interval $2 \le x \le 5$.
(c) $\int_1^2 \sin x^2\,dx > 3$.
(d) If $f' > g'$ for all $a < x < b$ then the left-hand Riemann sum approximation of $\int_a^b f\,dx$ will have larger error than the left-hand Riemann sum for $\int_a^b g\,dx$.

ANSWER:

(a) FALSE. Try, for example, $\int_{-1}^{1} |x|\,dx = 1$. $\mathrm{LEFT}(1) = 2$ is too big, but $\mathrm{LEFT}(2) = 1 + 0 = 1$ is exactly right.

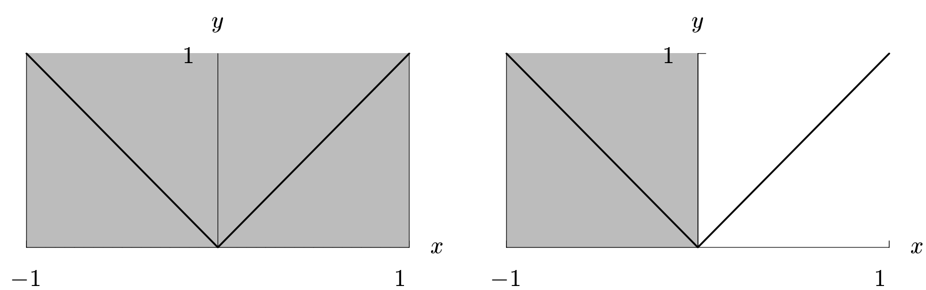
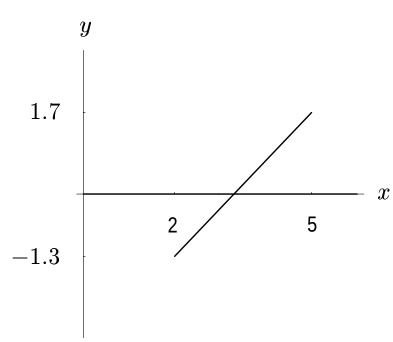


Figure 7.8.94: $\mathrm{LEFT}(1)$ and $\mathrm{LEFT}(2)$ for $f(x) = |x|$

(b) FALSE. Consider $f(x) = x - 3.3$. As can be seen from the figure below, the average value of $f(x)$ over the interval is between 0 and 1, but not all values of $f(x)$ lie between 0 and 1.



(c) FALSE. Since $|\sin(x^2)| \le 1$ for all x, $\displaystyle\int_1^2 \sin(x^2)\,dx \le 1 \cdot (2-1) = 1$.

(d) FALSE. Let $f(x) = 1$ and $g(x) = 2 - x$ on the interval $0 \le x \le 5$. Then $f'(x) = 0$, and $g'(x) = -1$, so $f'(x) > g'(x)$ in this interval. For f, $\mathrm{LEFT}(1) = 1 \cdot (5-0) = 5$, and for g, $\mathrm{LEFT}(1) = 2 \cdot (5-0) = 10$. Evaluating f and g directly, we get $\int_0^5 f(x)\,dx = 5$, and $\int_0^5 g(x)\,dx = -2.5$. So although $f'(x) > g'(x)$, the error in the left-hand Riemann sum is greater for g than for f.

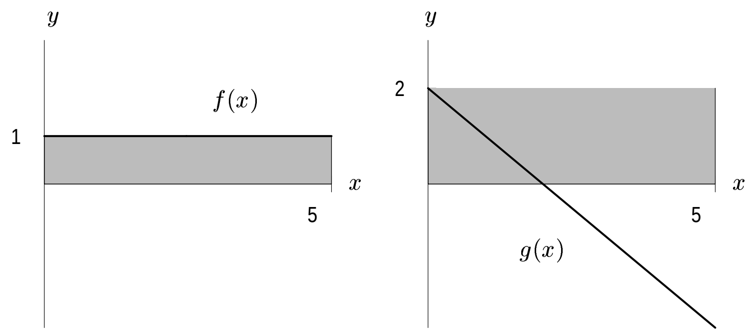


Figure 7.8.95: Graphs of $f(x)$ and $g(x)$; shaded areas show $\text{LEFT}(1)$

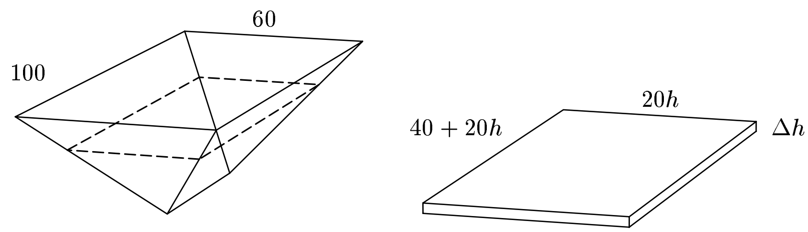
Chapter 8 Exam Questions

Questions and Solutions for Section 8.1

1. A rectangular lake is 100 km long and 60 km wide. The depth of the water at any point of the surface is a tenth of the distance from the *nearest* shoreline. How much water does the lake contain (in km^3)?

   ANSWER:

   We find the volume by slicing the lake horizontally.



   Note that at height h from the bottom, the volume of a slice of water of thickness Δh is approximately

$$(40 + 20h)(20h)\Delta h.$$

   Since the maximum depth is $h = \frac{30}{10} = 3$ km, summing up all these slices yields

$$\begin{aligned}\text{Volume} &= \int_0^3 (40 + 20h)20h\, dh \\ &= \int_0^3 400h^2 + 800h\, dh \\ &= \frac{400}{3}h^3 + 400h^2 \Big|_0^3 \\ &= 7200\,\text{km}^3.\end{aligned}$$

2. Find the area of the dumbbell shaped region bounded by the curve $y^2 = x^6(1 - x^2)$.

   [Hint 1: Sketch the graphs of $y = x^3\sqrt{1 - x^2}$ and $y = -x^3\sqrt{1 - x^2}$ (using your calculator) and use symmetry to decide what integral to evaluate.

   Hint 2: You may find that the substitution $x = \sin\theta$ is helpful in evaluating your integral, as well as IV-22 in the table of integrals.]

   ANSWER:

   See Figure 8.1.96. By symmetry, we see that Area = $4\int_0^1 x^3\sqrt{1 - x^2}\, dx$. Set $x = \sin\theta, dx = \cos\theta\, dt$ and substitute to get $4\int_0^{\frac{\pi}{2}} \sin^3\theta\cos^2\theta\, d\theta$. Now, we write

   $4\int_0^{\frac{\pi}{2}} \sin^3\theta\cos^2\theta\, d\theta = 4\int_0^{\frac{\pi}{2}} (1 - \cos^2\theta)\sin\theta\cos^2\theta\, d\theta = 4(-\frac{\cos^5\theta}{5} + \frac{\cos^3\theta}{3})\Big|_0^{\frac{\pi}{2}} = \frac{8}{15}.$

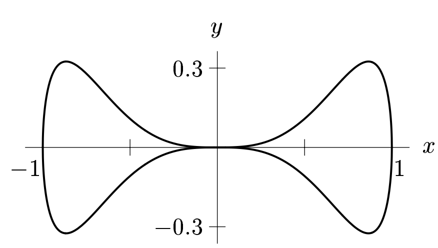
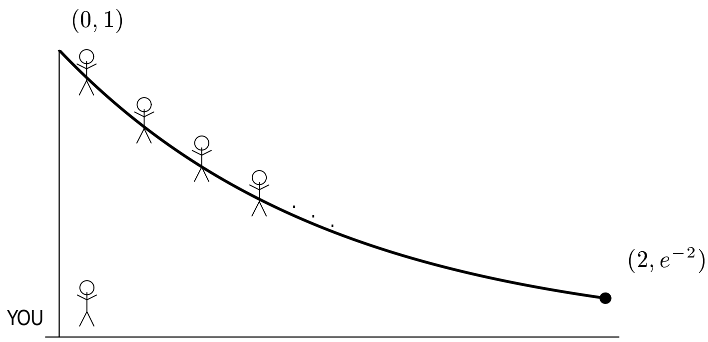


Figure 8.1.96

Focus on Engineering

3. It's time for the School of Engineering class picture and you are the photographer! You stand at the origin with your camera and your classmates are strung out along the curve $y = e^{-x}$ from $(0, 1)$ to $(2, e^{-2})$.



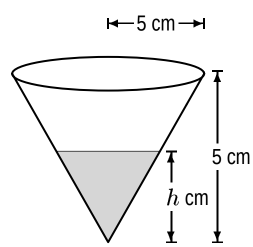
(a) As a function of x, what is your distance to your classmate at a point (x, y) on the curve?
(b) Write down an integral that gives the average value of the function in (a).
(c) Use your calculator to evaluate the integral in (b) to one decimal place. Say what you are doing.
(d) You focus your camera according to your answer in part (c). Who is more in focus, the person at $(0, 1)$ or the person at $(2, e^{-2})$?
(e) Approximately where on the curve should you tell your best friend to stand so that she will be in focus? (Use your calculator to solve graphically for her x-coordinate.)

ANSWER:

(a) Distance $= S(x) = \sqrt{x^2 + e^{-2x}}$
(b) $\overline{S(x)} = \frac{1}{2}\int_0^2 \sqrt{x^2 + e^{-2x}}\, dx$
(c) LEFT(400) gives 1.1955. RIGHT(400) gives 1.1980. So the true value is 1.197 to one decimal place.
(d) $S(0) = 1$; $S(2) = \sqrt{4 + e^{-4}} \approx 2.005$. Thus the person at (0,1) is more in focus.
(e) Graphing $\sqrt{x^2 + e^{-2x}} - 1.197$, we see that it is 0 at $x \approx 1.16$.

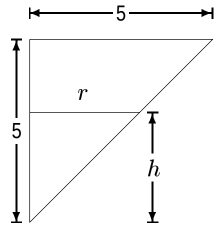
4. A coffee filter is in the shape of a cone, as shown below. When it is filled with water to a height h cm, the rate at which coffee flows out the hole at the bottom is given by

$$\begin{pmatrix}\text{Volume of coffee which}\\ \text{flows out per second}\end{pmatrix} = \sqrt{h}\ \text{cm}^3/\text{sec}.$$



(a) What is the radius of the surface of the coffee when it is at height of h?
(b) What is the approximate volume of the "slice" of coffee lying between h and $h + \Delta h$?
(c) Given that when the height is h, the coffee is leaving at a rate of $\sqrt{h}$ cm^3/sec, approximately how long does it take for the height of the coffee to fall from $h + \Delta h$ to h?
(d) Suppose the coffee filter starts full. Write an integral representing the total amount of time it takes for the coffee filter to empty, and hence find the time for the filter to empty. Give your answer to the nearest second.

ANSWER:

(a) 

By similar triangles $\frac{r}{h} = \frac{5}{5}$ so $r = h$. "Slice"is disc-shaped

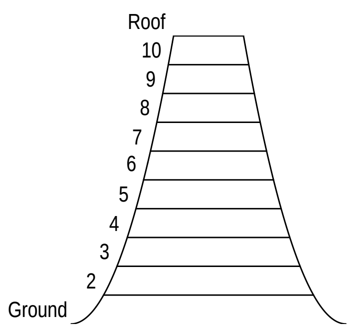
(b) Volume $=$ Area $\times \Delta h = \pi h^2 \Delta h$

(c) time $= \frac{\text{volume}}{\text{rate}} = \frac{\pi h^2 \Delta h}{\sqrt{h}} = \pi h^{\frac{3}{2}} \Delta h$

(d) $\int_0^5 \pi h^{\frac{3}{2}} dh = \frac{2\pi}{5} \left[h^{\frac{5}{2}} \right]_0^5 = 70$ seconds

5. Suppose that a new office building is being planned. The architect wants to design a building that is thick at the base and eventually tapers to a small flat roof at the top. The building is to have 10 floors, and each floor is to be 15 feet high. For purposes of air conditioning the building, an estimate of the total volume is needed. You have been hired as a consultant at an exorbitant salary to do this. You have been provided with the following information which shows how much area, in units of 100 square feet, each of the 10 floors will contain:

Floor #	Ground	2	3	4	5	6	7	8	9	10	Roof
Area	25	23	20	18	16	14	12	10	9	8	7



Earn your wage: find an approximate value for the volume of the entire building. Explain what you are doing.
ANSWER:
Since area is a decreasing function of height (floor #), starting from the ground, a left-hand sum gives an overestimate of the volume (area $\times$ height) and a right-hand sum gives an underestimate. Therefore,

$$\begin{aligned} \text{Overestimate} &= 15 \cdot (25 + 23 + 20 + 18 + 16 + 14 + 12 + 10 + 9 + 8) \cdot 100 \\ &= 232500 \text{ ft}^3 \\ \text{Underestimate} &= 15 \cdot (23 + 20 + 18 + 16 + 14 + 12 + 10 + 9 + 8 + 7) \cdot 100 \\ &= 205500 \text{ ft}^3 \end{aligned}$$

An approximate value for the volume of the entire building is given by the average of the overestimate and the underestimate, $\frac{232500 + 205500}{2} = 219000 \text{ ft}^3$.

6. If the volume of a cylinder with radius = 4 cm is 112π cm^3, find the height by writing a Riemann sum and then a definite integral representing the volume using slices.
ANSWER:
Each slice is a circular disk with radius = 4 cm.

$$\text{Volume of disk } = \pi r^2 \Delta x = 16\pi \Delta x \text{ cm}^3$$

Summing over all disks, we have

$$\text{Total volume } \approx \sum 16\pi\Delta x \text{ cm}^3.$$

Taking a limit as $\Delta x \to 0$, we get

$$\text{Total volume } = \lim_{\Delta x \to 0} \sum 16\pi\Delta x = \int_0^h 16\pi\, dx \text{ cm}^3$$

where h is the unknown height.
Evaluating gives

$$\text{Total volume } = 16\pi x\Big|_0^h = 16\pi h \text{ cm}^3.$$

Since volume $= 112\pi$ cm^3, we find $h = 7$ cm.

7. A cone has base radius = 10 cm and height = 5 cm. Use horizontal slicing to find the volume of the cone.
   ANSWER:
   Each slice is a circular disk of thickness Δh. The disk at height h_i above the base has radius $r_i = \dfrac{1}{2}w_i$. See Figure 8.1.97.

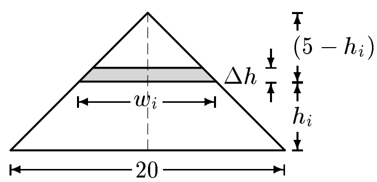


Figure 8.1.97

From similar triangles,

$$\frac{w_i}{20} = \frac{5-h_i}{5}$$

$$w_i = \frac{20}{5}(5-h_i) = 20 - 4h_i$$

$$r_i = 10 - 2h_i$$

$$\text{Volume of slice } \approx \pi r_i^2 \Delta h = \pi(10-2h_i)\Delta h \text{ cm}^3$$

Summing over all slices, we have

$$\text{Volume of cone } \approx \sum_{i=1}^{n} \pi(10-2h_i)^2 \Delta h \text{ cm}^3$$

Taking the limit as $n \to \infty$, so $h \to 0$, gives

$$\begin{aligned}
\text{Volume of cone } &= \lim_{n\to\infty} \sum_{i=1}^{n} \pi(10-2h_i)^2\Delta h \\
&= \int_0^5 \pi(10-2h)^2\, dh \text{ cm}^3 \\
&= \pi \int_0^5 (100 - 40h + 4h^2)\, dh \text{ cm}^3 \\
&= \pi\left(100h - 20h^2 + \frac{4}{3}h^3\right)\Big|_0^5 \\
&= \pi\left(500 - 20(5)^2 + \frac{4}{3}(5)^3\right) = \frac{500\pi}{3} \text{ cm}^3.
\end{aligned}$$

8. Set up and evaluate an integral that determines the volume of a cone of height H and radius R.
ANSWER:

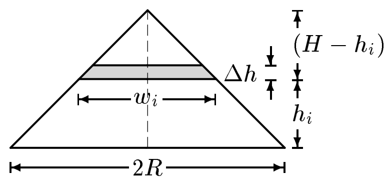


Figure 8.1.98

$$\frac{w_i}{2R} = \frac{H - h_i}{H}$$
$$w_i = (2R/H)(H - h_i)$$
$$r_i = (R/H)(H - h_i)$$

$$\begin{aligned}\text{Volume of cone} &= \lim_{n\to\infty} \sum_{i=1}^{n} \pi((R/H)(H-h_i))^2 \Delta h \\ &= \int_0^H \pi((R/H)(H-h))^2\, dh \\ &= \pi(R^2/H^2)\left(-\frac{(H-h)^3}{3}\right)\Big|_0^H = \frac{\pi}{3}R^2H.\end{aligned}$$

9. The integrals below represent the volume of either a hemisphere or a cone, and the variable of integration measures a length. In each case, say which shape is represented, and give the radius of the hemisphere or the height of the cone.
   (a)
$$\int_0^9 \pi(81 - h^2)\, dh$$
   (b)
$$\int_0^{18} \pi \left(\frac{x}{9}\right)^2 dx$$
   ANSWER:
   (a) Hemisphere with radius 9.
   (b) Cone with height 18 and radius 18/9=2.

10. A large chocolate bar has trapezoidal cross section shown in Figure 8.1.99.

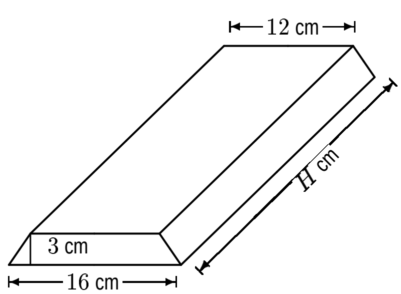


Figure 8.1.99

If you want the bar to have volume 1029 cm^3, what should H be? Set up and evaluate an appropriate integral.
ANSWER:

Volume of a trapezoid slice
$=\dfrac{16+12}{2}(3)\Delta h = 42\Delta h$ cm^3.

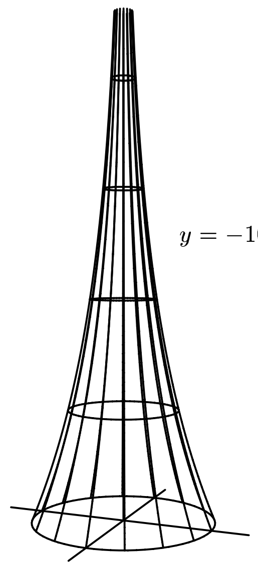
$$\begin{aligned}\text{Total volume} &= \lim_{\Delta h\to 0}\sum 42h\Delta h = \int_0^H 42h\,dh\\ &= \frac{42h^2}{2}\Big|_0^H = 21H^2 = 1029\\ H &= 7\text{ cm}\end{aligned}$$

Questions and Solutions for Section 8.2

1. Alice starts at the origin and walks along the graph of $y = \dfrac{x^2}{2}$ at a velocity of 10 units/second.
   (a) Write down the integral which shows how far Alice has traveled when she reaches the point where $x = a$.
   (b) You want to find the $x-$coordinate of the point Alice reaches after traveling for 2 seconds. Find upper and lower estimates, differing by less than 0.2, for this coordinate. Explain your reasoning carefully.

   ANSWER:
   (a) $L(a) = \displaystyle\int_0^a \sqrt{1+x^2}dx$
   (b) We want to find a such that $\displaystyle\int_0^a \sqrt{1+x^2}dx = 20$. Since $\sqrt{1+x^2}$ is increasing, the left hand side gives a lower estimate, the right hand side gives the upper estimate for this integral.
   With $n = 500$, RHS for $\displaystyle\int_0^{6.0} \sqrt{1+x^2}dx$ $=19.52\ldots$ so $\displaystyle\int_0^{6.0} \sqrt{1+x^2}dx < 20$.
   With $n = 500$, LHS for $\displaystyle\int_0^{6.1} \sqrt{1+x^2}dx = 20.07\ldots$ so $\displaystyle\int_0^{6.1} \sqrt{1+x^2}dx > 20$.
   Thus $6.0 < a < 6.2$ (more accurate than asked).

2. In a recent archaeological expedition, a scroll was discovered containing a description of a plan to build what appears to be the Tower of Babel. According to the manuscript, the tower was supposed to have a circular cross section and "go up to the heavens" (i.e., be infinitely high). A mathematician was consulted to solve some of the questions posed by the archaeologists. The mathematicians plotted half of the silhouette of the tower on a set of coordinate axis with the y–axis running through the center and discovered that it was approximated by the curve $y = -100\ \ln(\frac{x}{5})$. Please answer the questions posed by the archaeologists:
   (a) Would such a tower have finite volume? Justify your work completely.



(b) The manuscript mentions that 4200 cubic "shrims" (Babel's unit of length) of stones were available to build the tower. The base of the tower was to have radius 4 shrims. Did they have enough? Explain.

ANSWER:

(a) If we slice horizontally, the volume of one disk with radius x and thickness dy is $\pi(x^2)dy$. Now $y = -100\ln(\frac{x}{5})$, so $\frac{x}{5} = e^{-\frac{y}{100}}$ and $x = 5e^{-\frac{y}{100}}$. Consequently,

$$\begin{aligned}\text{Volume} &= \pi\int_0^\infty (5e^{-\frac{y}{100}})^2\,dy \\ &= \pi\int_0^\infty 25e^{-\frac{y}{50}}\,dy \\ &= -1250\pi e^{-\frac{y}{50}}\Big|_0^\infty \\ &= 1250\pi\end{aligned}$$

(b) If we had done the above integral in "shrims", then we would have found the volume of a tower with radius of 5 shrims at the base, since $-100\ln\dfrac{x}{5}$ passes through the x-axis at $x = 5$. Its volume would be $1250\pi < 4200$. Since a tower of base radius 4 shrims would require less than 1250π cubic shrims, they would have enough stone.

3. Consider the region bounded by $y = e^x$, the x-axis and the lines $x = 0$ and $x = 1$. Find the volume of the solid whose base is the given region and whose cross sections perpendicular to the x-axis are isosceles right triangles with hypotenuses lying in the region.

ANSWER:

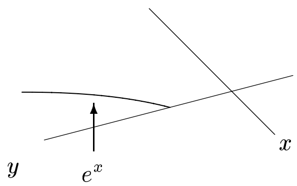


Figure 8.2.100

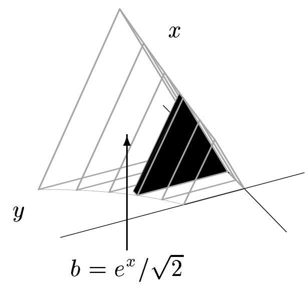


Figure 8.2.101

We slice perpendicular to the x-axis. As stated in the problem, the cross-sections obtained thereby are isosceles right triangles with hypotenuse length $y = e^x$. Let b be the length of the base. Then $b^2 + b^2 = e^{2x}$ by the Pythagorean Theorem, which means that $2b^2 = e^{2x}$, whence $b = \dfrac{e^x}{\sqrt{2}}$. Hence, the volume of one triangular section is $\dfrac{lw}{2}\,dx = \dfrac{b^2}{2}\,dx = \dfrac{e^{2x}}{4}\,dx$. Therefore,

$$\begin{aligned}\text{Volume} &= \int_0^1 \frac{e^{2x}}{4}\,dx \\ &= \left.\frac{e^{2x}}{8}\right|_0^1 = \frac{e^2-1}{8} \approx 0.80.\end{aligned}$$

4. (a) You love the function $y = \frac{2}{3}x^{\frac{3}{2}}$ and you also love the number 4. What is the arc length of this curve from $x = 0$ to $x = 4$?

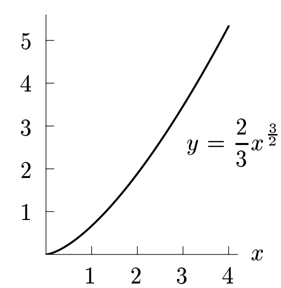


Figure 8.2.102

(b) You have a gold chain which is exactly 4 feet long. As a tribute to your favorite function you want to mount your chain in the shape of $y = \frac{2}{3}x^{\frac{3}{2}}$ from $x = 0$ to $x = 4$ on a beautiful rectangular piece of rosewood. If the lower left corner is labeled with the coordinate $(0, 0)$ and the upper right corner is labeled with the coordinate $(4, \frac{16}{3})$ and a unit on the x-axis and a unit on the y-axis represent the same number of feet, what are the dimensions of the piece of wood in feet?

ANSWER:

(a)

$$\begin{aligned}\text{Arclength} &= \int_0^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^4 \sqrt{1 + (\sqrt{x})^2}\,dx \\ &= \int_0^4 \sqrt{1+x}\,dx = \left.\frac{2}{3}(1+x)^{\frac{3}{2}}\right|_0^4 \\ &= \frac{2}{3}(5^{\frac{3}{2}} - 1) \\ &\approx 6.787\end{aligned}$$

(b) We have 4 feet for $\frac{2}{3}(5\sqrt{5}-1)$ units. Hence, the dimensions of the wood are $\dfrac{4}{\frac{2}{3}(5\sqrt{5}-1)}\cdot 4 \approx 2.36$ ft by $\dfrac{4}{\frac{2}{3}(5\sqrt{5}-1)}\left(\dfrac{2}{3}\right)\cdot 4^{\frac{3}{2}} = 3.143$ ft.

5. The circle $x^2+y^2=a^2$ is rotated around the y-axis to form a solid sphere of radius a. A plane perpendicular to the y-axis at $y=a/2$ cuts off a spherical cap from the sphere. What fraction of the total voume of the sphere is contained in the cap?

ANSWER:

A cross-section of the sphere is shown in Figure 8.2.103.

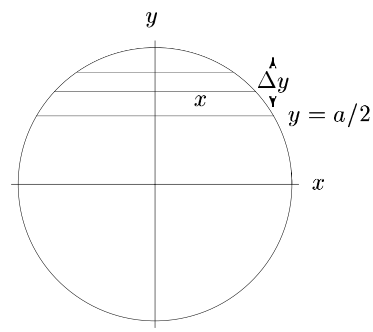


Figure 8.2.103

As can be seen from Figure 8.2.103,

$$\begin{aligned}\text{Volume of cap} &= \int_{a/2}^{a} \pi x^2 dy \\ &= \int_{a/2}^{a} \pi(a^2-y^2)dy \\ &= \pi\left[a^2y-\frac{y^3}{3}\right]\Bigg|_{a/2}^{a} \\ &= \frac{5\pi a^3}{24},\end{aligned}$$

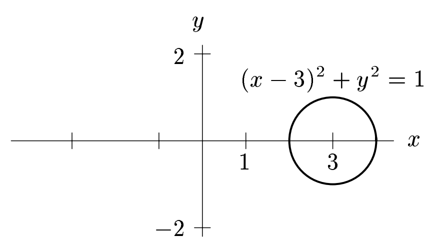
whereas the volume of the entire sphere is given by the formula

$$\text{Volume of sphere} = \frac{4}{3}\pi a^3.$$

So we have

$$\frac{\text{Volume of cap}}{\text{Total volume}} = \frac{5\pi a^3/24}{4\pi a^3/3} = \frac{5}{32}.$$

6. (a) Set up a Riemann sum approximating the volume of the torus (donut) obtained by rotating the circle $(x-3)^2+y^2=1$ about the y-axis.



(b) Write an integral representing the volume of the torus. (Do not evaluate the integral.)
ANSWER:

(a) Consider a horizontal slice at height y with thickness Δy, as shown in Figure 8.2.104.

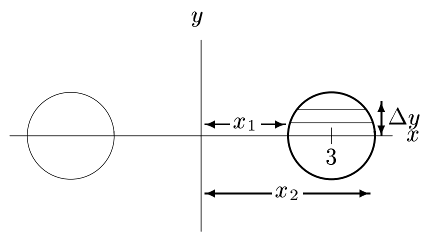


Figure 8.2.104

This slice will have volume of

$$\begin{aligned}\Delta V &\approx \pi(x_2^2 - x_1^2)\Delta y\\ &= \pi\left((3+\sqrt{1-y^2})^2 - (3-\sqrt{1-y^2})^2\right)\Delta y.\end{aligned}$$

So the entire volume will be the sum of all such slices,

$$\text{Volume} \approx \sum \pi\left((3+\sqrt{1-y^2})^2 - (3-\sqrt{1-y^2})^2\right)\Delta y.$$

(b) In the limit as the number of slices goes to infinity, we have

$$\text{Volume} = \int_{-1}^{1}\left((3+\sqrt{1-y^2})^2 - (3-\sqrt{1-y^2})^2\right)dy.$$

7. Find the volume of the solid obtained by rotating the region bounded by $y = x^3, x = 0, y = 0$, and $y = -8$ around $y = 3$.
ANSWER:

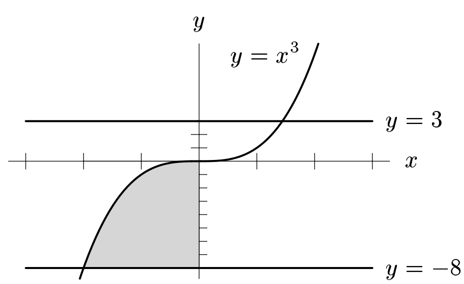


Figure 8.2.105

$$\begin{aligned}\text{Volume of slice} &\approx \pi r_{\text{out}}^2\Delta x - \pi r_{\text{in}}^2\Delta x\\ &= \pi(11)^2\Delta x - \pi(3-x^3)^2\Delta x\\ \text{Total volume} &= \int_{-2}^{0}(\pi(121) - \pi(9-6x^3+x^6))\,dx\\ &= \pi\int_{-2}^{0}(121-9+6x^3-x^6)\,dx\\ &= \pi\int_{-2}^{0}(112+6x^3-x^6)\,dx\\ &= \pi\left(112x + \frac{3x^4}{2} - \frac{x^7}{7}\right)\Big|_{-2}^{0} \approx 182\pi\end{aligned}$$

8. A plastic travel mug is made in two parts, the cup and the base. The cup part has outside shape $y = \sqrt{x}$ and inside shape $y = \sqrt[4]{2x}$, cut off at $x = 4$ as shown in Figure 8.2.106.

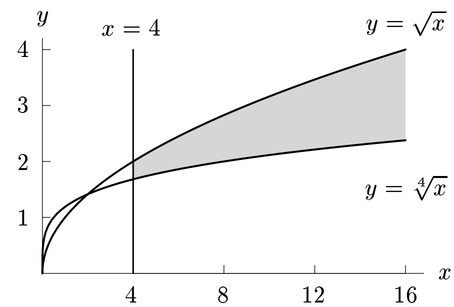


Figure 8.2.106

(a) If the cup is 12 cm tall, find the volume of plastic needed to make the cup.
(b) If the base is 7 cm in diameter and 1 cm thick, how much plastic is needed to make the entire mug?

ANSWER:

(a)

$$\begin{aligned}\text{Volume of cup } &= \int_4^{16} \pi \left((\sqrt{x})^2 - (\sqrt[4]{2x})^2 \right) \, dx \\ &= \int_4^{16} \pi (x - \sqrt{2x}) \, dx \\ &= \pi \left(\frac{x^2}{2} - \frac{(2x)^{3/2}}{3} \right) \Bigg|_4^{16} \approx 67.2\pi \approx 211.12 \text{ cm}^3\end{aligned}$$

(b) Base is a cylinder with radius = 3.5 cm and height = 1 cm.

$$\text{Volume } = \pi r^2 h = \pi (3.5)^2 (1) \approx 38.48 \text{ cm}^3$$

Total plastic needed for entire mug =

$$211.12 + 38.48 = 249.6 \text{ cm}^3.$$

9. (a) Find the length of the curve given parametrically as $x = 2\sin t$ and $y = 2\cos t$ for $0 \le t \le 2\pi$.
(b) How does the length change if the coefficients of x and y are doubled (meaning $x = 4\sin t$ and $y = 4\cos t$)?

ANSWER:

(a)

$$\text{Length of curve } = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

$$\frac{dx}{dt} = 2\cos t \qquad \frac{dy}{dt} = -2\sin t$$

$$\text{Length } = \int_0^{2\pi} \sqrt{4\cos^2 t + 4\sin^2 t} \, dt = \int_0^{2\pi} 2 \, dt = 2t\Big|_0^{2\pi} = 4\pi.$$

(b)

$$\frac{dx}{dt} = 4\cos t \qquad \frac{dy}{dt} = -4\sin t$$

$$\text{Length } = \int_0^{2\pi} \sqrt{16\cos^2 t + 16\sin^2 t} \, dt = \int_0^{2\pi} 4 \, dt = 4t\Big|_0^{2\pi} = 8\pi.$$

So the length is doubled when the coefficients are doubled.

Questions and Solutions for Section 8.3

1. When an oil well burns, sediment is carried up into the air by the flames and is eventually deposited on the ground. Less sediment is deposited further away from the oil well. Experimental evidence indicates that the density (in tons/square mile) at a distance r from the burning oil well is given by

$$\frac{7}{1+r^2}.$$

(a) Find a Riemann sum which approximates the total amount of sediment which is deposited within 100 miles of the well. Explain your work.

(b) Find and evaluate an integral which represents this total deposit.

ANSWER:

(a) The area where sediment is deposited is a disk with radius 100. The density of sediment deposited is a function of the distance from the burning oil well, so we can cut the disk into rings of radius r from the center with width Δr. For narrow enough rings, the density is nearly constant. Thus, the sediment in each ring equals the area of the ring, which is about $2\pi r\Delta r$, multiplied by its density (i.e. for a ring with radius r we get $\frac{7}{1+r^2}\cdot 2\pi r\Delta r$). We can add up the areas of these rings multiplied by their densities to find the total amount of sediment deposited. The Riemann sum we want is:

$$\sum \frac{14\pi r\Delta r}{1+r^2}.$$

(b) The integral is $\displaystyle\int_0^{100} \frac{14\pi r}{1+r^2}\,dr$. Set $u = 1+r^2$, $du = 2r dr$ and substitute to get

$$\begin{aligned}\int_0^{100} \frac{14\pi r}{1+r^2}\,dr &= \int_1^{10001} \frac{7\pi du}{u}\\ &= 7\pi \ln|u|\Big|_1^{10001}\\ &= 7\pi\ln(10001) \approx 202.5 \text{ tons.}\end{aligned}$$

2. A straight road goes through the center of a circular city of radius 5 km. The density of the population at a distance r (in km) from the road is well approximated by

$$D(r) = 20 - 4r$$

(in thousand people per km^2). Find the total population of the city.

ANSWER:

A strip of the city, of width Δr, that lies parallel to the road at distance r from it will have length $\sqrt{25-r^2}$ and hence area $\sqrt{25-r^2}\Delta r$. The population in the strip is thus $(20-4r)\sqrt{25-r^2}\Delta r$. The integral representing the population of the city is thus:

$$\text{Population } = 2\int_0^5 (20-4r)\sqrt{25-r^2}\,dr \approx 226{,}000 \text{ people}.$$

3. The globular cluster M13 is a spherical distribution of stars which orbits our galaxy. Suppose that the density of stars in the cluster is purely a function of distance r from the center of the cluster and is given as

$$\rho(r) = \left(1+\left(\frac{r}{100}\right)^3\right)^{-5} \frac{\text{stars}}{(\text{ly})^3}$$

where r is measured in light-years, and $0 \le r \le 100ly$. (One light-year is the distance light travels in one year; "light-year" is abbreviated as "ly".)

(a) Set up a Riemann sum which approximates the total number of stars in M13.

(b) Set up an integral whose value is the exact number of stars in M13.

(c) Evaluate the integral in part (b) to compute the total number of stars in M13. If you evaluate the integral numerically (e.g., on your calculator), you will get full credit for part (c) only if you:

(i) Give a number which is within 10,000 stars of the true value of the integral.

(ii) Give justification for your answer being accurate to within 10,000 stars.

ANSWER:

(a) To get full credit (12 points), you had to write:

$$\sum_{1\le n\le N} (4\pi \cdot r_n^2 \cdot \left(1+\left(\frac{r_n}{100}\right)^3\right)^{-5} \cdot \Delta r$$

where $\Delta r = \frac{100}{N}$, and where $r_n = n \cdot \frac{100}{N}$. By the way, this equation is derived by observing that the number of stars in a spherical shell at radius r and of thickness Δr is approximately equal to:

$$\text{(Volume of shell)} \times \left(1+\left(\frac{r}{100}\right)^3\right)^{-5},$$

where the volume of the shell is approximately Area$\times\Delta r$, where the Area $= 4\cdot\pi\cdot r^2$.

(b) For full credit (7 points), you had to write:

$$\int_0^{100} 4\cdot\pi\cdot r^2 \cdot \left(1+\left(\frac{r}{100}\right)^3\right)^{-5} \cdot dr.$$

(c) Full credit was (6 points). The integral could be computed with the substitution $u = \left(\frac{r}{100}\right)^3$. Then, $du = 3dr \cdot \frac{r^2}{(100)^3}$. Thus, the integral above is equal to

$$\left(4\cdot\frac{\pi}{3}\right)\cdot(100)^3\cdot\int_0^1 (1+u)^{-5}\cdot du.$$

This last integral can be done using the fundamental theorem of calculus with the observation that

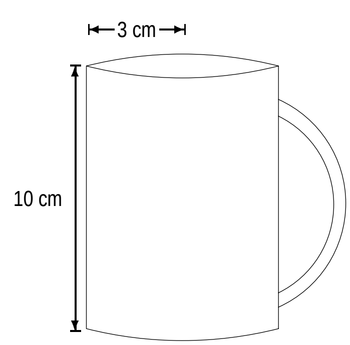
$$(1+u)^{-5} = \frac{d}{du}\left(-\frac{1}{4}\cdot(1+u)^{-4}\right).$$

Thus, the final answer is

$$\left(4\cdot\frac{\pi}{3}\right)\cdot\frac{(100)^3}{4}\cdot\left(1-\frac{1}{16}\right) = 10^6\cdot\frac{\pi}{3}\cdot\frac{15}{16} \approx 981,748.$$

Many people tried to do this problem numerically. Only two people gave a believable justification for their answer being within 10,000 of the correct value. The integrand is not always convex, nor concave nor increasing nor decreasing. You must split the integral into two or more pieces which are solely increasing or solely decreasing (or, solely concave or solely convex). Then, you can use left and right sums (or midpoint and trapezoid sums) to estimate the contribution from each piece.

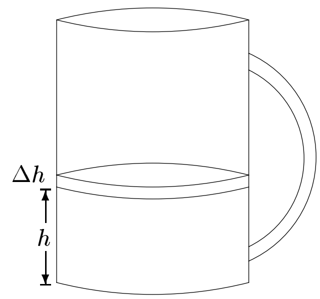
4. A cylindrically-shaped mug with a 3 cm radius and a 10 cm height is filled with tea. You have added some sugar to the tea, which tends to settle to the bottom of the mug. It turns out that the density ρ of sugar (in gm/cm^3) in the tea, as a function of the height, h, in cm, above the bottom of the mug, is given by the formula $\rho(h) = 0.01(10-h)$.



(a) Write a Riemann sum that approximates the total mass of sugar (in grams) in the mug of tea. Show your work clearly.
(b) Turn the Riemann sum into the integral that gives the exact amount of sugar in the mug, and evaluate the integral.

ANSWER:

(a) Slice horizontally.
Volume of slice at height h is $\pi(r^2)\Delta h$
Sugar in slice $= \pi(3^2)\Delta h(0.01(10-h))$
Total sugar $= \sum_{\text{all slices}} 9\pi(0.01)(10-h)\Delta h$



(b) Total sugar $= \int_0^{10} 0.09\pi(10-h)dh$

$= 0.09\pi \left[10h - \frac{h^2}{2}\right]_0^{10}$
$= 0.09\pi(50)$
$= 4.5\pi$ gm

5. After Mt. St. Helens erupted in 1980, it was found that ash was spread in decreasing density as a function of distance r from the center of the crater. Say that the density ρ of ash at a distance r (meters) from the center of the crater is given as follows:

$$\rho(r) = \frac{2000}{1+r^2} \quad \text{kg/m}^2$$

(a) Write a Riemann sum that approximates the total mass of ash deposited within a 1000-meter radius of the center of the crater.
(b) Turn your Riemann sum from part (a) into a definite integral and evaluate that integral to find the exact value of the total mass of ash within 1000 meters of the center of the crater.

ANSWER:

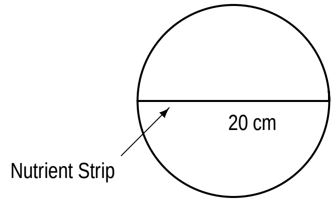
(a) Partition $0 \le r \le 1000$ into rings of width $\Delta r = \frac{1000}{N}$. The mass of ash in the i-th ring is

$$m_i = \underbrace{A_i}_{\text{Area}} \underbrace{\rho_i}_{\text{Density}} = (2\pi r_i \Delta r)\frac{2000}{1+r_i^2}.$$

$$\text{Total mass} = \sum_{i=1}^{N} \frac{4000\pi r_i}{1+r_i^2}\Delta r$$

(b) $\int_0^{1000} \frac{4000\pi r}{1+r^2}dr = 2000\pi \int_1^{1,000,001} \frac{dw}{w} = \left[2000\pi \ln w\right]_1^{1,000,001} \approx 86,805$ kg

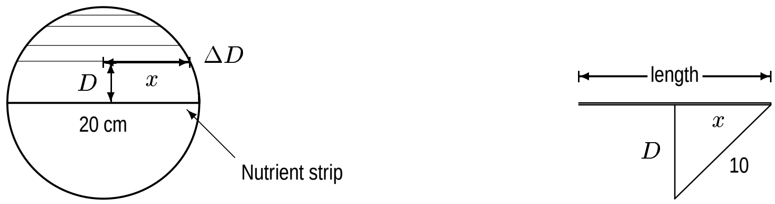
6. A thin strip of nutrients 20 cm long is placed in a circular petri dish of radius 10 cm, as shown. The population density of bacteria in the disk after 3 hours is given by $\frac{80}{D+4}$ bacteria/cm^2 where D is the distance (in cm) to the nutrient strip.



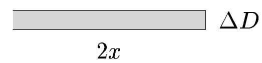
(a) Write a general Riemann sum that approximates the number of bacteria in the dish 3 hours after the nutrient strip is introduced. You must explain your reasoning carefully and clearly. Any variables you use must be clearly identified in words.
(b) Write an integral that gives the number of bacteria in the petri dish 3 hours after the nutrient strip has been introduced. You need not evaluate.

ANSWER:

(a) Partition $[0, 10]$ into n equal pieces each of width ΔD. Let D_i be in the i-th interval.



Use Pythagorean Theorem:
$x^2 + D^2 = 100$
$x = \sqrt{100 - D^2}$
or put axes through the center of the circle.
$x^2 + y^2 = 100$
$|y|$ corresponds to D,
$2|x|$ corresponds to length.



$$\begin{pmatrix}\text{\# of bacteria} \\ \text{in a strip}\end{pmatrix} \approx (\text{density})(\text{area})$$

$$= \frac{80}{D_i + 4}\frac{\text{bact}}{(\text{cm})^2}2x_i\Delta D(\text{cm})^2$$

$$= \frac{80}{D_i + 4}2\sqrt{100 - D_i^2}\Delta D$$

$$\text{Total \# of bacteria} \approx \underbrace{2}_{\substack{\text{for the other 1/2}\\ \text{of the petri dish}}} \sum_{i=1}^{n} \frac{80}{D_i + 4}2\sqrt{100 - D_i^2}\Delta D$$

Common errors:
(1) The area of a strip is NOT $2\pi r$.
(2) D is the distance from a line–so don't cut into concentric circles.
(3) A general Riemann sum should not be a sum from 1 to 10 or 0 to 9 or 1 to 3.
(4) Many people introduced new variables; didn't say what they were, and then lost track of them.

(b)
$$\lim_{n\to\infty} 2\sum_{i=1}^{n} \frac{80}{D_i + 4}2\sqrt{100 - D_i^2}\Delta D = 2\int_0^{10} \frac{80}{D + 4}2\sqrt{100 - D^2}dD = 320\int_0^{10} \frac{\sqrt{100 - D^2}}{D + 4}dD$$

Common error: You don't want to integrate from $0 \to 20$ since D can only be as big as 10.
An odd error: Some folks integrated from 0 to 3. I don't know why.

The integral should involve the variable D and no other variable.

7. The density of cars (in cars per mile) down a 20-mile stretch of the Massachusetts Turnpike starting at a toll plaza is given by

$$\rho(x) = 500 + 100\sin(\pi x)$$

where x is the distance in miles from the toll plaza and $0 \le x \le 20$.

(a) Write a Riemann sum which estimates the total number of cars down the 20-mile stretch. Explain your reasoning.
(b) Convert this sum to an integral and evaluate it.

ANSWER:

(a) Divide the 20-mile stretch into n pieces of length $\frac{20}{n} = \Delta x$. An estimate of the number of cars in any segment is simply $\rho(x_i)\Delta x$, where x_i is a point in the i^{th} segment. The total number of cars will therefore be approximately

$$\sum_{i=0}^{n-1} \rho(x_i)\Delta x.$$

(b) Letting n go to ∞, we get the integral

$$\begin{aligned}\int_0^{20} \rho(x)\,dx &= \int_0^{20} (500 + 100\sin(\pi x))\,dx \\ &= 500x - \frac{100}{\pi}\cos(\pi x)\Big|_0^{20} \\ &= 10000 - \frac{100}{\pi}\cos(20\pi) + \frac{100}{\pi} \\ &= 10000\end{aligned}$$

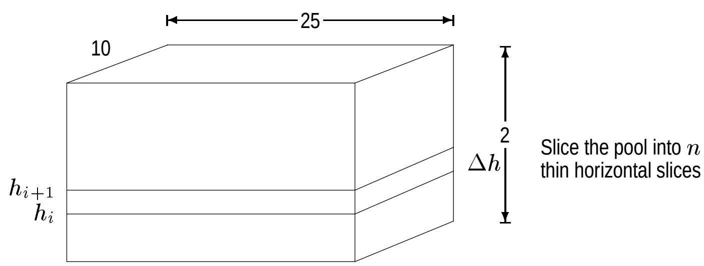
8. A chlorine solution is poured over the surface of a rectangular swimming pool that is 25 meters long, 10 meters wide and 2 meters deep everywhere. Before the circulating pumps in the pool are turned on, it is discovered that the density of the chlorine solution at a height h meters above the bottom of the pool is given by $\rho(h) = 100h$ gm/m^3. In other words, the chlorine solution has distributed itself so that its density increases linearly from the bottom of the pool.

(a) Write a Riemann sum that approximates the total mass of chlorine solution in the pool.

(b) Turn the Riemann sum in part (a) into a definite integral that gives the exact total mass of chlorine solution in the pool, and evaluate the integral.

ANSWER:

(a) Slice the pool into n thin horizontal slices.



Find the mass over the i-th slice (shaded above). We will assume that density is approximately constant over a very thin slice. So, over the ith slice, density $\approx \rho(h_i) = 100h_i$ gm/cm^3.
mass=density$\times$volume. The volume of the i-th slice is $\ell \cdot w \cdot h = 25 \cdot 10 \cdot \Delta h = 250\Delta h$.
So, the mass of the i-th slice $= (100h_i)(250\Delta h) = 25,000h_i\Delta h$.
To find total mass, we sum up the masses of all n slices:

$$\text{Mass}= \sum_{i=0}^{n-1} 25,000h_i\Delta h$$

Alternatively, if we had let h_i represent the <u>top</u> height of our slice, then the bottom height would be h_{i-1}, and our sum would be: $\sum_{i=1}^{n} 25,000h_i\Delta h$. (Either answer is fine.)

(b) As the number of slices approaches ∞, or $\Delta h \to 0$, we get:

$$\text{Mass}= \lim_{n\to\infty}\left(\sum_{i=0}^{n-1} 25,000h_i\Delta h\right) = \int_0^2 25,000h\,dh$$

Evaluate:

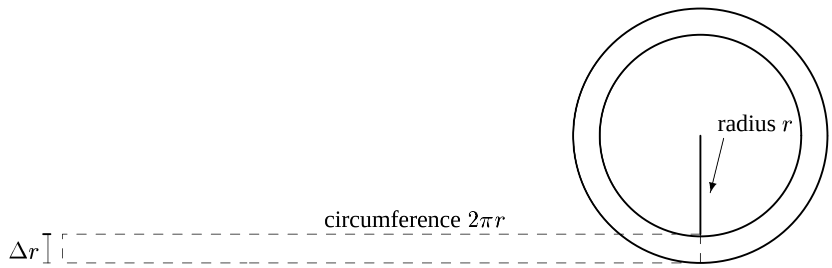
$$= 12,500h^2\Big|_0^2 = 50,000 \text{ g or } 50 \text{ kg}$$

9. Circle City is circular with a radius of four miles. Right in the center is a circular park with diameter one mile. No one lives in the park. Elsewhere the population density is $4000(5-r)$ people per square mile, where r is the distance from the center in miles.
   (a) What is the total population of Circle City? Explain how you get your answer.
   (b) What is the average density of population of the whole city?

   ANSWER:

   (a) Consider concentric slices of the city of width Δr. A slice at distance r from the center has approximate area $2\pi r\Delta r$. Assuming that the population is constant on each slice, the population of a slice at distance r from the center is about

$$\text{Density} \cdot \text{Area} = 4000(5-r)\cdot 2\pi r\Delta r.$$



So the total population is about

$$\sum 4000(5-r)\cdot 2\pi r\Delta r,$$

where r runs between 1 and 4. As $\Delta r \to 0$, this Riemann sum becomes the integral

$$\begin{aligned}\int_1^4 4000(5-r)2\pi r\,dr &= \int_1^4 (40000\pi r - 8000\pi r^2)\,dr \\ &= 20000\pi r^2 - \frac{8000\pi r^3}{3}\Big|_1^4 \\ &= 320000\pi - \frac{512000\pi}{3} - 20000\pi + \frac{8000\pi}{3} \\ &= 132000\pi \\ &\approx 415000.\end{aligned}$$

   (b) Average density of population $= \frac{\text{total population}}{\text{area of city}} = \frac{132000\pi}{\pi(4^2-1)} = 8800.$

10. A 5-gram drop of thick red paint is added to a large can of white paint. A red disk forms and spreads outward, growing lighter at the edges. Since the amount of red paint stays constant through time, the density of the red paint in the disk must vary with time. Suppose that its density p in gm/cm^2 is of the form

$$p = k(t)f(r)$$

for some functions $k(t)$ of time and $f(r)$ of the distance to the center of the disk.

   (a) Let $R(t)$ be the radius of the disk at time t. Write an integral that expresses the fact that there are 5 grams of red paint in the disk. Explain.
   (Hint: Divide the disk into thin concentric rings and ask yourself how much paint there is in each ring.)
   (b) For fixed r write down an integral for the average density of red paint at a distance r from the center of the disk from 0 to T seconds.

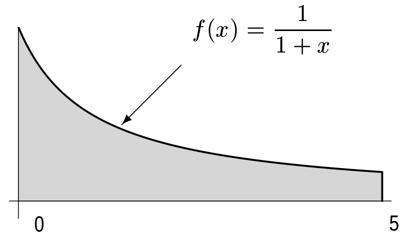
   ANSWER:

   (a) $5 = k(t)\displaystyle\int_0^{R(t)} f(r)2\pi r\,dr$, since each ring has area $2\pi r\,dr$.
   (b)

$$\text{average density} = \frac{1}{T}\int_0^T f(r)k(t)\,dt = \frac{f(r)}{T}\int_0^T k(t)\,dt.$$

We can factor out $f(r)$ from the integral because $f(r)$ doesn't depend on t.

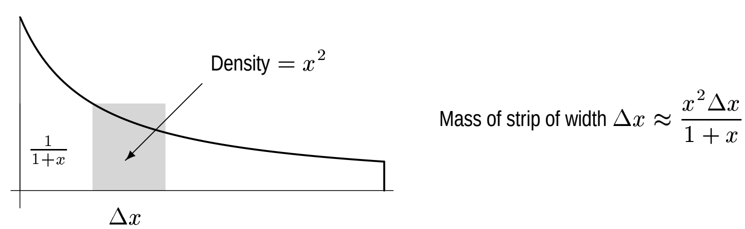
11. A flat metal plate is in the shape determined by the area under the graph of $f(x) = \dfrac{1}{1+x}$ between $x = 0$ and $x = 5$. The density of the plate x units from the y-axis is given by x^2 grams/cm^2.



(a) Write down a Riemann sum with 5 terms which approximates the total mass. Is your approximation an underestimate or an overestimate? Explain.
(b) Write down a definite integral which gives the exact value of the total mass of the plate.
(c) Evaluate the integral you found in part (b).

ANSWER:

(a) The height at distance x from the y-axis is $1/(1+x)$ cm, and the density is x^2 gm/cm^2, so the plate has “linear density” $x^2/(1+x)$ gm/cm at position x.



We can suppose, as an approximation, that this density is constant on each centimeter of the plate; so the total mass is approximately

$$\sum_{i=0}^{4} \frac{x_i^2}{1+x_i}(1) = 0 + 1/2 + 4/3 + 9/4 + 16/5 = 437/60 \approx 7.2833 \text{ gm.}$$

Since we have used $\text{LEFT}(5)$ on $\dfrac{x^2}{1+x}$, which is an *increasing* function for $x > 0$, our approximation is an *underestimate*.

(b) $M = \displaystyle\int_0^5 \frac{x^2}{1+x}\,dx$

(c) Divide the denominator into the numerator. Since

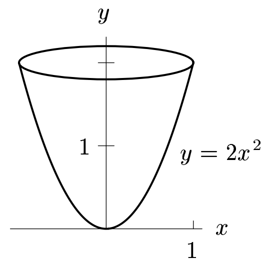
$$\frac{x^2}{1+x} = \frac{(x^2-1)+1}{x+1} = x - 1 + \frac{1}{1+x},$$

we find that

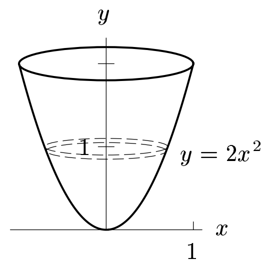
$$\int_0^5 \frac{x^2}{1+x}\,dx = \left(\frac{x^2}{2} - x + \ln(1+x)\right)\Bigg|_0^5 = \frac{15}{2} + \ln 6 \approx 9.292 \text{ gm.}$$

12. An object is in the shape drawn below; its boundary is obtained by rotating the parabola $y = 2x^2$ (for $0 \le x \le 1$) around the y axis. (Units are in centimeters.) Suppose that the density of this object varies with height according to the rule $\rho(y) = 8 \cdot (2 - y)$ grams/cm^3.

(a) Set up a Riemann sum which computes (approximately) the weight in grams of this object.
(b) Compute the exact weight in grams of this object.



ANSWER:



(a) $\displaystyle\sum_{y=0}^{2} \pi \cdot \frac{y}{2} \cdot 8(2-y)\Delta y$

(b) $$\begin{aligned}\int_0^2 \pi(8y-4y^2)dy &= \pi\left(4y^2 - \frac{4}{3}y^3\right)\Big|_0^2 \\ &= \pi\left(16 - \frac{4}{3}\cdot 8\right) \\ &= \frac{16}{3}\pi\end{aligned}$$

13. The density of a compressible liquid is $40(5-h)$ kg/m^3 at a height of h meters above the bottom.
    (a) The liquid is put in the container shown below, whose cross sections are isosceles triangles. It has straight sides, and looks like a triangular prism. How many kg will it hold when placed as shown in Figure 8.3.107, resting on the triangular side?
    (b) How many kg will it hold if it is placed (with some support, of course) as shown in Figure 8.3.108?

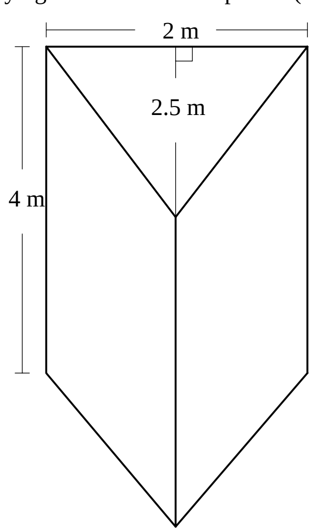


Figure 8.3.107: Container on End

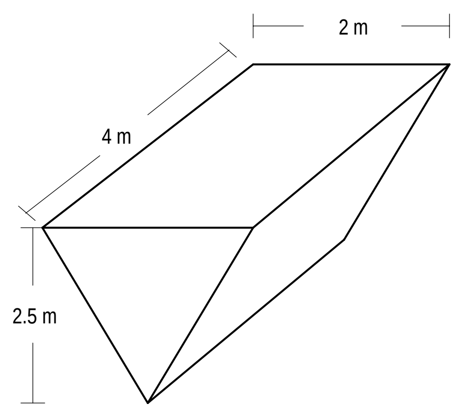


Figure 8.3.108: Container on Side

ANSWER:

(a) The volume of a single horizontal slice is Area $\times \Delta h = \frac{1}{2}(2)(2.5)\Delta h$. The mass of such a slice is simply $40(5-h)(2.5)\Delta h$ kg. Hence

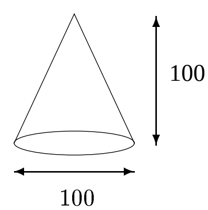
$$\begin{aligned}\text{The total mass} &= \int_0^4 (2.5)(40)(5-h)\,dh\\ &= \int_0^4 (500-100h)\,dh\\ &= 500h - 50h^2\Big|_0^4 = 2000-800\\ &= 1200 \text{ kg}\end{aligned}$$

(b) The volume of a single horizontal slice at height h is $4\cdot\dfrac{2}{2.5}\cdot h\cdot\Delta h$. The mass of such a slice will be $128(5-h)\Delta h$. Hence,

$$\begin{aligned}\text{The total mass} &= \int_0^{2.5} (128(5-h)h\,dh\\ &= \int_0^{2.5} (640h-128h^2)\,dh\\ &= \left(320h^2 - \frac{128}{3}h^3\right)\Big|_0^{2.5}\\ &= 2000 - \frac{2000}{3}\\ &\approx 1333\text{kg}\end{aligned}$$

Questions and Solutions for Section 8.4

1. The Great Cone of Haverford College is a monument built by freshmen during a customs week long, long ago. It is 100 ft. high and its base has a diameter of 100 ft. It has been built from bricks (purportedly made of straw) which weigh 2 lbs/ft^3. Use a definite integral to approximate the amount of work required to build the Cone.



ANSWER:

Work = Force × Distance, so the amount of work necessary to raise a volume of brick to height x above the ground is $2x$ foot-pounds per cubic foot. We now think of the cone as being made of a series of thin horizontal layers, each of height Δx. Such a layer, at height x, would have radius $r(x) = 50 - x/2$; its area would therefore be $\pi r(x)^2$, and its volume approximately $\pi r(x)^2\Delta x$, since Δx is small. Raising a layer of radius $r(x)$ to height x would thus take $2x\pi r(x)^2\Delta x$ foot-pounds; so if we have n such layers, we can construct a Riemann sum approximation of the total work done as follows:

$$W \approx \sum_{i=0}^{n-1} 2\pi x_i r(x_i)^2\Delta x = \sum_{i=0}^{n-1} 2\pi x_i\left(50-\frac{x_i}{2}\right)^2\Delta x \quad \text{foot-pounds}$$

The corresponding definite integral is

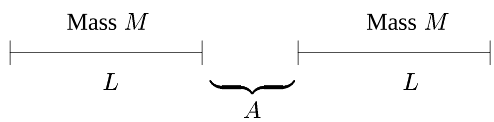
$$\begin{aligned}\int_0^{100} 2x\pi\left(50-\frac{x}{2}\right)^2\,dx &= 2500\pi x^2 - \frac{100}{3}\pi x^3 + \frac{\pi}{8}x^4\Big|_0^{100}\\ &= 12500000\frac{\pi}{3}\\ &\approx 1.3\times 10^7 \quad \text{foot-pounds.}\end{aligned}$$

2. The force of gravitational attraction between a thin rod of mass M and length L and a particle of mass m lying on the same line as the rod at a distance of A from one of the ends is

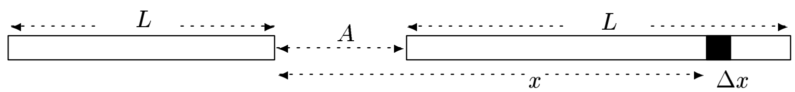
$$\frac{GmM}{A(L+A)}.$$

Use this result to set up an integral for the total force due to gravity between two thin rods, both of mass M and length L, lying along the same line and separated by a distance A. You need not evaluate the integral. Explain what you are doing.

[Hint: Divide one of the rods into small pieces, each of length dx and mass $\frac{M}{L}\,dx$. Apply the formula above to each of the pieces, then form a Riemann sum. You know the rest…]



ANSWER:



Consider a small chunk of the right-hand rod, of length Δx and at distance x from the right end of the left rod; it has mass $\dfrac{M}{L}\Delta x$. By the formula given, the gravitational attraction between the left rod and this piece is approximately $\dfrac{GM^2\Delta x}{Lx(L+x)}$. We can therefore approximate the gravitational attraction between the two rods by adding up the contributions from each of the n pieces of the right rod:

$$F \approx \sum_{i=0}^{n-1} \frac{GM^2\Delta x}{Lx_i(L+x_i)}, \qquad \text{where } \Delta x = \frac{L}{n}.$$

The corresponding definite integral is then:

$$F = \int_A^{A+L} \frac{GM^2\,dx}{Lx(L+x)}.$$

3. Find the work done lifting a 10-lb bag of sugar 3.5 feet off the floor.

   ANSWER:

   Work = force · distance = 10 lbs · 3.5 ft = 35 ft-lbs.

4. If it is known to take 147 joules of work to lift a box 2.5 meters off the floor, how heavy must the box be?

   ANSWER:

   Let x = weight of the box.

   Force due to gravity is mg where $g = 9.8$ m/sec^2.

   Work = $F \cdot g = x$ kg · 9.8 m/sec^2 · 2.5 m = 147 joules.

   So the box weighs 6 kg.

5. A bridge worker needs to pull a 15-meter uniform cable with mass 4 kg/meter up to the work platform. How much work is needed?

   ANSWER:

   Gravitational force per meter of cable is

$$(4\text{ kg})(9.8\text{ m/sec}^2) = 39.2\text{ newtons}$$

If $\Delta y =$ the length of a small section of cable, work done on the small section

$$\approx (39.2\Delta y\text{ newtons})(y\text{ meters}) = 39.2y\Delta y\text{ joules}$$

$$\begin{aligned}\text{Work done} &\approx \sum 39.2y\Delta y\text{ joules}\\ &= \int_0^{15} (39.2y)\,dy = \frac{39.2y^2}{2}\bigg|_0^{15}\\ &= 4410\text{ joules}\end{aligned}$$

6. If the worker in Exercise 5 now needs to pull an identical cable up twice the original distance, will the work be twice the original work?
   ANSWER:

$$\text{Work done} = \int_0^{30} (39.2y)\,dy = \frac{39.2}{2}y^2\Big|_0^{30}$$
$$= 17640 \text{ joules}$$

   No, the work is four times as much if the distance is doubled.

7. A swimming pool has shape as shown in Figure 8.4.109.

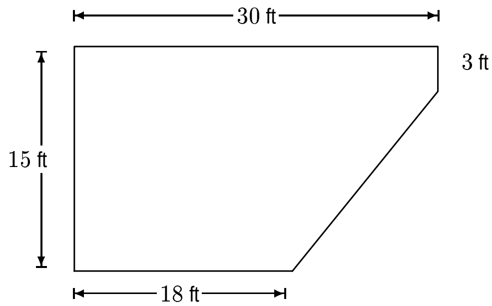


Figure 8.4.109

   If the pool is 10 ft deep, how much work does it take to pump all the water out?

   (Note: water weighs 62.4 pounds/ft^3.)
   ANSWER:
   Consider lifting a rectangular slab of water h feet from the bottom up to the top. The area of such a slab can be found as follows:

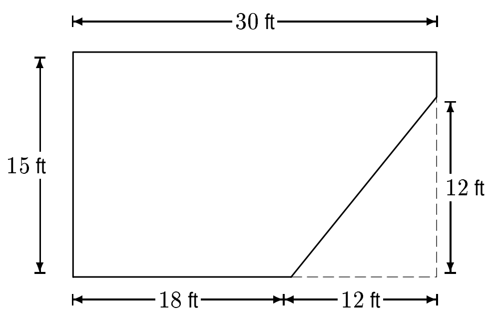


Figure 8.4.110

   Area of slab $= (30 \cdot 15) - \left(\frac{12 \cdot 12}{2}\right) = 378 \text{ ft}^2$.

   If the thickness is dh, then the volume of such a slab is 378 dh cubic feet. Such a slab weighs

$$(378\,dh)(62.4) = 23,587.2\,dh \text{ pounds}$$

   To lift that much water h feet requires 23587.2 dh foot-pounds of work. To lift the whole tank, we lift one slab at a time and sum. Taking the limit as dh goes to zero gives:

$$\int_0^{10} 23587.2\,dh = \frac{23587.2h^2}{2}\Big|_0^{10}$$
$$= 1,179,360 \text{ foot} - \text{pounds}.$$

Questions and Solutions for Section 8.5

1. A study of the costs to produce airplanes in World War II led to the theory of "learning curves," the idea of which is that the marginal cost per plane decreases over the duration of a production run. In other words, with experience, staff on an assembly line can produce planes with greater efficiency. The 90% learning curve describes a typical situation where the marginal cost, MC, to produce the x^{th} plane is given by

$$MC(x) = M_0 x^{\log_2 0.9},$$

where $M_0 =$ marginal cost to produce the first plane.

[Note: You may use the fact that $\log_2 x = \dfrac{\ln x}{\ln 2}$.]

(a) If a plant produces planes with a 90% learning curve on production costs, and the marginal cost for the first plane is $500,000, then what is the marginal cost to produce the second plane? The fourth plane?

(b) Recall that marginal cost is related to total cost as follows:

$$MC(x) = C'(x),$$

where $C(x) =$ total cost to produce x units. Given this, and the formula for $MC(x)$ with $M_0 =$ $500,000, find a formula for $C(x)$. What, physically, is the meaning of the constant in you formula for $C(x)$?

(c) If the constant for $C(x)$ is $20 million, and $M_0 =$ $500 thousand, then what, approximately, is $C(50)$?

ANSWER:

(a) $M_0 = 0.5$ (million dollars). So,
$MC(2) = (0.5)2^{\log_2 0.9} \approx (0.5)2^{-0.152} \approx 0.450$ (million dollars).
$MC(4) = (0.5)4^{\log_2 0.9} \approx (0.5)2^{2(-0.152)} \approx 0.405$ (million dollars).

(b)

$$\begin{aligned} C(x) &= \int MC(x)\,dx \\ &= \int (0.5)x^{\log_2 0.9}\,dx \\ &= \frac{0.5}{1+\log_2 0.9}x^{1+\log_2 0.9} + K \\ &\approx 0.590x^{0.848} + K. \end{aligned}$$

K is the cost if no planes are produced; it represents the costs of setting up the plant for production.

(c) Given $K = 20$ million, we have

$$C(50) \approx (0.590)50^{0.848} + 20 \approx 36.277 \text{ (millions)},$$

so the cost is approximately 36,277,000 dollars.

2. Rank in order of increasing present value, assuming 7% interest compounded continuously. No work need be shown.

(a) $1000, paid today.
(b) $1050, paid six months from now.
(c) $1085, paid a year from now.
(d) $1050, paid continuously over the next year.

ANSWER:

For (a), the present value is just $1000. The present value of $1050 paid six months from now is $\$1050e^{(-0.07)(0.5)} = \$1013.89$. For $1085 paid in a year, the present value is $\$1085e^{-0.07} = \$1011.65$. The present value of (d) is

$$\int_0^1 1050e^{-0.07t}\,dt = \$1014.09.$$

Thus, $(\text{a}) < (\text{c}) < (\text{b}) < (\text{d})$.

3. On each of January 1, 1991 and January 1, 1992, a person deposits 1000 dollars in a savings bank.

(a) On the last day of each year, the bank deposits interest in the account at a rate of 8%, compounded annually. Write down a sum that gives the size of the bank account after the second interest deposit (at the end of 1992). Do not evaluate the sum.

(b) Write down a sum that gives the size of the bank account at the end of 1992 if the interest is compounded 4 times per year. Do not evaluate the sum.

(c) Write down a sum and/or integral for the size of the bank account at the end of 1992 if the interest is compounded continuously. Do not evaluate your expression.

ANSWER:

(a) $[1000 + (1000(1 + 0.08))](1 + 0.08)$

(b) $\left[1000 + \left(1000\left(1 + \frac{0.08}{4}\right)^4\right)\right]\left(1 + \frac{0.08}{4}\right)^4$

(c) $[1000 + 1000e^{0.08}]e^{0.08}$

Remarks

- The answer should be about 2000.
- No integral in (c): this is not an income stream.

4. By the year 1996 you will have made your first million dollars. You invest it in a new company on January 1, 1997. The new company starts to earn a profit six months later. Thus, starting July 1, 1997, you receive income from the company in a continuous stream at a constant rate of $\frac{1}{2}$ million dollars per year.

   Your bank offers interest at a nominal rate of 8% per year, compounded continuously.

(a) When will you have received an income of $1 million from the company? (Do not take into account the bank's interest; this question is simply asking when the total income you have received will reach $1 million.) Give an exact date as an answer.

(b) Consider your answer to (a): Is your investment just paid off at that time? If not, is your investment paid off at a later date or is it paid off at an earlier date? Explain your answer.

Suppose T is measured in years from January 1, 1997.

(c) What is the future value of your original investment of $1 million at time T?

(d) What is the future value at time T of the income that you have received by that time?

(e) After how many years, T, will your investment have paid off? During what month of what year will this happen?

ANSWER:

(a) July 1, 1999.

(b) No. The present value of the earned $1 million is less than $1 million, so it will pay off later.

(c) $B = 10^6 e^{0.08T}$

(d) Future value at $T = \displaystyle\int_{\frac{1}{2}}^{T} \frac{10^6}{2} e^{0.08(T-t)}\,dt = 6,004,934e^{0.08T} - 6,250,000.$

(e) It will be paid off when future values equal $10^6 e^{0.08T} = 6,004,934e^{0.08T} - 6,250,000.$

$T = \dfrac{1}{0.08}\ln\left(\dfrac{6,250,000}{5,004,934}\right) \approx 2.78$ years.

Since 0.78 years = 9.36 months, this is October 1999.

5. Somebody offers to pay you money in one of the following ways:

- Two $54 payments, one six months from now and one twelve months from now.
- Payment in a continuous cash flow over the next year at a constant rate of $107 per year.

These payments are to be deposited into a bank account that earns 10% interest compounded continuously. You want to determine which plan is preferable, i.e., which plan has a larger present value.

(a) Find the present value of payment plan (i).

(b) Write a Riemann sum that approximates the present value of the second payment plan. Show all your work clearly.

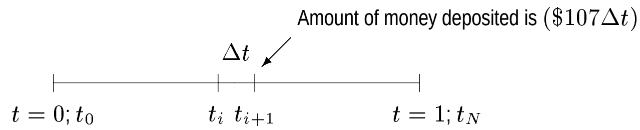
(c) Turn your Riemann sum in part (b) into a definite integral that gives the exact present value of payment plan (ii), and evaluate the integral. Which payment plan has the larger present value?

ANSWER:

(a) $P = 54e^{-0.1(0.5)} + 54e^{-0.1(1)} = \$100.23$

$t = 0.5$ year is 6 months from now. $t = 1$ year is the payment 1 year from now.

(b)



Portion $0 \le t \le 1$ year into N subintervals of width $\Delta t = \frac{1}{N}$. On each subinterval, you are paid $(107\Delta t)$ dollars. On the i-th subinterval, as pictured above, the present value of the payment will be approximately $(107\Delta t)e^{-0.1t_i}$. The total present value of the payments is therefore approximated by the Riemann sum

$$\sum_{i=1}^{N} 107e^{-0.1t_i}\Delta t$$

(c) $P = \int_0^1 107e^{-0.1t}dt$

$$= -1070e^{-0.1t}\Big|_0^1$$

$$= -1070(e^{-0.1} - 1)$$

$$= \$101.82$$

The second payment plan is better because it has the larger present value.

6. You have a bank account that earns 8% nominal annual interest compounded continuously, and you want to have \$80,000 in the bank account in five years so that you can buy a brand new Porsche.
   (a) How much money would you have to deposit in one lump sum today so that the account balance would be \$80,000 in five years?
   (b) If you instead deposit money in the account at a constant continuous rate of K dollars per year, then write a Riemann sum in terms of K that approximates the balance of the account after five years.
   (c) Turn the Riemann sum from part (b) into a definite integral that gives the exact balance after five years. Evaluate the integral in terms of the constant K.
   (d) At what constant continuous rate K dollars per year would you have to deposit money so that the balance of the account would be \$80,000 after five years?

   ANSWER:

   (a) There are two ways of looking at this. You can either find the necessary size of a deposit P dollars so that the future value F of the deposit in five years is \$80,000; or you can realize that this is equivalent to finding the *present value* of \$80,000:

$$P = Fe^{-it} = (\$80,000)e^{-0.08(5)} = \$53,625.60$$

   (b) Partition the time interval $0 \le t \le 5$ into N subintervals of width $\Delta t = \frac{5}{N}$. The amount of money deposited during the i-th time subinterval at time t_i is $K\Delta t$ dollars, and the future value of this amount of money is $Ke^{0.08(5-t_i)}\Delta t$ dollars. Therefore, the total bank balance after five years will be approximated by $\sum_{i=1}^{N} Ke^{0.08(5-t_i)}\Delta t$

   (c) $w = 0.08(5 - t) \Rightarrow dw = -0.08dt \Rightarrow dt = -12.5dw$

$$F = \int_0^5 Ke^{0.08(5-t)}dt = -12.5K\int_{0.4}^{0} e^w dw = -12.5K\left[e^w\right]_{0.4}^{0} = 12.5K(e^{0.4} - 1)$$

   (d) $12.5K(e^{0.4} - 1) = 80,000 \Rightarrow K = \dfrac{80,000}{12.5(e^{0.4} - 1)}$ = \$13,012.77 per year

   As an aside, notice that the total amount of money deposited would be $5K = 65,063.83$, which is larger than the lump sum deposit in part (a). This is to be expected, since the money will be in the bank for less time overall.

7. It is estimated that in fifteen years, it will cost \$200,000 to send a child to a four-year college.
   (a) Find the present value of a college education that will cost \$200,000 in fifteen years, assuming you could get 6% nominal annual interest on your money compounded continuously. Based on the current cost of a four-year college education, does the estimated \$200,000 in fifteen years sound high? Say you want to set up an account at a

bank that offers 6% nominal annual interest compounded continuously, so that fifteen years from today, the account has $200,000 in it for your child's college education. In parts (b) through (d) you will determine at what constant continuous rate K dollars per year you would need to deposit money.

(b) Set up a differential equation for the rate of change of your bank balance, where $B = f(t)$ is your bank balance at time t.

(c) Solve this differential equation for an initial balance of zero.

(d) Use your solution to part (c) to find K.

ANSWER:

(a) $\dfrac{200,000}{e^{.06(15)}}$ =$\$81,313.93$. This is ~ $20,000 per year (present value) which is about the current cost.

(b) $\dfrac{dB}{dt} = 0.06B + K$

(c) $\dfrac{dB}{dt} = 0.06\left(B + \dfrac{K}{0.06}\right)$, so $\displaystyle\int \frac{dB}{B + \frac{K}{0.06}} = \int 0.06dt$. $\ln\left|B + \dfrac{K}{0.06}\right| = 0.06t + C$, so

$B + \dfrac{K}{0.06} = Ae^{0.06t}$. When $t = 0$, $B = 0$ so $A = \dfrac{K}{0.06}$ $B = \dfrac{K}{0.06}(e^{0.06t} - 1)$

(d) We want to find K so that $B = 200,000$ when $t = 15$.

$200,000 = \dfrac{K}{0.06}(e^{.06(15)} - 1)$ and $K = \$8221.41$ dollars per year.

8. An insurance salesman offers you a life insurance policy with the following terms. You are to make payments at a rate of $1000 per year until age 70. If you pass away at any time, the policy will pay $150,000. Consider the payments to be made at a constant continuous rate of $1000 per year.
You are 30 years old, and you have a bank account that you know will offer you 5% nominal annual interest compounded continuously for an indefinite amount of time.

(a) Let's say you're feeling unlucky, and you think that you will die at age 70 (40 years from the time you start making payments for this insurance policy). If you had deposited your payments in the bank account, then what would be your balance at the time of your death?

(b) Based on your answer to part (a), if you die at age 70, are you better off depositing your money in the bank or buying the insurance policy?

(c) If you were to stop making payments in the bank account after 40 years, when you turn age 70, just as you would stop making payments on the life insurance policy, then you would simply earn interest on the bank balance that you calculated in part (a) until you died. How many years after age 70 would it be until that balance was $150,000?

(d) Based on your answer to part (c), if you thought you would live until age 80, are you better off depositing your money in the bank or buying the insurance policy?

ANSWER:

(a) $FV = \displaystyle\int_0^{40} 1000e^{0.05(40-t)}dt$. Let $w = 0.05(40 - t) \Rightarrow dw = (-20)^{-1}dt$.

$$= 1000\int_2^0 e^w(-20dw) = -20,000e^w\Big|_2^0 = 20,000(e^2 - 1) = \$127,781.12$$

(b) You are better off buying the insurance policy because the future value of your payments is less than the $150,000 future value of the insurance policy.

(c) Find T such that the present value of $150,000 is $127,781.12:

$$127,781.12 = 150,000e^{-0.05T} \Rightarrow T = -20\ln\left(\frac{127,781.12}{150,000}\right) = 3.21 \text{ years}$$

(d) The result in part (c) means that if you live more than 3.21 years past the age of 70, you would have been better off depositing your money for 40 years in the bank account. This is because at age 73.21, the bank balance reaches $150,000, and obviously after that point, the bank balance exceeds the $150,000 that your beneficiaries could collect from the insurance policy.

9. You have $100,000 that you want to invest. Some "business men" are willing to sell you a machine for your $100,000 that prints money. You figure that every day you can print $300 with the machine, and you would deposit the $300 each day in a "special" bank account at BCCI. Your friends at BCCI will only be able to offer you 5% nominal annual interest, compounded continuously, due to the "sensitive nature" of the transaction. It would be your intention to print money each day for one year.

(a) Write a sum that gives the exact value of your bank balance after one year. Do not attempt to evaluate the sum.

(b) If you were depositing the money that you printed in a continuous stream at a constant rate of $300 per day into the same bank account, then what definite integral would give your balance after one year? Evaluate the definite integral. (This result is very close to the numerical value of the sum you wrote in part (a).)

(c) If you had just taken the original $100,000 and placed it in a regular bank account that compounds interest annually, then what interest rate would you have had to earn in order for this option to be more profitable than the money machine (legal concerns aside)?

ANSWER:

(a) Each daily deposit earns at a daily interest rate of $\frac{0.05}{365}$ for a period of $365 - t$ days. So at the end of the year, a deposit of $300 made at time t has increased to $300e^{\frac{0.05}{365}[365-t]}$. The sum for the entire year is

$$\text{Balance} = \sum_{i=1}^{365} \underbrace{300}_{\text{units=\$/day}} e^{\frac{0.05}{365}[365-t_i]}\Delta t (\Delta t = 1 \text{ day})$$

This sum evaluated turns out to be $112,276.01.

(b)

$$\begin{aligned}\text{Balance} &= \int_0^{365} 300e^{\frac{0.05}{365}[365-t]}dt \\ &= 300\left(-\frac{365}{0.05}\right)e^{\frac{0.05}{365}[365-t]}\Big|_0^{365} \\ &= 300\left(-\frac{365}{0.05}\right)[e^0 - e^{0.05}] = \$112,283.70\end{aligned}$$

(c) For $100,000 to become greater than the amount earned from the money machine (part (b)), then the annual (compounded once) interest rate, r, is needed such that:

$$\begin{aligned}112,283.70 &= 100,000(1+r) \\ 1.1228370 &= 1 + r \\ .1228370 &= r \Rightarrow 12.28\%\end{aligned}$$

10. The *capital value* of an asset such as a machine is sometimes defined as the present value of all future net earnings of the asset. The actual lifetime of the asset may not be known, and since some assets last indefinitely, the capital value of the asset may be written in the form

$$\int_0^\infty K(t)e^{-rt}\,dt,$$

where $K(t)$ is the annual rate of earnings produced by the asset at time t, and r is the annual interest rate, compounded continuously. Find the capital value of an asset that generates income at a rate of $500 per year, with an interest rate of 10%.

ANSWER:

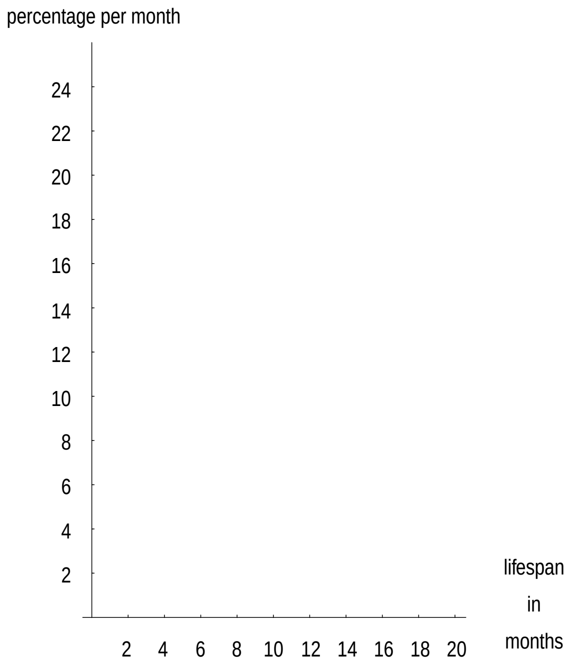
$K(t)$ is constant, $500/yr and $r = 0.1$. So

$$\begin{aligned}\text{capital value} &= \int_0^\infty 500e^{-0.1t}\,dt \\ &= 500 \cdot \left(-\frac{1}{0.1}e^{-0.1t}\right)\Big|_0^\infty \\ &= \frac{500}{0.1} \\ &= \$5000\end{aligned}$$

Questions and Solutions for Section 8.6

1. A lightbulb company is interested in the lifespan of their lightbulbs. They have 10,000 lightbulbs burning and have collected the following information.

After 2 months, 98% of the bulbs were still working.
After 8 months, 80% of the bulbs were still working.
We summarize all the data collected below: (Read carefully: the data was not collected at regular intervals.)

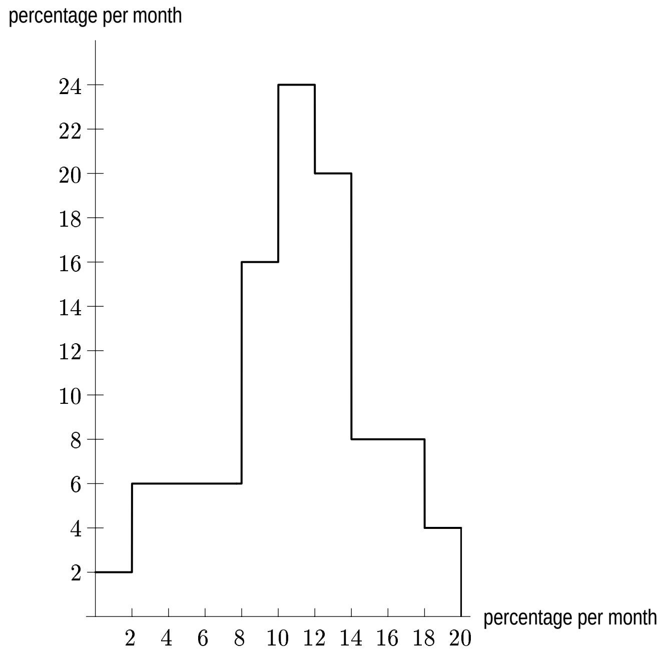


# of months	% of bulbs still burning
2	98
4	92
8	80
10	64
12	40
14	20
18	4
20	0

(a) How many bulbs out of the original $10{,}000$ burned out during the first 4 months?
(b) Use the axes above to draw a histogram for the lifespan of a bulb reflecting all the information in the table. (Do not smooth out the graph.)
(c) Approximate the average lifespan of a lightbulb. Explain your reasoning clearly.

ANSWER:

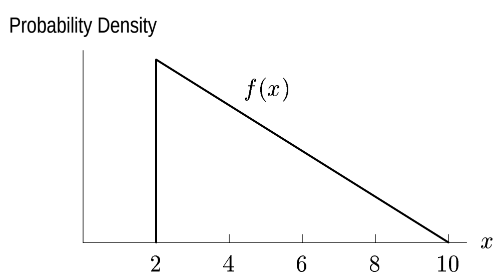
(a) After four months, 92% of the original $10{,}000$ lightbulbs were still burning. This means that 8%, or $10{,}000 \cdot 0.08 = 800$ lightbulbs, had burned out already.
(b) Since 2% of the bulbs die out within the first two months, we assume that 1% of the bulbs die out per month during this period. Between 2 and 4 months, 6% of the bulbs die out, or 3% per month. Between 4 and 8 months, 12% die out, or 3% a month. Between 8 and 10 months, $\frac{80-64}{2} = 8\%$ a month die out. Between 10 and 12 months, $\frac{24}{2} = 12\%$ a month die out. Between 12 and 14 months, $\frac{20}{2} = 10\%$ a month die out. Between 14 and 18 months, $\frac{16}{4} = 4\%$ a month die out, and between 18 and 20 months, $\frac{4}{2} = 2\%$ a month die out.



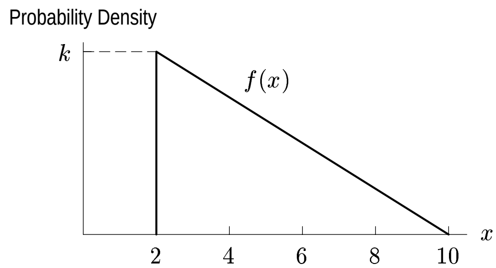
(c) Since 1% of the bulbs die per month over the first two months, 2% of the bulbs have an average lifespan of 1 month. Between 2 and 8 months, 3% die out per month, so 18% have an average lifespan of 5 months. Between 8 and 10 months, 8% die out per month, so 16% have an average lifespan of 9 months, and so on. Thus:

$$\begin{aligned}\text{Average Lifespan} &\approx 1(2\%) + 5(18\%) + 9(16\%) + \\ &\quad 11(24\%) + 13(20\%) + 16(16\%) + 19(4\%) \\ &= 10.92 \text{ months.}\end{aligned}$$

2. The probability density function $f(x)$ shown below describes the chances that a computer circuit board will cost a manufacturer more than a certain number of dollars to produce. In this case, the cost of the circuit board, x, is measured in thousands of dollars.



(a) What is the probability that the circuit board will cost more than \$10 thousand to produce? What is the probability that the circuit board will cost less than \$2 thousand to produce? What is the probability that the circuit board will cost between \$2 thousand and \$10 thousand to produce? Is it more likely that the circuit board will cost more or less than \$6 thousand?
(b) Show on the graph below, and describe in words, the geometrical interpretation of the probability that the circuit card will cost between \$2 thousand and some amount \$$b$ thousand. (Assume that b is between 2 and 10.)



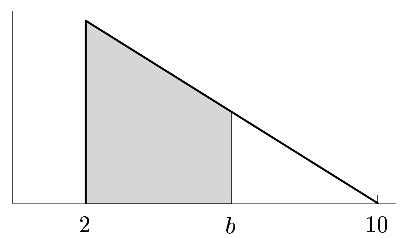
(c) Write a definite integral in terms of $f(x)$ that gives the probability that the circuit card will cost between \$2 thousand and some amount \$$b$ thousand. Do not attempt to evaluate the integral.

(d) Given your answers above, find the value of k, the height of the triangle that describes the probability density function.

ANSWER:

(a) Probability more than \$10 thousand=probability less than \$2 thousand= 0.
Probability between \$2 and \$10 thousand= 1.
More likely less than \$6 thousand.

(b) Probability between \$2 and \$$b$ thousand=Area under $f(x)$ between \$2 and \$$b$ thousand.



(c) $= \int_2^b f(x)dx$

(d) Area of $\Delta = 1$, so $\frac{1}{2} \cdot 8k = 1 k = \frac{1}{4}$

Questions and Solutions for Section 8.7

1. Suppose that the distribution of family sizes in the city of Boston in the year 1956 was given by:

Size:	2	3	4	5	6	≥ 7
# of Families:	13 921	9770	8955	5251	2520	2426

Represent this data on a histogram as a density distribution function.

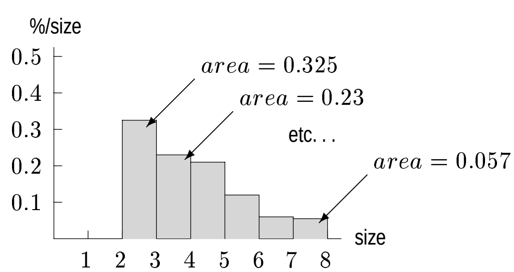
ANSWER:

The total number of families in Boston, according to this data, was: $13,921+9770+8955+5251+2520+2426 = 42843$.

The percentage of families by sizes are therefore:

Size:	2	3	4	5	6	≤ 7
%:	32.5	23	21	12	6	5.5

and gives the following histogram:



We assumed that there are no families with size ≥ 8. If the biggest family had F individuals, then the "tail" from 7 to F must have area $= 0.055$.

2. Using the data in Problem 1, find the mean of family sizes in the city of Boston in the year 1956. Assume that all of the families with 7 or more members had precisely 7 members. (You do not need to have answered Problem 1 in order to answer this question.)

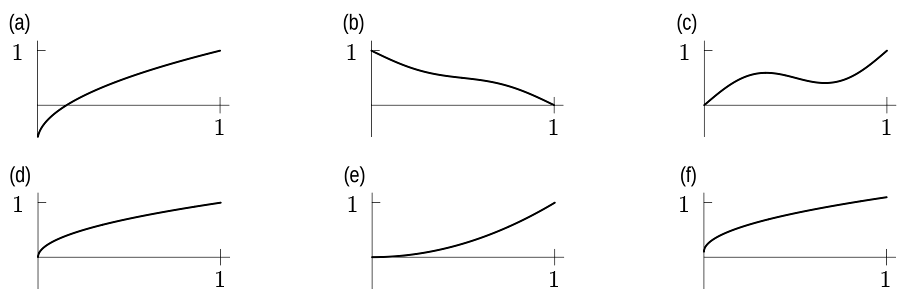
   ANSWER:

   Let

$$\begin{aligned} N_i &= \#\text{ of families with } i \text{ people} \\ N &= \text{ total \# of families} \\ P_i &= \frac{N_i}{N} \end{aligned}$$

$$\Rightarrow \text{mean} = \sum_{i=1}^{7} i \cdot P_i \approx 3.53 \text{ people}$$

3. (a) Let $p(t)$ be a probability density which is defined for $0 \leq t \leq 1$. Which of the following could be the cumulative distribution function for p? (Remember that the cumulative distribution function at time t is the integral of p from 0 to t.)

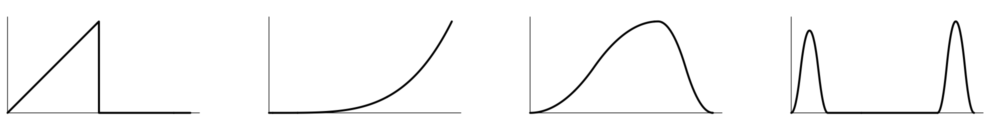


   (b) Draw a probability density function whose mean is substantially smaller than its median. (Make the difference unambiguous to get full credit.)

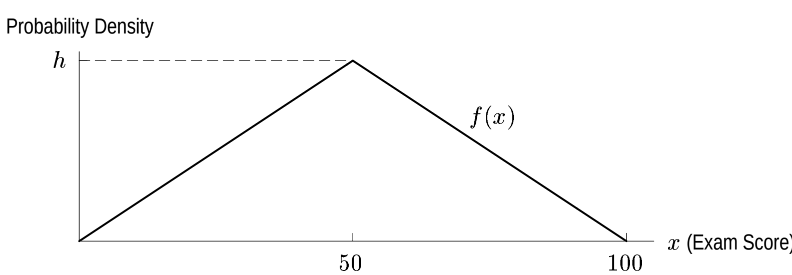
   ANSWER:

   (a) Diagrams (d) and (e). In (a), the function is negative, in (f) it exceeds 1, and in (b) and (c) the function decreases.

   (b) Here are some possible examples:



4. A professor gives the same 100-point final exam year after year and discovers that this students' scores tend to follow the triangular probability density function $f(x)$ pictured below:



(All persons, places, and events in this story are fictitious. Any similarity to real persons or situations are purely coincidental.)

   (a) TRUE or FALSE: The median, mean, and mode all describe the same point on this probability density function.

(b) What is the probability that a student's score will lie in the range $0 \le x \le 100$? Use this fact to find the value of the height h of the triangular probability density function.
(c) Find the equation of the probability density function $f(x)$ in the range $0 \le x \le 50$.
(d) What fraction of the students would you expect to score below 25 points on the exam? What fraction of the students would you expect to score below 75 points on the exam?

ANSWER:

(a) TRUE
(b) $\text{Prob}(0 \le x \le 100) = 1$

$$\text{Prob}(0 \le x \le 100) = \frac{1}{2}\underbrace{(100-0)}_{\text{Base}}h = 50h = 1 \Rightarrow h = \frac{1}{50} = 0.02$$

(c) Slope$= \dfrac{\Delta y}{\Delta x} = \dfrac{h}{50-0} = \dfrac{\frac{1}{50}}{50} = \dfrac{1}{2500}$

$f(x) = y - mx$ (line through origin)

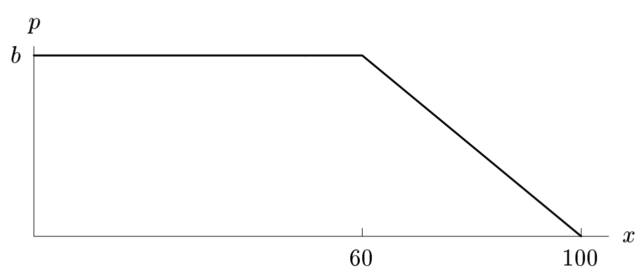
$$f(x) = \frac{1}{2500}x$$

(d) $\text{Prob}(0 \le x < 25) = \text{Prob}(75 < x \le 100)$ (since $f(x)$ symmetrical)

$$= \int_0^{25} f(x)dx = \int_0^{25} \frac{1}{2500}x\,dx = \left[\frac{x^2}{5000}\right]_0^{25} = 0.125 = \frac{1}{8}$$

The fraction of students that would be expected to score below 75 is 7/8.

5. The distribution of people's ages in the United States is essentially constant, or uniform, from age 0 to age 60, and from there it decreases linearly until age 100. This distribution $p(x)$ is shown below, where x is age in years, and p measures probability density. Such a probability distribution is called *trapezoidal*.

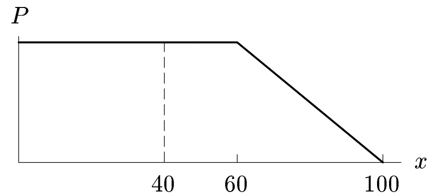


(a) According to this simplified model of the distribution of people's ages in the United States, what fraction of the population is older than 100? What fraction is between 0 and 100 years old?
(b) In terms of the length of the base, b, of the trapezoidal distribution (notice that the base of the trapezoid lies along the p-axis), find the fraction of the population that is between 0 and 60 years old.
(c) In terms of b, find the fraction of the population that is between 60 and 100 years old.
(d) Use the results of parts (a), (b), and (c) to find the value of b.
(e) Find the median age of the United States population.

ANSWER:

(a) 0% are older than 100, 100% are between 0 and 100.
(b) $\displaystyle\int_0^{60} p(x)dx =$ area of rectangle $= 60b$
(c) $\displaystyle\int_{60}^{100} p(x)dx =$ area of triangle from 60 to 100 $= \frac{1}{2} \cdot 40 \cdot b = 20b$
(d) Total area under the probability distribution $p(x) = 100\% = 1$. So $60b + 20b = 1 \Rightarrow 80b = 1 \Rightarrow b = \dfrac{1}{80}$

(e)



Let a = median age. This value a should be in the middle, i.e., half the population should be older and half younger. Thus, the area under $p(x)$ to the left of a should equal the area under $p(x)$ to the right of a. Or, the area under $p(x)$ to the left of a should be $\frac{1}{2}$ the total, or $\frac{1}{2} \cdot 1 = \frac{1}{2}$.

$$\int_0^a p(x)dx = \text{area of rectangle} = ab = a \cdot \frac{1}{80} = \frac{1}{2} \Rightarrow a = 40 \text{ years}$$

6. In a hydrogen atom in the unexcited state, the probability of finding the sole electron within x meters of the nucleus is given by

$$F(x) = \frac{4}{(a_0)^3} \int_0^x r^2 e^{\frac{-2r}{a_0}}\, dr, \qquad x \geq 0,$$

where $a_0 \approx 5.29 \times 10^{-11}$ meters.

(a) $F(x)$, as given above, is a cumulative probability distribution function. What is its corresponding probability density function $f(x)$? Sketch a graph of $y = f(x)$. What happens to $f(x)$ as $x \to \infty$ and what is $f(0)$? [Hint: Find $f'(x)$ to locate any local maxima or minima.]

(b) Carry out the integration given in the definition of $F(x)$ to find a more likable formula for $F(x)$. Simplify your formula. (Remember to evaluate the integral between 0 and x.)

(c) What is the probability that the electron will be found within a sphere of radius a_0?

(d) What is the probability that the electron will be found within $\frac{3}{2}a_0$ meters of the nucleus?

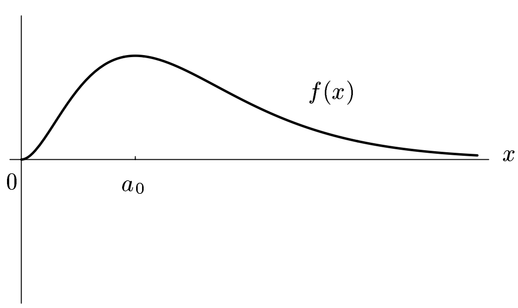
ANSWER:

(a) The probablity density function is $f(x) = F'(x) = \frac{4}{a_0^3}x^2 e^{-\frac{2x}{a_0}}$. It's easy to see that $f(0) = 0$. As $x \to \infty$, $f(x) \to 0$

$$\begin{aligned} f'(x) &= \frac{4}{a_0^3}\left(2xe^{-\frac{2x}{a_0}} - \frac{2}{a_0}x^2 e^{-\frac{2x}{a_0}}\right) \\ &= \frac{8x}{a_0^3} e^{-\frac{2x}{a_0}}\left(1 - \frac{x}{a_0}\right). \end{aligned}$$

Therefore $f'(x) = 0$ for $x = 0$ and $x = a_0$. Additionally, $f(x) \geq 0$ for all x. Since $f(0) = 0$ and $f(x) \to 0$ as $x \to \infty$, we deduce that $x = a$ is a maximum (both local and global) and $x = 0$ is a minimum (both local and global).

(b)



(c)

$$\begin{aligned} F(x) &= \frac{4}{a_0^3}\int_0^x r^2 e^{-\frac{2r}{a_0}}\, dr \\ &= \frac{4}{a_0^3}\left(-\frac{a_0}{2}r^2 e^{-\frac{2r}{a_0}}\Big|_0^x + \int_0^x a_0 r e^{\frac{-2r}{a_0}}\, dr\right) \end{aligned}$$

$$= \frac{4}{a_0^3}\left(-\frac{a_0}{2}x^2e^{-\frac{2x}{a_0}} + a_0 r\left(-\frac{a_0}{2}\right)e^{-\frac{2r}{a_0}}\Big|_0^x + \frac{1}{2}\int_0^x a_0^2 e^{\frac{-2r}{a_0}}\,dr\right)$$
$$= \frac{4}{a_0^3}\left(-\frac{1}{2}a_0 e^{\frac{-2x}{a_0}}x^2 - \frac{1}{2}a_0^2 e^{\frac{-2x}{a_0}}x - \frac{1}{4}a_0^3 e^{\frac{-2x}{a_0}} + \frac{1}{4}a_0^3\right)$$
$$= e^{\frac{-2x}{a_0}}\left(-\frac{2x^2}{a_0^2} - \frac{2x}{a_0} - 1\right) + 1.$$

(d) Probability of finding an electron in a shell of radius a_0 is

$$F(a_0) = e^{-\frac{2a_0}{a_0}}\left(-\frac{2a_0^2}{a_0^2} - \frac{2a_0}{a_0} - 1\right) + 1$$
$$= 1 - \frac{5}{e^2}.$$

(e) Probability of finding an electron in a shell of radius $\frac{3}{2}a_0$ is

$$F\left(\frac{3}{2}a_0\right) = e^{-\frac{3a_0}{a_0}}\left(-\frac{2(\frac{9}{4}a_0^2)}{a_0^2} - \frac{3a_0}{a_0} - 1\right) + 1$$
$$= 1 - \frac{17}{2e^3}.$$

7. In March 1995 the space shuttle carried an experiment designed by a Harvard student who studies the growth of crystals. Suppose the *probability density* function of the length, x cm, of a crystal grown in space is modeled by

$$p(x) = xe^{-x} \quad \text{for } x \geq 0.$$

The *cumulative distribution* function giving the probability that a crystal has length $\leq t$ cm is represented by $P(t)$.

(a) Which of the quantities (i) - (x) below best approximates the probability that a crystal has length between 2 cm and 2.01 cm?

(b) Which of the quantities (i) - (x) represents precisely the probability that a crystal has length less than 2.01 cm?

Possible answers for parts (a) and (b):

(i)	$p(2)$	(vi)	$P(2)$
(ii)	$p(2.01)-p(2)$	(vii)	$P(2.01)$
(iii)	$p(2)(0.01)$	(viii)	$P(2)(0.01)$
(iv)	$\frac{p(2)}{0.01}$	(ix)	$\frac{P(2)}{0.01}$
(v)	$\frac{p(2.01)-p(2)}{0.01}$	(x)	$\frac{P(2.02)-P(2)}{0.01}$

(c) Find a formula for the cumulative distribution function $P(t)$ for $t \geq 0$.

(d) What is the median crystal length that this model predicts? Give your answer to two decimal places.

(e) Set up an integral giving the mean crystal length predicted by this model.

(f) Calculate the mean crystal length. Give an exact answer.

ANSWER:

(a) Choice (iii) since

$$\begin{pmatrix}\text{Probability that crystal has}\\ \text{length between 2 and 2.01 cm}\end{pmatrix} = \begin{pmatrix}\text{Area under } p(x)\\ \text{between 2 and 2.01}\end{pmatrix} \approx p(2)\cdot(0.01).$$

(b) Choice (vii) since by the definition of $P(t)$, $P(2.01)$ is the probability that crystal has length up to 2.01.

(c) By definition,

$$P(t) = \int_{-\infty}^{t} p(x)dx.$$

Here the lower limit can be replaced by 0 since crystals cannot have negative length, and $p(x) = xe^{-x}$ for $x \geq 0$, so

$$\begin{aligned} P(t) &= \int_0^t xe^{-x}dx \\ &= -te^{-t} - e^{-t} + 1, \quad \text{through integration by parts} \\ &= 1 - \frac{t+1}{e^t}. \end{aligned}$$

(d) The median T is the value of length such that

$$\begin{aligned} \int_0^T p(x)dx &= \frac{1}{2} \\ P(t) &= \frac{1}{2} \\ -Te^{-T} - e^{-T} + 1 &= \frac{1}{2} \\ e^{-T}(T+1) &= \frac{1}{2} \\ e^{-T}(T+1) - \frac{1}{2} &= 0. \end{aligned}$$

Using the calculator to graph this equation and tracing, we get that $T \approx 1.68$.

(e) By definition,

$$\text{Mean} = \int_{-\infty}^{\infty} xp(x)dx = \int_0^{\infty} x(xe^{-x})dx.$$

(f) Evaluating the integral from part (e),

$$\begin{aligned} \text{Mean} &= \lim_{b\to\infty} \int_0^b x(xe^{-x})dx \\ &= \lim_{b\to\infty} \left[\left(-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right) \Big|_0^b \right], \quad \text{from tables} \\ &= \lim \left[\left(-b^2e^{-b} - 2be^{-b} - 2e^{-b}\right) - (0 - 0 - 2) \right] \\ &= 2. \end{aligned}$$

Review Questions and Solutions for Chapter 8

1. The price of crude oil in the recent past was well approximated by $P(t) = 40 - (t-4)^2$, where $P(t)$ is measured in $US/barrel, and time t is measured in months, with $t = 0$ on July 1, 1990. In the same time period, Saudi Arabia produced oil at a rate well approximated by $R(t) = 160 + 30\arctan(t-3)$ (measured in million barrels per month). Assume that the oil is sold continuously two months after its production. How much did Saudi Arabia get for the oil it produced in the second half of 1990?

ANSWER:

If $P(t) = 40 - (t-4)^2$ is the price per barrel at time t, and $R(t) = 160 + 30\arctan(t-3)$ is the number of barrels produced in millions of barrels/month, then $P(t)R(t-2)$ is the rate at which money is made in millions of dollars/month, since oil is not sold until 2 months after its production. The total amount of money made, in millions of dollars, is given by

$$\int_0^6 (40 - (t-4)^2)(160 + 30\arctan(t-5))\, dt \approx 29608.$$

So $29.6 billion is made in the second half of 1990.

Chapter 9 Exam Questions

Questions and Solutions for Section 9.1

1. A radioactive isotope is released into the air as an industrial by-product. This isotope is not very stable due to radioactive decay. Two-thirds of the original radioactive material loses its radioactivity after each month. If 10 grams of this isotope are released into the atmosphere at the end of the first and every subsequent month, then
   (a) how much radioactive material is in the atmosphere at the end of the twelfth month? If the answer involves a sum, write it in closed form.
   (b) In the long run, i.e., if the situation goes on *ad infinitum*, what will be the amount of this radioactive isotope in the atmosphere at the end of each month?

   ANSWER:

   (a)

Month $\#n$	Amount in atmosphere at end of month
1	10
2	$\underbrace{\frac{1}{3}(10)}_{\text{from previous month}} + \underbrace{10}_{\text{newly released}}$
3	$\frac{1}{3}\left(\frac{1}{3}10+10\right)+10=\left(\frac{1}{3}\right)^2\cdot 10+\left(\frac{1}{3}\right)10+10$
$\vdots$	$\vdots$
n	$10+\left(\frac{1}{3}\right)10+\left(\frac{1}{3}\right)^2 10+\left(\frac{1}{3}\right)^3+\ldots+\left(\frac{1}{3}\right)^{n-1}10$

   So after 12 months:

$$S_{12} = 10+\frac{1}{3}10+\left(\frac{1}{3}\right)^2 10+\ldots+\left(\frac{1}{3}\right)^{11}\cdot 10$$

$$\frac{1}{3}S_{12}=\frac{1}{3}10-\left(\frac{1}{3}\right)^2 10+\ldots+\left(\frac{1}{3}\right)^{11}\cdot 10+\left(\frac{1}{3}\right)^{12}\cdot 10$$

$$S_{12}-\frac{1}{3}S_{12}=10-\left(\frac{1}{3}\right)^{12}\cdot 10 \Rightarrow S_{12}=\frac{10-\left(\frac{1}{3}\right)^{12}\cdot 10}{1-\frac{1}{3}}$$

   (b) In the long run, i.e., as $u\to\infty$, we need the sum of the infinite geometric series. Here $a=10, r=\frac{1}{3}$ so the series converges to $\frac{a}{1-r}=\frac{10}{1-\frac{1}{3}}=15$.

2. (a) Find the exact value of the following:

$$\frac{3}{7}+\left(\frac{3}{7}\right)^2+...+\left(\frac{3}{7}\right)^{100}$$

   (b) Find the exact value of the infinite product

$$e^{1/2}\cdot e^{1/4}\cdot e^{1/8}\cdot e^{1/16}\cdot ...\cdot e^{1/2n}\cdot ....$$

   ANSWER:

   (a) We can rearrange the given sum to apply the formula for a geometric series:

$$\begin{aligned}\frac{3}{7}+...+\left(\frac{3}{7}\right)^{100} &= \frac{3}{7}\left(1+\frac{3}{7}+...+\left(\frac{3}{7}\right)^{99}\right)\\ &= \frac{3}{7}\left(\frac{1-\left(\frac{3}{7}\right)^{100}}{1-\frac{3}{7}}\right)\\ &= \frac{3}{4}\left(1-\left(\frac{3}{7}\right)^{100}\right).\end{aligned}$$

(b) We can write the given product as follows:

$$\begin{aligned} e^{1/2}\cdot e^{1/4}\cdot e^{1/8}\cdot \ldots &= e^{1/2+1/4+1/8+\ldots+1/2^n+\ldots} \\ &= e^1, \quad \text{since } \tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\ldots = 1 \\ &= e. \end{aligned}$$

3. Suppose the government spends \$1 million on highways. Some of this money is earned by the highway workers who in turn spend \$500,000 on food, travel, and entertainment. This causes \$250,000 to be spent by the workers in the food, travel, and entertainment industries. This \$250,000 causes another \$125,000 to be spent; the \$125,000 causes another \$62,500 to be spent, and so on. (Notice that each expenditure is half the previous one.) Assuming that this process continues forever, what is the total spending generated by the original \$1 million expenditure? (Include the original \$1 million in your total.)

ANSWER:

Total spending (in millions of dollars) is given by

$$\text{Total spending} = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots.$$

This is a convenient geometric series with $a=1$, $x=\frac{1}{2}$, so its sum will be given by

$$S=\frac{a}{1-x}=\frac{1}{1-\frac{1}{2}}=2 \text{ million dollars.}$$

4. Does the infinite series

$$4+\frac{4}{\sqrt{3}}+\frac{4}{3}+\frac{4}{3^{3/2}}+\frac{4}{3^2}+\cdots$$

converge or diverge?

ANSWER:

$$4+\frac{4}{\sqrt{3}}+\frac{4}{3}+\frac{4}{3^{3/2}}+\frac{4}{3^2}+\cdots$$

is

$$a+ax+ax^2+ax^3+\cdots$$

with $a=4$ and $x=\dfrac{1}{\sqrt{3}}$.

Since $\left|\dfrac{1}{\sqrt{3}}\right|<1$, it converges.

5. Find the sum of the first 5 and the sum of the first 10 terms of the series in Exercise 4.

ANSWER:

$$S_5=\frac{4\left(1-\left(\frac{1}{\sqrt{3}}\right)^5\right)}{1-\frac{1}{\sqrt{3}}}\approx 8.86$$

$$S_{10}=\frac{4\left(1-\left(\frac{1}{\sqrt{3}}\right)^{10}\right)}{1-\frac{1}{\sqrt{3}}}\approx 9.425$$

6. Find the sum of the series

$$\sum_{n=5}^{15}\left(\frac{4}{3}\right)^n$$

ANSWER:

$$\sum_{n=5}^{15}\left(\frac{4}{3}\right)^n=\left(\frac{4}{3}\right)^5+\left(\frac{4}{3}\right)^6+\cdots$$

$$= \left(\frac{4}{3}\right)^5 \left(1 + \frac{4}{3} + \left(\frac{4}{3}\right)^2 + \cdots + \left(\frac{4}{3}\right)^{10}\right)$$
$$= \frac{\left(\frac{4}{3}\right)^5 \left(1 - \left(\frac{4}{3}\right)^{11}\right)}{1 - \frac{4}{3}} \approx 286.68$$

7. A ball is dropped from a height of 14 feet and bounces. Each bounce is $\frac{2}{3}$ of the height of the bounce before.

   (a) Find an expression for the height to which the ball rises after it hits the floor for the h^{th} time.
   (b) Find the total vertical distance the ball has traveled when it hits the floor for the 4^{th} time.

   ANSWER:

   (a) Let h_n be the height of the n^{th} bounce after the ball hits the floor for the n^{th} time. Then from Figure 9.1.111,

$$h_0 = \text{ height before first bounce } = 14 \text{ feet}$$
$$h_1 = \text{ height after first bounce } = 14\left(\frac{2}{3}\right) \text{ feet}$$
$$h_0 = \text{ height after second bounce } = 14\left(\frac{2}{3}\right)^2 \text{ feet}$$

Generalizing gives $h_n = 14\left(\frac{2}{3}\right)^n$.

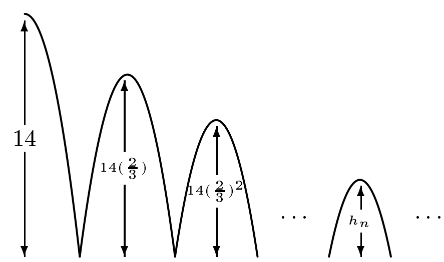


Figure 9.1.111

   (b) Total vertical distance when the ball hits the floor for the 4^{th} time is

$$14 + 2\left(14\left(\frac{2}{3}\right)\right) + 2\left(14\left(\frac{2}{3}\right)^2\right) + 2\left(14\left(\frac{2}{3}\right)^3\right) \approx 53.41 \text{ feet}$$

8. A tennis ball is dropped from a height of 40 feet and bounces. Each bounce is $\frac{1}{2}$ the height of the bounce before. A superball has a bounce $\frac{3}{4}$ the height of the bounce before, and is dropped from a height of 30 feet. Which ball bounces a greater total vertical distance?

   ANSWER:

   Tennis ball:

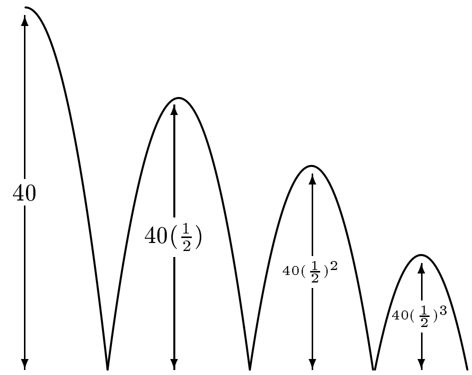


Figure 9.1.112

After the first drop of 40 feet, the total vertical distance is a geometric series with $a = 40$ and $r = \frac{1}{2}$:

$$40 + 2 \cdot 40(\frac{1}{2}) + 2 \cdot 40(\frac{1}{2})^2 + 2 \cdot 40(\frac{1}{2})^3 + \cdots$$

which converges to $40 + 40/(1 - \frac{1}{2}) = 120$ feet.

Superball:

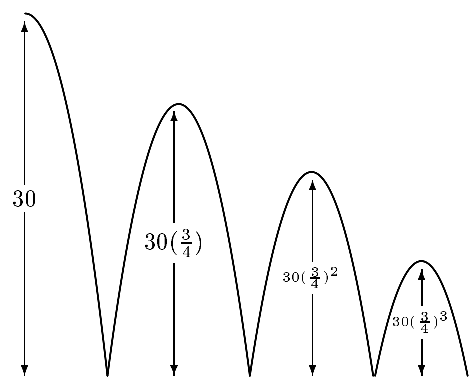


Figure 9.1.113

After the first drop of 30 feet, the total vertical distance is a geometric series with $a = 45$ and $r = \frac{3}{4}$:

$$30 + 2 \cdot 30(\frac{3}{4}) + 2 \cdot 30(\frac{3}{4})^2 + 2 \cdot 30(\frac{3}{4})^3 + \cdots$$

which converges to $30 + 45/(1 - \frac{3}{4}) = 210$ feet. So the superball travels more vertical distance than the tennis ball.

9. Decide which of the following are geometric series. For those which are, give the first term and the ratio between successive terms. For those which are not, explain why not.

(a) $2 + 2a + 2a^2 + 2a^3 + \cdots$
(b) $2 + 4a + 6a^2 + 8a^3 + \cdots$
(c) $2 + 2ak + 2a^2k^2 + 2a^3k^3 + \cdots$

ANSWER:

(a) Yes. First term = 2, ratio = $\dfrac{2a}{2} = a$

(b) No. Ratio between successive terms is not constant: $\dfrac{4a}{2} = 2a$ while $\dfrac{6a^2}{4a} = \dfrac{3}{2}a$

(c) Yes. First term = 2, ratio = $\dfrac{2ak}{2} = ak$.

10. Find the sum $\sum_{n=2}^{\infty}\left(\frac{4}{5}\right)^n$

ANSWER:

$$\begin{aligned}\sum_{n=2}^{\infty}\left(\frac{4}{5}\right)^n &= \left(\frac{4}{5}\right)^2+\left(\frac{4}{5}\right)^3+\cdots\\ &= \left(\frac{4}{5}\right)^2\left(1+\left(\frac{4}{5}\right)+\left(\frac{4}{5}\right)^2+\cdots\right)\\ \text{Sum} &= \frac{\left(\frac{4}{5}\right)^2}{1-\frac{4}{5}}=\frac{\left(\frac{4}{5}\right)^2}{\frac{1}{5}}=\frac{16}{5}\end{aligned}$$

Questions and Solutions for Section 9.2

1. (a) Show that $\int_1^{\infty}\frac{1}{x}\,dx$ does not converge.
   (b) Use part (a) to show that the harmonic series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots$$

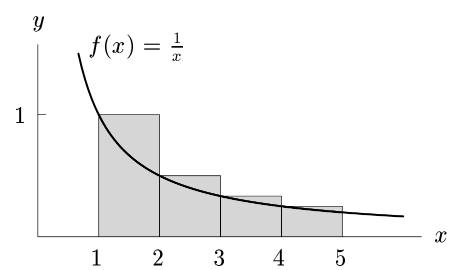
   does not converge. (Hint: Consider a left hand sum of $f(x)=\frac{1}{x}$ with $\Delta x=1$.)

   ANSWER:

   (a) $\int_1^{\infty}\frac{1}{x}\,dx=\lim_{b\to\infty}\ln x\Big|_1^b=\lim_{b\to\infty}\ln b$. As $b\to\infty$, $\ln b\to\infty$, so $\int_1^{\infty}\frac{1}{x}\,dx$ diverges.

   (b) Since $\frac{1}{x}$ is a decreasing function for $x\geq 1$, any left-hand sum over the interval $[1,\infty)$ is greater than $\int_1^{\infty}\frac{1}{x}\,dx$, which diverges. Consider a left-hand sum with $\Delta x=1$.

$$\begin{aligned}\text{left-hand sum} &= 1\cdot 1+1\cdot\frac{1}{2}+1\cdot\frac{1}{3}+\cdots\\ &= 1+\frac{1}{2}+\frac{1}{3}+\cdots.\end{aligned}$$



   Thus the harmonic series does not converge.

2. Use the integral test to decide whether the series $\sum_{n=1}^{\infty}\frac{(\ln n)^2}{n}$ converges or diverges.

   ANSWER:

$$\begin{aligned}\lim_{b\to\infty}\int_1^b\frac{(\ln x)^2}{x}\,dx &= \lim_{b\to\infty}\frac{(\ln x)^3}{3}\Big|_1^b\\ &= \lim_{b\to\infty}\left[\frac{(\ln b)^3}{3}-\frac{(\ln 1)^3}{3}\right]=\infty\end{aligned}$$

   So the series diverges.

3. Do these series converge or diverge?

   (a) $\sum_{n=0}^{\infty}\frac{2n}{\sqrt{2+n^2}}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{3n^2+2}{n^3+2n+5}$

ANSWER:

(a) We use the integral test and calculate the corresponding improper integral,

$$\int_0^{\infty} \frac{2x}{\sqrt{2+x^2}}\,dx = \lim_{b\to\infty}\int_0^b \frac{2x}{\sqrt{2+x^2}}\,dx = \lim_{b\to\infty} 2(2+x^2)^{1/2}\Big|_0^b$$
$$= \lim_{b\to\infty}\left(2(2+b^2)^{1/2} - 2(2)^{1/2}\right)$$

Since the limit does not exist (it is ∞) the integral diverges, so the series diverges.

(b) Similarly,

$$\int_0^{\infty} \frac{3x^2+2}{x^3+2x+5}\,dx = \lim_{b\to\infty}\int_0^b \frac{3x^2+2}{x^3+2x+5}\,dx = \lim_{b\to\infty} \ln|x^3+2x+5|\Big|_0^b$$
$$= \lim_{b\to\infty}\left(\ln|b^3+2b+5| - \ln|5|\right)$$

Since the limit does not exist, the integral diverges and so does the series.

4. Does

$1+\frac{1}{5}+\frac{1}{11}+\frac{1}{17}+\cdots+\frac{1}{6n-1}+\cdots$ converge or diverge?

ANSWER:

Let $f(x) = \dfrac{1}{6x-1}$ and use the integral test.

$$\int_1^n f(x)\,dx = \int_1^n \frac{dx}{6x-1} = \frac{1}{6}\ln(6x-1)\Big|_1^n$$
$$= \frac{1}{6}\ln(6n+5) - \frac{1}{6}\ln(5)$$

Since this has no limit as n grows, the series is not convergent.

5. Are the following statements true or false?

(a) If $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges, then $\displaystyle\sum_{n=1}^{\infty} ka_n$ diverges ($k \neq 0$).

(b) If $\displaystyle\lim_{n\to\infty} a_n \neq 0$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ does not converge.

ANSWER:

(a) True

(b) True

Questions and Solutions for Section 9.3

1. TRUE/FALSE questions. For each statement, write whether it is true or false and provide a short explanation or counterexample.

(a) If $\sum a_k$ is the sum of a series of numbers, and $\lim_{k\to\infty} a_k = 0$, then the series converges.

(b) If a series of constants $\sum a_k$ converges, then $\sum |a_k|$ converges.

(c) If a series of constants $\sum a_k$ diverges, then $\sum |a_k|$ diverges.

ANSWER:

(a) FALSE. $\displaystyle\lim_{k\to\infty}\frac{1}{k} = 0$, but $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$, is equal to $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$, which diverges.

(b) FALSE. The alternating series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$, converges (by the alternating series test). However, the harmonic series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$, diverges.

(c) TRUE. For each k, $a_k \leq |a_k|$ and since the series $\sum a_k$ diverges, $\sum |a_k|$ must also diverge. This statement is equivalent to the fact that absolute convergence implies convergence.

2. Use the comparison test to determine whether $\displaystyle\sum_{n=2}^{\infty} \frac{1}{4n^4+e^n}$ converges.

ANSWER:
Let $a_n = \dfrac{1}{4n^4+e^n}$. Since $4n^4 + e^n > n^4, n > 2$, we have

$$\frac{1}{4n^4+e^n} < \frac{1}{n^4}$$

so $0 < a_n < \dfrac{1}{n^4}$.

Since $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^4}$ converges, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{4n^4+e^n}$ also converges.

3. Use the alternating series test to decide if $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n^2}$ converges.

ANSWER:
Let $a_n = \dfrac{1}{2n^2}$. Then $a_{n+1} = \dfrac{1}{2(n+1)^2}$

Since $2(n+1)^2 > 2n^2$, we have

$$0 < a_{n+1} = \frac{1}{2(n+1)^2} < \frac{1}{n^2} = a_n$$

We also have $\lim\limits_{n\to\infty} a_n = 0$. So, we know the series converges.

4. Estimate the error in approximating the sum of the alternating series $S = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{1+2^n}$ by the sum of the first ten terms.

ANSWER:
The error using S_{10} to approximate S is less than the magnitude of the first term of the series which is omitted in the approximation. Thus the error is less than $\dfrac{(-1)^{n-1}}{1+2^n}$ for $n = 11$.

$$\frac{(-1)^{11-1}}{1+2^{11}} \approx 0.0005$$

5. For which of the following series does the ratio test fail to determine whether or not the series is convergent or divergent?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{\sqrt{n}}$

ANSWER:

(a) $a_n = \dfrac{1}{n^3},\ a_{n+1} = \dfrac{1}{(n+1)^3}$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|n^3|}{|(n+1)^3|} = 1$$

Therefore the ratio test does not tell us anything about the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$.

(b) $a_n = \dfrac{n}{2^n},\ a_{n+1} = \dfrac{(n+1)}{2^{n+1}}$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)}{2^{n+1}} \cdot \frac{2^n}{n} = \lim_{n\to\infty} \frac{(n+1)}{n} \cdot \frac{1}{2} = \frac{1}{2}$$

Since $\dfrac{1}{2} < 1$, $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges.

(c) $a_n = \dfrac{(-3)^{n-1}}{\sqrt{n}},\ a_{n+1} = \dfrac{(-3)^n}{\sqrt{n+1}}$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|(-3)^n|}{|\sqrt{n+1}|} \cdot \frac{|\sqrt{n}|}{|(-3)^{n-1}|} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} \cdot 3 = 3$$

So, $\sum a_n$ diverges.

Questions and Solutions for Section 9.4

1. TRUE/FALSE questions. For each statement, write whether it is true or false and provide a short explanation or counterexample.
   (a) If a power series $\sum a_k x^k$ converges at $x = 1$ and $x = 2$ then it converges at $x = -1$.
   (b) If a power series $\sum a_k x^k$ diverges at $x = c$ then it also diverges at $x = -c$.

   ANSWER:
   (a) TRUE. Since the series is centered at $x = 0$, its interval of convergence is centered at $x = 0$. Hence it will converge for at least all x with $|x| < 2$, in particular $x = -1$.
   (b) FALSE. The points c and $-c$ may be the endpoints of the interval of convergence for the given power series (note that the interval of convergence is centered at $x = 0$). In this case, divergence (or convergence) at $x = c$ does not guarantee divergence (or convergence) at $x = -c$. Such is the case, for example, for the power series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$, which diverges for $c = -1$ but converges for $-c = 1$.

2. Find the radius of convergence and the interval of convergence for $\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n+1}}$.

   ANSWER:
   $\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} C_n(x^n)$ where $C_n = \frac{1}{\sqrt{n+1}} \neq 0$.

   So, use $a_n = \frac{x^n}{\sqrt{n+1}}$ and $a_{n+1} = \frac{x^{n+1}}{\sqrt{n+2}}$ and the ratio test:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x^{n+1}|}{|\sqrt{n+2}|} \cdot \frac{|\sqrt{n+1}|}{|x^n|}$$

$$= |x| \lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n+2}} = |x|.$$

   The radius of convergence is $R = 1$. It converges for $|x| < 1$, so the interval of convergence is $-1 < x < 1$.

3. Find an expression for the general term of the series $\frac{x}{2} + \frac{x^2}{9} + \frac{x^3}{28} + \frac{x^4}{65} + \cdots$

   ANSWER:
   The general term is written $\frac{x^n}{n^3+1}$ for $n \geq 1$.

4. Use the ratio test to find the radius of convergence of $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{2^{n+1}}$.

   ANSWER:

$$a_n = \frac{n(x+2)^n}{2^{n+1}}, \qquad a_{n+1} = \frac{(n+1)(x+2)^{n+1}}{2^{n+2}}$$

$$\lim_{x\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{x\to\infty} \frac{|(n+1)(x+2)^{n+1}|}{|2^{n+2}|} \cdot \frac{|2^{n+1}|}{|n(x+2)^n|}$$

$$= \lim_{n\to\infty} \frac{(n+1)}{n} \cdot \frac{|(x+2)|}{2} = \frac{1}{2}|x+2|$$

   The series converges when $\frac{1}{2}|x+2| < 1$, or $|x+2| < 2$. Radius of convergence = 2.

5. If the series $\sum C_n x^n$ has a radius of convergence of 4 and $\sum D_n x^n$ has a radius of convergence of 6, what is the radius of convergence of $\sum (C_n + D_n)x^n$?

   ANSWER:
   The radius of convergence of $\sum (C_n + D_n)x^n$ will be the smaller of the radii of convergence of $\sum C_n x^n$ and $\sum D_n x^n$. So the radius is 4.

6. Find the radius of convergence of $x + \frac{3x^2}{4} + \frac{4x^3}{6} + \frac{5x^4}{8} + \cdots$

   ANSWER:

The coefficient of the n^{th} term is $C_n = \dfrac{n+1}{2n}$. So

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)+1}{2(n+1)}\right| \cdot \left|\frac{2n}{n+1}\right| \cdot \left|\frac{x^{n+1}}{x^n}\right| = \frac{(n+2)(2n)}{(n+1)(2n+2)}|x| \to |x| \text{ as } n \to \infty$$

Thus, the radius fo convergence is $R = 1$.

7. Find the radius and interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{n}{4^n}(2x-1)^n$.

ANSWER:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)}{4^{n+1}}(2x-1)^{n+1}\right| \cdot \left|\frac{4^n}{n} \cdot \frac{1}{(2x-1)^n}\right|$$
$$= \frac{(n+1)}{4n} \cdot |2x-1| \to \frac{|2x-1|}{4} \text{ as } n \to \infty.$$

The series converges if $\dfrac{|2x-1|}{4} < 1,\ |2x-1| < 4$.

Radius of convergence is $R = 2$. The interval of convergence is $(-\frac{3}{2}, \frac{5}{2})$.

Review Questions and Solutions for Chapter 9

1. True or false?

(a) If $\sum C_n 3^n$ is convergent, then $\sum C_n(-3)^n$ is also convergent.
(b) If $a_n > a_{n+1} > 0$ and $\sum a_n$ converges, then $\sum(-1)^n a_n$ converges.

ANSWER:

(a) False. For example, if $C_n = 1/(n(-3)^n)$, then the first series is an alternating harmonic series (which converges), and the second series is a harmonic series, which diverges.
(b) True. The convergence of the first series implies that $a_n \to 0$ so the alternating series test applies.

2. Find the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{2^n(x-3)^n}{\sqrt{n+3}}$.

ANSWER:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}(x-3)^{n+1}}{\sqrt{n+1+3}}\right| \cdot \left|\frac{\sqrt{n+3}}{2^n(x-3)^n}\right|$$
$$= 2|x-3| \cdot \frac{\sqrt{n+3}}{\sqrt{n+4}} \to 2|x-3| \text{ as } n \to \infty$$

$2|x-3| < 1$ for it to converge. Radius of convergence is $R = \dfrac{1}{2}$.

3. Do the following converge or diverge?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\sin n}$
(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 \sin n}{n^4+2}$

ANSWER:

(a) Writing $a_n = \dfrac{1}{1+\sin n}$ we get $\lim\limits_{n\to\infty} a_n \neq 0$, so the series diverges.
(b) Since $|\sin n| < 1$, we know that

$$\left|\frac{n^2 \sin n}{n^4+2}\right| \leq \frac{n^2}{n^4+2} \leq \frac{n^2}{n^4} = \frac{1}{n^2}$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, comparison test tells us that $\displaystyle\sum_{n=1}^{\infty} \left|\frac{n^2 \sin n}{n^4+2}\right|$ converges so $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 \sin n}{n^4+2}$ converges.

4. True or false?

(a) The ratio test can be used to determine whether $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

(b) If the power series $\sum C_n x^n$ converges for $x = a, a > 0$, then it converges for $x = \frac{a}{2}$.

ANSWER:

(a) False. $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{1}{(n+3)^3} \cdot \frac{n^3}{1}\right| \to 1$ as $n \to \infty$. So the test is inconclusive.

(b) True.

Chapter 10 Exam Questions

Questions and Solutions for Section 10.1

1. Construct the Taylor polynomial approximation of degree 3 to the function $f(x) = \arctan x$ about the point $x = 0$. Use it to approximate the value $f(0.25)$. How does the approximation compare to the actual value?

   ANSWER:

$$\begin{aligned} f(x) &= \arctan x \quad f(0) = 0 \\ f'(x) &= \frac{1}{1+x^2} \quad f'(0) = 1 \\ f''(x) &= \frac{-2x}{(1+x^2)^2} \quad f''(0) = 0 \\ f'''(x) &= \frac{-2(1+x^2)^2 + (2x)2(1+x^2)(2x)}{(1+x^2)^4} = \frac{6x^2-2}{(1+x^2)^3} \quad f'''(0) = -2 \end{aligned}$$

   The third-degree Taylor polynomial approximation for $\arctan x$ around 0 is

$$\arctan x \approx P_3(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = x - \frac{1}{3}x^3,$$

   so $f(.25) \approx .25 - (.25)^3/3 = 0.24479$. The actual value of $f(0.25)$ is ≈ 0.24498, so the Taylor approximation is accurate to three decimals.

2. (a) Estimate the value of $\displaystyle\int_0^1 e^{-x^2}\,dx$ using both left- and right-hand Riemann sums with $n = 5$ subdivisions.
   (b) Approximate the function $f(x) = e^{-x^2}$ with a Taylor polynomial of degree 6.
   (c) Estimate the integral in (a) by integrating the Taylor polynomial approximation from (b).
   (d) Indicate briefly how you could improve the results in both cases.

   ANSWER:

   (a) With $n = 5$ subdivisions, and $\Delta x = 1/n = 1/5$,

$$\text{LEFT}(5) = \sum_{i=0}^{4} \frac{1}{5}e^{-x_i^2} = \frac{1}{5}\left(1 + e^{-\frac{1}{25}} + e^{-\frac{4}{25}} + e^{-\frac{9}{25}} + e^{-\frac{16}{25}}\right) \approx 0.80758$$

$$\text{RIGHT}(5) = \sum_{i=1}^{5} \frac{1}{5}e^{-x_i^2} = \frac{1}{5}\left(e^{-\frac{1}{25}} + e^{-\frac{4}{25}} + e^{-\frac{9}{25}} + e^{-\frac{16}{25}} + e^{-1}\right) \approx 0.68116$$

   Because e^{-x^2} is monotone decreasing between 0 and 1, the true value of $\int_0^1 e^{-x^2}\,dx$ is less than the left-hand sum, and greater than the right-hand sum.

   (b) The Taylor polynomial around 0, to degree 3, for e^x is:

$$e^x \approx P_3(x) = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3.$$

   Substitute $-x^2$ for x into the above expression to obtain:

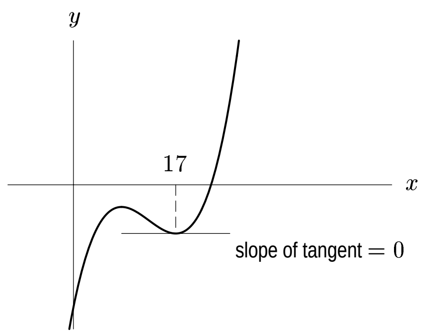
$$e^{-x^2} \approx 1 - \frac{1}{1!}x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6.$$

   (c) Integrating the polynomial above from 0 to 1 gives:

$$\int_0^1 e^{-x^2}\,dx \approx \left(x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7\right)\Bigg|_0^1 \approx 0.74286.$$

   (d) We could improve the estimate in part (a) by using more subdivisions, or by using a more sophisticated scheme, such as Simpson's rule, for estimating the integral. We could improve the estimate in part (b) by using a higher-degree Taylor polynomial.

3. The graph of $y = f(x)$ is given below.



Suppose we approximate $f(x)$ near $x = 17$ by the second degree Taylor polynomial centered about 17,

$$a + b(x - 17) + c(x - 17)^2.$$

Determine the sign of a, b, and c and circle the correct answer.

(a) a is positive negative zero
Reasoning:
(b) b is positive negative zero
Reasoning:
(c) c is positive negative zero
Reasoning:

ANSWER:

(a) negative
"a" is a constant, equal to $f(17)$, not $f(0)$, which is negative according to the graph.
(b) zero
"b" corresponds to $f'(17)$, which is 0.
(c) positive
"c" corresponds to $f''(17)$, which is > 0 since $f(x)$ is concave up at $x = 17$.

4. Write down the fourth degree Taylor polynomial for $\cos(3x^2)$ about $x = 0$.
ANSWER:
The Taylor expansion for $\cos t$ about $x = 0$ is

$$\cos t = 1 - \frac{t^2}{2!} + \ldots.$$

Substituting $3x^2$ for t gives

$$\begin{aligned}\cos(3x^2) &= 1 - \frac{(3x^2)^2}{2!} + \ldots \\ &= 1 - \frac{9}{2}x^4 + \ldots\end{aligned}$$

5. Suppose a function satisfies $f(2) = 4$, $f'(2) = 3$, $f''(2) = -5$, $f'''(2) = 12$. Write down the third degree Taylor polynomial for f about $x = 2$.
ANSWER:
In general,

$$f(x) \approx f(a) + f'(a)(x - 1) + \frac{f''(a)(x-a)^2}{2!} + \ldots + \frac{f^{(n)}(a)(x-a)^n}{n!}.$$

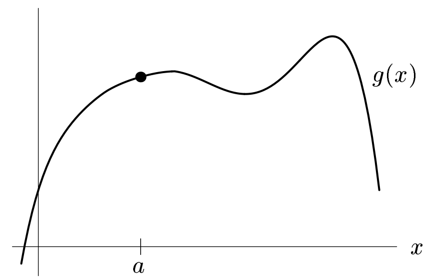
Here, $a = 2$. Making this substitution gives

$$\begin{aligned}f(x) \approx P_3(x) &= 4 + 3(x - 2) - \frac{5}{2}(x-2)^2 + \frac{12}{6}(x-2)^3 \\ &= 4 + 3(x-2) - \frac{5}{2}(x-2)^2 + 2(x-2)^3.\end{aligned}$$

6. The function g has the Taylor approximation

$$g(x) \approx c_0 + c_1(x-a) + c_2(x-a)^2,$$

and the graph given below:



What can you say about the signs of c_0, c_1, and c_2? (Circle your answers; no reasons need be given.)

(a) c_0 is negative zero positive
(b) c_1 is negative zero positive
(c) c_2 is negative positive

ANSWER:
From the picture, we see that $g(a) > 0$, $g'(a) > 0$, and $g''(a) < 0$. Since

$$g(x) \approx g(a) + g'(a)(x-a) + \frac{g''(a)}{2!}(x-a)^2,$$

we can differentiate to get $c_0 = g(a)$, $c_1 = g'(a)$, and $c_2 = g''(a)$. So in fact c_0 is positive, c_1 is positive, and c_2 is negative.

7. (a) Find the Taylor polynomial of degree 3 around $x = 0$ for the function

$$f(x) = \sqrt{1-x}.$$

(b) Use your answer to part (a) to give approximate values to $\sqrt{\frac{1}{2}}$ and $\sqrt{0.9}$.
(c) Which approximation in part (b) is more accurate? Explain why.
ANSWER:

(a)

$$\begin{aligned} f(x) &= \sqrt{1-x} & f(0) &= 1 \\ f'(x) &= -\frac{1}{2}(1-x)^{-\frac{1}{2}} & f'(0) &= -\frac{1}{2}(1-0)^{-\frac{1}{2}} = -\frac{1}{2} \\ f''(x) &= -\frac{1}{4}(1-x)^{-\frac{3}{2}} & f''(0) &= -\frac{1}{4}(1-0)^{-\frac{3}{2}} = -\frac{1}{4} \\ f'(x) &= -\frac{3}{8}(1-x)^{-\frac{5}{2}} & f'''(0) &= -\frac{3}{8}(1-0)^{-\frac{5}{2}} = -\frac{3}{8} \end{aligned}$$

$$\text{So} \quad P_3(x) = f(0) + \frac{f'(0)}{1}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$
$$= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3$$

(b) Use $P_3(x)$ obtained above with $x = \frac{1}{2}$:

$$\sqrt{1-\frac{1}{2}} = \sqrt{\frac{1}{2}} \approx 1 - \frac{1}{2}\left(\frac{1}{2}\right) - \frac{1}{8}\left(\frac{1}{2}\right)^2 - \frac{1}{16}\left(\frac{1}{2}\right)^3 \approx 0.7109.$$

Use $P_3(x)$ obtained above with $x = 0.1$:

$$\sqrt{1-0.1} = \sqrt{0.9} \approx 1 - \frac{1}{2}(0.1) - \frac{1}{8}(0.1)^2 - \frac{1}{16}(0.1)^3 \approx 0.9487$$

(c) We expect the approximation to be more accurate for $\sqrt{0.9}$ because 0.1 is significantly closer to 0 than 0.5 is. The actual value for $\sqrt{0.9}$ is ≈ 0.9487, while the actual value for $\sqrt{0.5}$ is ≈ 0.7071, so our expectations are correct.

8. Explain why $\lim_{x\to 0}\frac{\sin x}{2x}=\frac{1}{2}$ using a Taylor approximation for $\sin x$.
ANSWER:
Use the Taylor approximation $\sin x \approx x-\frac{x^3}{3!}$, for x near 0.

$$\lim_{x\to 0}\frac{\sin x}{2x}=\lim_{x\to 0}\frac{1}{2}\frac{\sin x}{x}=\frac{1}{2}\lim_{x\to 0}\frac{\sin x}{x}=$$

$$\frac{1}{2}\lim_{x\to 0}\frac{x-\frac{x^3}{3!}}{x}=\frac{1}{2}\lim_{x\to 0}\left(1-\frac{x^2}{3!}\right)=\frac{1}{2}.$$

9. Estimate $\int_1^2 \ln t\,dt$ using a 4^{th} degree Taylor Polynomial for $\ln t$ about $t=1$.
ANSWER:
Let $f(t)=\ln t$ for t near 1

$$\begin{aligned}
\text{Then } f(t)&=\ln t & \text{so } f(1)&=\ln(1)=0\\
f'(t)&=\frac{1}{t} & f'(1)&=1\\
f''(t)&=-\frac{1}{t^2} & f''(1)&=-1\\
f'''(t)&=\frac{2}{t^3} & f'''(1)&=2\\
f^{(4)}(t)&=-\frac{6}{t^4} & f^{(4)}(1)&=-6
\end{aligned}$$

Therefore $\ln t\approx (t-1)-\frac{(t-1)^2}{2}+\frac{(t-1)^3}{3}-\frac{(t-1)^4}{4}$ for t near 1.

$$\int_1^2 \ln t\,dt \approx \int_1^2\left((t-1)-\frac{(t-1)^2}{2}+\frac{(t-1)^3}{3}-\frac{(t-1)^4}{4}\right)dt$$

$$=\frac{(t-1)^2}{2}-\frac{(t-1)^3}{6}+\frac{(t-1)^4}{12}-\frac{(t-1)^5}{20}\Big|_1^2$$

$$\frac{1}{2}-\frac{1}{6}+\frac{1}{12}-\frac{1}{20}\approx 0.367$$

10. True or false? The Taylor polynomial of degree 6 for e^{-x^2} for x near 0 is

$$1-\frac{x^2}{1!}+\frac{x^4}{2!}-\frac{x^6}{3!}.$$

ANSWER:

$$e^{-x^2}=\sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!}=\sum_{n=1}^{\infty}(-1)^n\frac{x^{2n}}{n!}=1-\frac{x^2}{1!}+\frac{x^4}{2!}-\frac{x^6}{3!}+\cdots$$

So it is true that
$1-\frac{x^2}{1!}+\frac{x^4}{2!}-\frac{x^6}{3!}$ is the 6^{th} degree Taylor Polynomial for e^{-x^2} for x near 0.

Questions and Solutions for Section 10.2

1. Find the first four terms of the Taylor series for the following function: $f(x)=\sin x$ about $x=\pi/6$.
ANSWER:

$$\begin{aligned}
f(x)&=\sin x & f\left(\frac{\pi}{6}\right)&=\frac{1}{2}\\
f'(x)&=\cos x & f'\left(\frac{\pi}{6}\right)&=\frac{\sqrt{3}}{2}\\
f''(x)&=-\sin x & f''\left(\frac{\pi}{6}\right)&=-\frac{1}{2}\\
f'''(x)&=-\cos x & f'''\left(\frac{\pi}{6}\right)&=-\frac{\sqrt{3}}{2}
\end{aligned}$$

So the Taylor series is

$$\frac{1}{2}+\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)-\frac{1}{2}\frac{\left(x-\frac{\pi}{6}\right)^2}{2!}-\frac{\sqrt{3}}{2}\frac{\left(x-\frac{\pi}{6}\right)^3}{3!}+\cdots$$

2. Find the first four terms of the Taylor series for the following function: $f(x) = \ln(x+2)$ about $x = 2$.
   ANSWER:

$$\begin{array}{ll} f(x)=\ln(x+2) & f(2)=\ln 4 \\ f'(x)=\dfrac{1}{x+2} & f'(2)=\dfrac{1}{4} \\ f''(x)=-\dfrac{1}{(x+2)^3} & f''(2)=-\dfrac{1}{16} \\ f'''(x)=\dfrac{2}{(x+2)^3} & f'''(2)=\dfrac{2}{64}=\dfrac{1}{32} \end{array}$$

So the Taylor series is

$$\ln 4+\frac{(x-2)}{4}-\frac{(x-2)^2}{32}+\frac{(x-2)^3}{192}+\cdots$$

3. Find an expression for the general term of the series $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$ and give the starting value of the index.
   ANSWER:
   The general term can be written as

$$\frac{(-1)^k x^{2k}}{(2k)!} \text{ for } k \geq 0$$

4. Recognize $3-\frac{3^2}{2}+\frac{3^3}{3}-\frac{3^4}{4}+\cdots$ as a Taylor series evaluated at a particular value of x and find the sum.
   ANSWER:
   This is the series for $\ln(1+x)$ with x replaced by 3, so the series converges to $\ln 4$.
5. Recognize $1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}+\cdots$ as a Taylor series evaluated at a particular values of x and find the sum.
   ANSWER:
   This is the series for $\sin x$ with x replaced by 1, so the series converges to $\sin 1$.
6. Solve exactly for the variable:
   (a) $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots=e^5$
   (b) $1+x+x^2+x^3+\cdots=7$

   ANSWER:
   (a) The series is the Taylor series for e^x, so $x = 5$.
   (b) Since $1+x+x^2+x^3+\cdots=\frac{1}{1-x}$ is a geometric series, we solve $\frac{1}{1-x}=7$ giving $x=\frac{6}{7}$.
7. Suppose that you are told that the Taylor series of $f(x) = e^{-x^2}$ about $x = 0$ is

$$1-\frac{x^2}{1!}+\frac{x^4}{2!}-\frac{x^6}{3!}+\cdots$$

   find $\frac{d}{dx}\left(e^{-x^2}\right)\Big|_{x=0}$ and
   $\frac{d^4}{dx^4}\left(e^{-x^2}\right)\Big|_{x=0}$
   ANSWER:
   Let C_n be the coefficient of the n^{th} term in the series. Note that

$$0=C_1=\frac{d}{dx}\left(e^{-x^2}\right)\Big|_{x=0}$$

   and since

$$\frac{1}{2}=C_4=\frac{\frac{d^4}{dx^4}\left(e^{-x^2}\right)\Big|_{x=0}}{4!}$$

So

$$\frac{d^4}{dx^4}\left(e^{-x^2}\right)\Big|_{x=0} = \frac{4!}{2} = 12$$

8. Use the binomial series with $p = 4$ to expand $(1+x)^4$.

ANSWER:

$$(1+x)^4 = 1 + 4x + \frac{4(4-1)}{2!}x^2 + \frac{4(4-1)(4-2)}{3!}x^3 + \frac{4(4-1)(4-2)(4-3)}{4!}x^4$$
$$= 1 + 4x + 6x^2 + 4x^3 + x^4$$

Questions and Solutions for Section 10.3

1. (a) Use the formula for the Taylor polynomial approximation to the function $g(x) = e^x$ about $x_0 = 0$ to construct a polynomial approximation of degree 6 to $f(x) = e^{x^2}$.
   (b) Use the approximation you constructed in (a) to estimate the value of $e^{(0.2)^2}$.
   (c) What is the error in this approximation?

   ANSWER:

   (a) The third degree Taylor polynomial about $x = 0$ for e^x is $P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$. Replace x by x^2:

$$e^{x^2} \approx 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!}.$$

   (b) Substitute $x = 0.2$:

$$e^{(0.2)^2} \approx 1 + (0.2)^2 + \frac{(0.2)^4}{2!} + \frac{(0.2)^6}{3!}$$
$$= 1.040810666\ldots$$

   (c) The true value of $e^{(0.2)^2}$ is $1.040810774\ldots$ The error is thus less than 1.1×10^{-7}.

2. Answer the following questions about Taylor series. If you are asked to find a Taylor series, you may start with a series that you already know and modify it or you may derive the series "from scratch." Also, if you are asked to find a Taylor series, either give the answer in summation notation, or give at least the first four non-zero terms so that the pattern is apparent.
   (a) Consider the function $f(x) = 1 - \cos x$.
      (i) Find the Maclaurin series for $f(x)$.
      (ii) Based on the Maclaurin series for $1 - \cos x$, what do you conclude about the value of the following limit:

$$\lim_{x\to 0} \frac{1-\cos x}{x^2}$$

   (b) As you know, the function $f(x) = e^{\frac{-x}{2}}$ gives the form of the normal probability density function (or bell-shaped curve).
      (i) Find the Maclaurin series for $f(x)$.
      (ii) Find the Maclaurin series for the indefinite integral of $f(x)$ by integrating term-by-term the Maclaurin series you obtained above for $f(x)$.

   ANSWER:

   (a) (i) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$

$$1 - \cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \ldots$$

$\Rightarrow$

$$= \sum_{i=1}^{\infty} \frac{(-1)^{i+1} x^{2i}}{(2i)!}$$

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\left[\frac{\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\frac{x^8}{8!}+\dots}{x^2}\right]$$

(ii)

$$=\lim_{x\to 0}\left[\frac{1}{2}-\underbrace{\frac{x^2}{4!}+\frac{x^4}{6!}-\frac{x^8}{8!}}_{\text{All terms that have a power of } x\to 0}+\dots\right]=\frac{1}{2}$$

(b) (i) $e^y=1+y+\frac{y^2}{2!}+\frac{y^3}{3!}+\dots$ Let $y=-\frac{x^2}{2}$.

$$e^{-\frac{x^2}{2}}=1-\frac{x^2}{2}+\frac{x^4}{2^2(2!)}-\frac{x^6}{2^3(3!)}+\dots=\sum_{i=0}^{\infty}\frac{(-1)^i x^{2i}}{2^i(i!)}$$

(ii) $$\int e^{-\frac{x^2}{2}}dx=\int\left[1-\frac{x^2}{2}+\frac{x^4}{2^2(2!)}-\frac{x^6}{2^3(3!)}+\dots\right]dx$$

$$=x-\frac{x^3}{3(2)}+\frac{x^5}{5(2^2)(2!)}-\frac{x^7}{7(2^3)3!}+\dots+C$$

$$=\sum_{i=0}^{\infty}\frac{(-1)^i x^{2i+1}}{(2i+1)2^i(i!)}+C$$

3. There is no closed-form antiderivative to the function $f(x)=\sin(x^2)$, but a numerical approximation to the following definite integral is desired: $\int_0^1 \sin(x^2)dx$.

(a) Find the Taylor series centered at $a=0$ (i.e., the Maclaurin series) for $f(x)=\sin(x^2)$. Either express the series in summation notation, or show enough terms so that the pattern is apparent (at least three non-zero terms).

(b) Find the Maclaurin series for an antiderivative $F(x)$ of the function $f(x)=\sin(x^2)$. Do this by integrating the Maclaurin series from part (a) term-by-term. Again, either express your answer in summation notation, or show enough terms so that the pattern is apparent.

(c) Using the series from part (b) for $F(x)=\int \sin(x^2)dx$ and the fundamental theorem of calculus, estimate the value of the following definite integral correct to three decimal places, i.e., correct to the thousandths place: $\int_0^1 \sin(x^2)dx$. In doing this, be sure to demonstrate how you know you have three decimal places of accuracy.

ANSWER:

(a) Use the series for $\sin x$, except plug in x^2 for x: $\sin x\approx x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots=\sum_{i=0}^{\infty}\frac{(-1)^i x^{2i+1}}{(2i+1)!}$

So $\sin(x^2)\approx x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}-\dots=\sum_{i=0}^{\infty}\frac{(-1)^i(x^2)^{2i+1}}{(2i+1)!}=\sum_{i=0}^{\infty}\frac{(-1)^i x^{4i+2}}{(2i+1)!}$

(b)

$$\int\sin(x^2)dx\approx\int\left(x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}-\dots\right)dx=\frac{x^3}{3}-\frac{x^7}{7\cdots 3!}+\frac{x^{11}}{11\cdots 5!}=\dots$$

$$=\sum_{i=0}^{\infty}\frac{(-1)^i x^{4i+3}}{(4i+a)\cdot[(2i+1)!]}$$

(c) Approximate $F(x)=\int\sin x^2dx\approx P_7(x)=\frac{x^3}{3}-\frac{x^7}{7\cdot 3!}$. Then

$$\int_0^1\sin(x^2)dx=F(1)-F(0)\approx P_7(1)-P_7(0)=\frac{13}{42}\approx 0.30952$$

round off to 3 decimal places =0.310.

Now approximate $f(x) \approx P_{11}(x) = \dfrac{x^3}{3} - \dfrac{x^7}{7 \cdot 3!} + \dfrac{x^{11}}{11 \cdot 5!}$. Then

$$\int_0^1 \sin(x^2)dx = F(1) - F(0) \approx P_{11}(1) - P_{11}(0) = \frac{1}{3} - \frac{1}{42} + \frac{1}{1320}$$
$$= \frac{13}{42} + \frac{1}{1320} = \frac{2860}{9240} + \frac{7}{9240} = \frac{2867}{9240} \approx 0.310281.$$

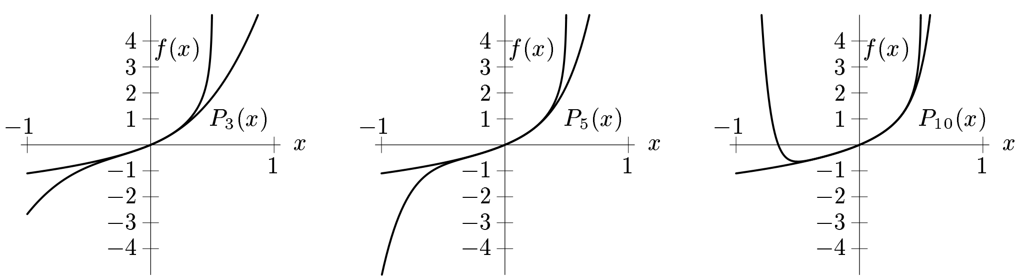
These approximations agree to the first 3 decimal places, so our answer is 0.310.
(The exact answer to 10 decimal places is 0.3102683017.)

4. (a) Find the Taylor expansion for $f(x) = -\ln(1-2x)$ by substituting into the series for $\ln(1+x)$.
(b) Plot both $f(x)$ and its Taylor polynomials of various degrees and use the graphs to guess what the interval of convergence is.
[Hint: Begin with the 3$^{\text{rd}}$ degree approximation. It's a good idea to use approximations as high as 10$^{\text{th}}$ degree!]
ANSWER:

(a) We know that $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$. So

$$\begin{aligned} -\ln(1-2x) &= -(-2x - 2x^2 - \frac{8}{3}x^3 - 4x^4 + \cdots) \\ &= 2x + 2x^2 + \frac{8}{3}x^3 + 4x^4 + \cdots. \end{aligned}$$

(b) We examine the behavior of $P_n(x)$ as n gets bigger:



As n gets large, the $P_n(x)$ seem to converge on the interval $[-\frac{1}{2}, \frac{1}{2}]$. In fact, it can be shown algebraically that $P_n(x)$ converges to $-\ln(1-2x)$ as $n \to \infty$ for $-\frac{1}{2} \le x < \frac{1}{2}$.

5. (a) Write down the Taylor series for $\cos x$ at $x = 0$.
(b) Use part (a) to write down the Taylor series for $\cos(\sqrt{x})$ at $x = 0$.
(c) To what number does the series

$$1 - \frac{2}{2!} + \frac{4}{4!} - \frac{8}{6!} + \frac{16}{8!} - \cdots$$

converge?
ANSWER:

(a) Let $f(x) = \cos x$. Then

$$\begin{array}{lll} f(0) & = \cos 0 & = 1 \\ f'(0) & = -\sin 0 & = 0 \\ f''(0) & = -\cos 0 & = -1 \\ f'''(0) & = \sin 0 & = 0 \\ f^{(4)}(0) & = \cos 0 & = 1 \\ \vdots & \vdots & \vdots \end{array}$$

Therefore, $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$.

(b) Replacing x by $\sqrt{x}$ in the series, we get

$$\cos(\sqrt{x}) = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots.$$

(c) Note that $\cos(\sqrt{2}) = 1 - \frac{2}{2!} + \frac{4}{4!} - \frac{8}{6!} + \cdots$, so the series converges to $\cos\sqrt{2} \approx 0.1559\ldots$

6. (a) Write the Taylor series about 0 for $\frac{1}{1-x}$.

(b) Use the derivative of the series you found in part (a) to help you calculate the Taylor series about 0 for $\frac{x}{(1-x)^2}$.

(c) Use your answer to part (b) to calculate the exact value of

$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \frac{6}{64} + \ldots.$$

ANSWER:

(a) The Taylor series for $1/(1-x)$ about $x = 0$ is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots.$$

(b) We differentiate the equation in (a) as follows:

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{1-x}\right) &= \frac{d}{dx}(1 + x + x^2 + x^3 + \ldots)\\ \frac{1}{(1-x)^2} &= 1 + 2x + 3x^2 + 4x^3 + \ldots.\end{aligned}$$

We can then multiply both sides by x to get the Taylor series for $x/(1-x)^2$:

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \ldots.$$

(c) Substituting $x = 1/2$ into the equation in part (b) gives

$$\begin{aligned}\frac{\frac{1}{2}}{(1-\frac{1}{2})^2} &= \frac{1}{2} + 2\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3 + 4\left(\frac{1}{2}\right)^4 + \ldots\\ 2 &= \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32}.\end{aligned}$$

7. According to the theory of relativity, the energy, E, of a body of mass m is given as a function of its speed, v, by

$$E = mc^2\left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right)$$

where c is a constant, the speed of light.

(a) Assuming $v < c$, expand E as a series in v/c, as far as the second nonzero term.

(b) Explain why the series shows you that if v/c is very small, E can be well approximated as follows:

$$E \approx \frac{1}{2}mv^2.$$

(c) Part (a) approximates E using two terms; part (b) uses one term. You will now compare the accuracy of the two approximations. If $v = 0.1c$, by what percentage do the approximations in parts (a) and (b) differ?

ANSWER:

(a) The expansion of E is given by

$$\begin{aligned}E &= mc^2\left[\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1\right]\\ &= mc^2\left[1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{(-1/2)(-3/2)}{2!}\left(\frac{-v^2}{c^2}\right)^2 \ldots - 1\right]\\ &= mc^2\left[\frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4}\ldots\right].\end{aligned}$$

(b) When v/c is very small, we can ignore all but the first term, so

$$E \approx mc^2 \cdot \frac{1}{2}\frac{v^2}{c^2} = \frac{1}{2}mv^2.$$

(c) Approximating E using two terms gives

$$E = mc^2 \cdot \frac{1}{2}\frac{v^2}{c^2}\left[1 + \frac{3}{4}\frac{v^2}{c^2} + ...\right] = \frac{1}{2}mv^2\left[1 + \frac{3}{4}\frac{v^2}{c^2}\right]$$

So if $v/c = 0.1$, the two approximations differ by $(3/4)(0.1)^2 = 0.0075 = 0.75\%$.

8. (a) Find the Taylor series for $f(x) = \sin 2x$ about $x = 0$.
   (b) Use the result from (a) to find the Taylor series for $g(x) = \cos 2x$ about $x = 0$.
   ANSWER:

(a)

$$\begin{aligned} f(x) &= \sin 2x & f(0) &= 0 \\ f'(x) &= 2\cos 2x & f'(0) &= 2 \\ f''(x) &= -4\sin 2x & f''(0) &= 0 \\ f'''(x) &= -8\cos 2x & f'''(0) &= -8 \\ f^{(4)}(x) &= 16\sin 2x & f^{(4)}(0) &= 0 \\ f^{(5)}(x) &= 32\cos 2x & f^{5)}(0) &= 32 \end{aligned}$$

The Taylor series is $2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \cdots$

$$= 2x - \frac{4x^3}{3} + \frac{4x^5}{15} - \cdots$$

(b) Since $g(x) = \frac{f'(x)}{2}$, we obtain

$$\cos(2x) \approx \frac{1}{2}\left(2 - \frac{3 \cdot 8}{3!}x^2 + \frac{5 \cdot 32}{5!}x^4 - \cdots\right)$$

$$= 1 - 2x^2 + \frac{2}{3}x^4 - \cdots$$

9. Use the fact that the Taylor series for $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ about $x = 0$ to find the Taylor series about $x = 0$ for $e^{x/2}$.

   ANSWER:

   If $f(x) = e^{x/2}$, then $f'(x) = \frac{1}{2}e^{x/2}$ and $f''(x) = \frac{1}{2} \cdot \frac{1}{2}e^{x/2}$. So to obtain the Taylor series for $e^{x/2}$ from the one for e^x, the coefficient becomes $\frac{1}{2^n n!}$ and the series is

$$e^{x/2} = 1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \cdots$$

10. Consider the functions $y = e^{-x^2}$ and $y = \cos x$ for $-1 \le x \le 1$.
    Write a Taylor expansion for the two functions about $x = 0$. What is similar about the two series? What is different?
    ANSWER:

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

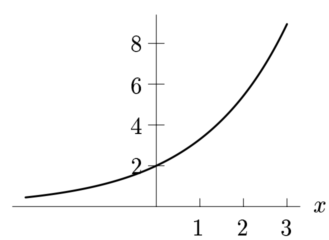
The first term is the same for both series. The same signs and powers of x exist in each series, but the coefficients of the terms (after the first one) in $\cos x$ are less than those for e^{-x^2}.

Questions and Solutions for Section 10.4

1. (a) Find the 12-th degree Taylor polynomial for $x\sin(x^2)$ centered at $x = 0$.
   (b) Suppose you use the first two non-zero terms of the series to approximate $x\sin(x^2)$ for $0 < x < 1$.
      (i) Is your approximation too big or too small? Explain.
      (ii) Is the magnitude of the error always less than 0.1667? Yes No
      (iii) Is the magnitude of the error always less than 0.0084? Yes No
   (c) Suppose you use the first two non-zero terms of the series to approximate $x\sin(x^2)$ for $-1 < x < 0$. Is your approximation too big or too small?
   ANSWER:
   (a) We know: $\sin u = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \ldots$
   so
   $$\sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} \ldots$$
   $$= x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} \ldots$$
   so
   $$x\sin(x^2) = x\left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} \ldots\right)$$
   $$= x^3 - \frac{x^7}{3!} + \frac{x^{11}}{5!} - \ldots$$
   N.B. The next nonzero term contains x^{15} so the degree would be too high.
   N.B. Degree of poly = highest power of variable, so you do <u>not</u> need 12 terms in expansion.
   (b) (i) The series above is alternating, terms are decreasing monotonically so the sign of the error = sign of first term omitted. So for x positive (e.g., $x \in (0,1)$), the sign of error is the same as sign of $\frac{x^{11}}{5!}$, i.e., positive, so approximation is too small.
      (ii) The error is always less than $\frac{x^{11}}{5!}$ in size. The error is largest for $x \approx 1$ so error $\leq \frac{1}{5!}$. Hence yes and
      (iii) yes.
   (c) Again, the sign of error = the sign of first term omitted, i.e., $\frac{x^{11}}{5!}$. For $x \in (-1, 0)$, $\frac{x^{11}}{5!}$ is <u>negative</u> so the approximation is an overestimate.

2. The function $h(x)$ is a continuous differentiable function whose graph is drawn below. The accompanying table provides some information about $h(x)$ and its derivatives.

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$
0	2	1	0.50	0.25
1	3.29	1.64	0.82	0.41
2	5.43	2.71	1.35	0.67
3	8.96	4.48	2.24	1.12



   (a) Which of the following is closest to $h(2.1)$?
      (i) $2 + 2.1 + \frac{0.5}{2}(2.1)^2 + \frac{0.25}{6}(2.1)^3$
      (ii) $2 + 2.1 + 0.5(2.1)^2 + 0.25(2.1)^3$
      (iii) $2 + 2.1 + \frac{0.5}{2}(2.1)^2$
      (iv) $5.43 + 2.71(2.1) + \frac{1.35}{2}(2.1)^2 + \frac{0.67}{6}(2.1)^3$

(v) $5.43 + 2.71(0.1) + \frac{1.35}{2}(0.1)^2 + \frac{0.67}{6}(0.1)^3$

(vi) $5.43 + 2.71(2.1) + 1.35(2.1)^2 + .67(2.1)^3$

(vii) $3.29 + 1.64(1.1) + \frac{0.82}{2}(1.1)^2 + \frac{0.41}{6}(1.1)^3$

(b) $h(x)$, $h'(x)$, $h''(x)$ and $h'''(x)$ are all increasing functions. Suppose we use a tangent line approximation at zero to approximate $h(0.2)$. Find a good upper bound for the error.

ANSWER:

(a) (v)

We want to approximate $f(2.1)$. Center our Taylor polynomial about a nearby point at which we know h and its derivatives:

$h(x) \approx h(2) + h'(2)(x-2) + \frac{h''(1)}{2}(x-2)^2 + \frac{h'''(2)}{6}(x-2)^3$ for x near 2.

$$h(2.1) \approx 5.43 + (2.71)(2.1-2) + \frac{1.35}{2}(2.1-2)^2 + \frac{67}{6}(2.1-2)^3$$

Common errors:

(iv) Look at the size of this answer: it's much larger than $h(3)$!

(i) It's much better to center your polynomial closer to the x- value in question.

(b) The tangent line approximation is a linear approximation: degree 1.

Error $= E_1(.2) \leq \left|\frac{h''(c)}{2!}(.2)^2\right|$ for some $c \in [0, .2]$

Error $\leq \left|\frac{h''(c).04}{2}\right| = h''(c)(.02) < .82(.02)$ Choose .82 NOT .5. $.5 \leq h''(c)$ and we want something $\geq h''(c)$. $h''(x)$ is increasing.

Error $< .0164$

$$\text{since} \begin{cases} |E_n(b)| \leq \dfrac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1} \quad \text{for some } c \text{ between } a \text{ and } b \\ b = .2 \\ a = \text{center} = 0 \\ n = \text{degree of poly.} = 1 \end{cases}$$

Note: The error is definitely positive–the tangent line lies below the curve for $x > 0$ (since the curve is concave up).

Common errors:

1) A tangent line is linear, i.e., degree 1.

2) |Error| is not automatically less than the size of the first unused term. This is true if a series has terms which are alternating in sign, decreasing in magnitude, and going to zero. Otherwise, often |Error| is about the same size as the first unused term–but not always–and not necessarily less than that term.

3) Finding a lower bound for $h''(c)$ instead of an upper bound.

3. Estimate the magnitude of the error in approximating the following quantity using a third-degree Taylor polynomial about $x = 0$.

$$\ln(.5)$$

ANSWER:

Let $f(x) = \ln(1+x)$. The error bound in the Taylor approximation of degree 3 about $x = 0$ is

$$|E_3| = |f(-0.5) - P_3(-0.5)| \leq \frac{M \cdot |-0.5-0|^4}{4!}$$

where $|f^{(4)}| \leq M$ for $-0.5 \leq x \leq 0$.

$f^{(4)}(x) = \dfrac{3!}{(1+x)^4}$, so $|f^{(4)}(x)| \leq 3!$ and

$$|E_4| \leq \frac{3!(-0.5)^4}{24} \approx 0.016$$

4. Estimate the magnitude of the error in approximating the following quantity using a third-degree Taylor polynomial about $x = 0$.

$$\sin 2$$

ANSWER:
Let $f(x) = \sin x$. The error bound is

$$|E_3| = |f(2) - P_3(2)| \leq \frac{M \cdot |2-0|^4}{4!}$$

where $|f^{(4)}| \leq M$ for $0 \leq x \leq 2$.

$f^{(4)}(x) = \sin x$, so $|f^{(4)}(x)| \leq 1$ and

$$|E_4| \leq \frac{1 \cdot (2)^4}{2!} = 8$$

This is not a very useful error bound for this function.

5. Give a bound for the maximum possible error for the n^{th} degree Taylor polynomial about $x = 0$ approximating $\sin \dfrac{x}{2}$ on the interval $[0, 1]$.

ANSWER:
The maximum possible error for the n^{th} degree Taylor polynomial about $x = 0$
approximating $\sin \dfrac{x}{2}$ is

$$|E_n| \leq \frac{M \cdot |x-0|^{n+1}}{(n+1)!}, \text{ where } \sin^{(n+1)} \frac{x}{2} \leq M \text{ for } 0 \leq x \leq 1$$

The derivatives of $\sin \dfrac{x}{2}$ never take on values greater than 1, so

$$|E_n| \leq \frac{|x|^{n+1}}{(n+1)!} \leq \frac{1}{(n+1)!}$$

6. What degree Taylor polynomial about $x = 0$ do you need to calculate $\sin 2$ to four decimal places?

ANSWER:
The error is at most $\dfrac{1}{(n+1)!}$. To get an answer that is correct to four decimal places, the error must be less than 0.00005. We need to find n such that $\dfrac{1}{(n+1)!} < 0.00005$. $n = 7$ works, so the 7^{th}-degree polynomial works.

7. Show that the Taylor series about 0 for $\sin x$ converges to $\sin x$ for every x.

ANSWER:
To do this, we need to show that the error $E_n(x) \to 0$ as $n \to \infty$. If $f(x) = \sin x$, the derivatives will all be ≤ 1, so $M = 1$.

We have

$$|E_n(x)| = |\sin x - P_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$$

for every n. To show the errors go to zero, we must show that for a fixed x,

$$\frac{|x|^{n+1}}{(n+1)!} \to 0 \text{ as } n \to \infty$$

Choose an arbitrary $|x|$. For $n > 2|x|$,

$$\frac{x^{n+1}}{(n+1)!}, \frac{x^{n+2}}{(n+2)!}, \frac{x^{n+3}}{(n+3)!}, \ldots$$

converges to zero because each term is obtained from its predecessor by multiplying by a number less than 1/2. So, the Taylor series does converge to $\sin x$.

Questions and Solutions for Section 10.5

1. Construct the first three Fourier approximations to the function

$$f(x) = \begin{cases} 1 & -\pi \le x \le 0 \\ \dfrac{1}{2} & 0 \le x \le \pi \end{cases}$$

ANSWER:

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} 1\,dx + \int_{0}^{\pi} \frac{1}{2}\,dx\right] = \frac{3}{4} \\ a_1 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} \cos x\,dx + \int_{0}^{\pi} \frac{1}{2}\cos x\,dx\right] \\ &= \frac{1}{\pi}\left[\sin x\Big|_{-\pi}^{0} + \frac{1}{2}\sin x\Big|_{0}^{\pi}\right] = 0 \end{aligned}$$

a_2 and a_3 are also 0.

$$\begin{aligned} b_1 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} \sin x\,dx + \int_{0}^{\pi} \frac{1}{2}\sin x\,dx\right] \\ &= \frac{1}{\pi}\left[-\cos x\Big|_{-\pi}^{0} - \frac{1}{2}\cos x\Big|_{0}^{\pi}\right] = \frac{1}{2\pi} \\ b_2 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin 2x\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} \sin 2x\,dx + \int_{0}^{\pi} \frac{1}{2}\sin 2x\,dx\right] \\ &= \frac{1}{\pi}\left[-\frac{1}{2}\cos 2x\Big|_{-\pi}^{0} - \frac{1}{4}\cos 2x\Big|_{0}^{\pi}\right] = 0 \\ b_3 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin 3x\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} \sin 3x\,dx + \int_{0}^{\pi} \frac{1}{2}\sin 3x\,dx\right] \\ &= \frac{1}{\pi}\left[-\frac{1}{3}\cos 3x\Big|_{-\pi}^{0} - \frac{1}{6}\cos 3x\Big|_{0}^{\pi}\right] = -\frac{1}{3\pi} \end{aligned}$$

$$\begin{aligned} \text{Thus } F_1(x) &= \frac{3}{4} + \frac{1}{2\pi}\sin x \\ F_2(x) &= \frac{3}{4} + \frac{1}{2\pi}\sin x \\ F_3(x) &= \frac{3}{4} + \frac{1}{2\pi}\sin x - \frac{1}{3\pi}\sin 3x \end{aligned}$$

2. Construct the first three Fourier approximations to the function

$$f(x) = \begin{cases} -2 & -\pi \le x \le 0 \\ 0 & 0 \le x \le \pi \end{cases}$$

ANSWER:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} 2\,dx + \int_{0}^{\pi} 0\,dx\right]$$

$$= \frac{1}{2\pi}\left[-2x\Big|_{-\pi}^{0}\right] = -1$$

$$a_1 = \frac{1}{\pi}\int f(x)\cos x\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -2\cos x\,dx = \frac{1}{\pi}\left[-2\sin x\Big|_{-\pi}^{0}\right] = 0$$

$$a_2 = \frac{1}{\pi}\int f(x)\cos 2x\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -2\cos 2x\,dx = \frac{1}{\pi}\left[-\sin 2x\Big|_{-\pi}^{0}\right] = 0$$

$$a_3 = 0$$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -2\sin x\,dx = \frac{1}{\pi}(2\cos x)\Big|_{-\pi}^{0} = 4$$

$$b_2 = \frac{1}{\pi}\int_{-\pi}^{0} -2\sin 2x\,dx = \frac{1}{\pi}(\cos 2x)\Big|_{-\pi}^{0} = 0$$

$$b_3 = \frac{1}{\pi}\int_{-\pi}^{0} -2\sin 3x\,dx = \frac{1}{\pi}\cdot\frac{2}{3}\cos 3x\Big|_{-\pi}^{0} = \frac{2}{3\pi}(1-1) = 0$$

$$\text{Thus } F_1(x) = -1$$
$$F_2(x) = -1 + 4\sin 2x$$
$$F_3(x) = -1 + 4\sin 2x$$

3. Find a_0 and the first two harmonics of the function

$$g(x) = \begin{cases} -1 & -\pi \le x \le 0 \\ 1 & 0 \le x \le \pi \end{cases}$$

ANSWER:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\,dx = \frac{1}{2\pi}\left[\int_0^{\pi} -1\,dx + \int_{-\pi}^{0} 1\,dx\right]$$
$$= \frac{1}{2\pi}\left[-x\Big|_0^{\pi} + x\Big|_{-\pi}^{0}\right] = 0$$
$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos kx\,dx = \frac{1}{\pi}\left[\int_0^{\pi} -\cos kx\,dx + \int_{-\pi}^{0}\cos kx\,dx\right]$$
$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin kx\,dx = \frac{1}{\pi}\left[\int_0^{\pi} -\sin kx\,dx + \int_{-\pi}^{0}\sin kx\,dx\right]$$

$a_1 = 0$ $\qquad b_1 = -\frac{4}{\pi}$

$a_2 = 0$ $\qquad b_2 = 0$

The first harmonic is $-\frac{4}{\pi}\sin x$ and the second is 0.

4. Find a_0 and the first two harmonics of the function

$$h(x) = \begin{cases} \pi & -\pi \le x \le 0 \\ 0 & 0 \le x \le \pi \end{cases}$$

ANSWER:

$$a_1 = \frac{1}{\pi}\int_{-pi}^{\pi} h(x)\cos(kx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} \pi\cos x\,dx = \frac{1}{\pi}(\pi\sin x)\Big|_{-\pi}^{0} = 0$$

$$a_2 = 0$$
$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} h(x)\sin(kx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} \pi\sin x\,dx = \frac{1}{\pi}(-\pi\cos x)\Big|_{-\pi}^{0} = -2$$
$$b_2 = \frac{1}{\pi}\int_{-\pi}^{0} \pi\sin 2x\,dx = \frac{1}{\pi}(-\frac{\pi}{2}\cos 2x)\Big|_{-\pi}^{0} = 0$$

The first harmonic is $-2\sin x$ and the second is 0.

5. Find the third-degree Fourier polynomial for

$$f(t) = \begin{cases} 0 & -2 \le t \le 0 \\ c & 0 \le t \le 2 \end{cases}$$

where c is a constant by writing a new function, $g(x) = f(t)$ with period 2π.

ANSWER:

Since $f(t)$ has period $b = 4$, we let $t = \dfrac{bx}{2\pi} = \dfrac{2x}{\pi}$ and use the function $g(x) = f\left(\dfrac{2k}{\pi}\right)$ which has period 2π.

$$g(x) = \begin{cases} 0 & -\pi \le x \le 0 \\ c & 0 \le x \le \pi \end{cases}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\,dx = \frac{1}{2\pi}\int_{0}^{\pi} c\,dx = \frac{1}{2\pi}(cx)\Big|_{0}^{\pi} = \frac{c}{2}$$
$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos x\,dx = \frac{1}{\pi}\int_{0}^{\pi} c\cos x\,dx = \frac{1}{\pi}c\sin x\Big|_{0}^{\pi} = 0$$
$$\text{Similarly, } a_2 = 0$$
$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin x\,dx = \frac{1}{\pi}\int_{0}^{\pi} c\sin x\,dx = \frac{1}{\pi}(-c\cos x)\Big|_{0}^{\pi} = \frac{2c}{\pi}$$
$$b_2 = \frac{1}{\pi}\int_{0}^{\pi} c\sin 2x\,dx = \frac{1}{\pi}\left(-\frac{c\cos 2x}{2}\right)\Big|_{0}^{\pi} = 0$$
$$b_3 = \frac{1}{\pi}\int_{0}^{\pi} c\sin 3x\,dx = \frac{1}{\pi}\left(-\frac{c\cos 3x}{3}\right)\Big|_{0}^{\pi} = \frac{2c}{3\pi}$$

Therefore the Fourier polynomial of degree three is given by

$$g(x) \approx \frac{c}{2} + \frac{2c}{\pi}\sin x + \frac{2c}{3\pi}\sin 3x$$

Substituting $x = \dfrac{\pi t}{2}$,

$$f(t) = g(x) = g\left(\frac{\pi t}{2}\right) \approx \frac{c}{2} + \frac{2c}{\pi}\sin\left(\frac{\pi t}{2}\right) + \frac{2c}{3\pi}\sin\left(\frac{3\pi t}{2}\right)$$

Review Questions and Solutions for Chapter 10

1. Suppose that g is the pulse train of width 0.5.

(a) What fraction of energy of g is contained in the constant term of its Fourier series?

(b) What fraction of the energy is contained in the constant term and the first harmonic together?

ANSWER:

(a) The energy of the pulse train g is

$$E = \frac{1}{\pi}\int_{-\pi}^{\pi} [g(x)]^2\,dx = \frac{1}{\pi}\int_{-1/4}^{1/4} 1^2\,dx = \frac{1}{\pi}\left(\frac{1}{4} - \left(-\frac{1}{4}\right)\right) = \frac{1}{2\pi}$$

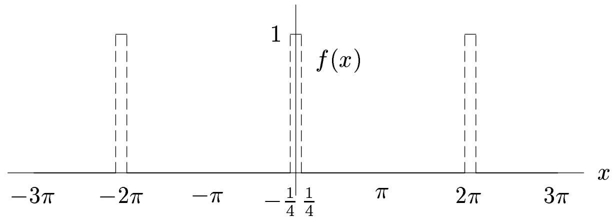


Figure 10.5.114

Find the coefficients.

$$a_0 = \text{ average value of } g \text{ on } [-\pi,\pi] = \frac{1}{2\pi}(\text{area}) = \frac{1}{2\pi}\left(\frac{1}{2}\right) = \frac{1}{4\pi}$$
$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos kx\,dx = \frac{1}{\pi}\int_{-1/4}^{1/4}\cos kx\,dx = \frac{1}{k\pi}\sin kx\Big|_{-1/4}^{1/4}$$
$$a_1 = \frac{1}{\pi}\left(\sin\left(\frac{1}{4}\right) - \sin\left(-\frac{1}{4}\right)\right) = 0.158$$
$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin kx\,dx = \frac{1}{\pi}\int_{-1/4}^{1/4}\sin kx\,dx = -\frac{1}{k\pi}\cos kx\Big|_{-1/4}^{1/4}$$
$$b_1 = -\frac{1}{\pi}\left(\cos\left(\frac{1}{4}\right) - \cos\left(-\frac{1}{4}\right)\right) = 0$$

The energy of g contained in the constant term is

$$A_0^2 = 2a_0^2 = 2\left(\frac{1}{4\pi}\right)^2 = \frac{1}{8\pi^2}$$

Which is $\frac{A_0^2}{E} = \frac{\frac{1}{8\pi^2}}{\frac{1}{2\pi}} = \frac{1}{4\pi} \approx 0.0796 = 7.96\%$ of the total.

(b) The fraction of energy contained in the first harmonic is

$$\frac{A_1^2}{E} = \frac{a_1^2}{E} \approx 0.157 = 15.7\%.$$

The fraction of energy contained in both the constant term and the first harmonic together is

$$\frac{A_0^2}{E} + \frac{A_1^2}{E} \approx 23.66\%$$

2. Find the Taylor series for $\sin x - \cos x$ about $x = 0$.
ANSWER:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$
$$\sin x - \cos x = -1 + x + \frac{x^2}{2!} - \frac{x^3}{3!} - \frac{x^4}{4!} + \frac{x^5}{5!} - \cdots$$

3. Use the binomial series to expand $\frac{1}{(1+2x)^4}$ as a power series. State the radius of convergence.
ANSWER:

$$\frac{1}{(2x+4)^4} = \sum_{n=0}^{\infty}(-1)^n\frac{(n+1)(n+2)(n+3)2^n}{6}x^n.$$

Radius of convergence is $R = \frac{1}{2}$.

4. Find the second and third degree Taylor Polynomial approximations of $\dfrac{1}{1+x^2}$ about $x = 1$.

ANSWER:
At $x = 1$, $\dfrac{1}{1+x^2} = \dfrac{1}{2} = f(0)$.

$$\begin{aligned}
f(x) &= (1+x^2)^{-1} & & \\
f'(x) &= -2x(1+x^2)^{-2} & \qquad f'(1) &= -\frac{1}{2} \\
f''(x) &= 8x^2(1+x^2)^{-3} & \qquad f''(1) &= 1 \\
f'''(x) &= -48x^3(1+x^2)^{-4} & \qquad f'''(1) &= -3
\end{aligned}$$

Second-degree approximation is $f(a) + f'(a)(x-a) + f''(a)(x-a)^2$

$$= \frac{1}{2} - \frac{1}{2}(x-1) + \frac{(x-1)^2}{2!}.$$

Third-degree approximation is

$$\frac{1}{2} - \frac{1}{2}(x-1) + \frac{(x-1)^2}{2!} - \frac{3(x+1)^3}{3!}.$$

Chapter 11 Exam Questions

Questions and Solutions for Section 11.1

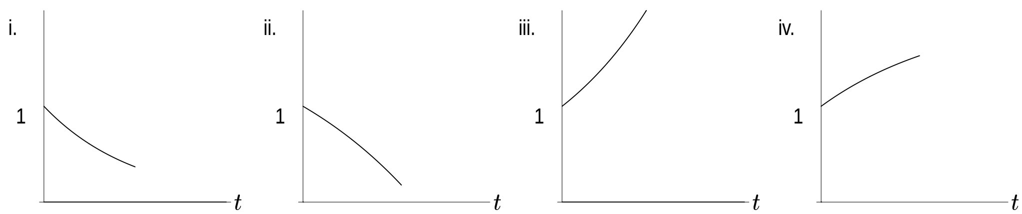
1. Suppose that the function $P(t)$ satisfies the differential equation

$$P'(t) = P(t)(4 - P(t))$$

with the initial condition $P(0) = 1$. Even without knowing an explicit formula for $P(t)$ we can find many of its properties. For example, note first that

$$P'(0) = P(0)(4 - P(0)) = 1(4 - 1) = 3.$$

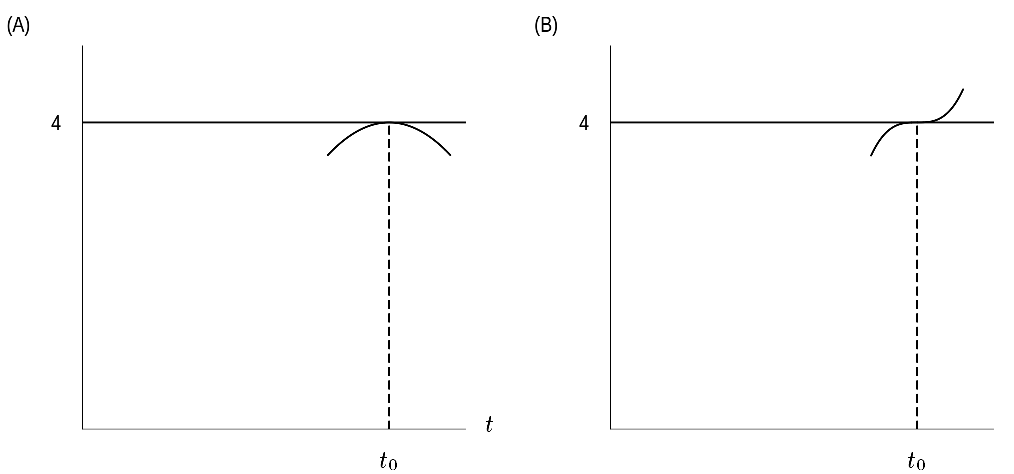
(a) Find $P''(t)$ in terms of $P(t)$. Find $P''(0)$.
(b) Which of the following is a possible graph for $P(t)$ for small $t > 0$? Explain.



(c) Since $P(0) = 1$, the function $P(t)$ starts out less than 4. If it reaches 4, that is, if there is a first time t_0 where $P(t_0) = 4$ then

$$P'(t_0) = P(t_0)(r - P(t_0)) = 0.$$

So near t_0 the graph of P would look like either of the graphs below:



Is either of these consistent with $P(t)$ satisfying the equation $P'(t) = P(t)(4 - P(t))$? Explain.
(d) Sketch the complete graph of the function $P(t)$, $t > 0$. Explain any critical points, inflection points, concavity, and the behavior of $P(t)$ as $t \to \infty$.

ANSWER:

(a)

$$P''(t) = \frac{d}{dt}(P'(t)) = \frac{d}{dt}\left(P(t)(4 - P(t))\right)$$

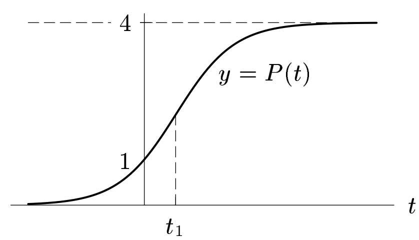
$$
\begin{aligned}
&= \left(\frac{d}{dt}P(t)\right)(4 - P(t)) + P(t)\frac{d}{dt}(4 - P(t)) \\
&= 4P'(t) - P'(t)P(t) - P(t)P'(t) \\
&= P'(t)(4 - 2P(t)) \\
&= P(t)(4 - P(t))(4 - 2P(t)) \\
P''(0) &= P(0)(4 - P(0))(4 - 2P(0)) \\
&= 6
\end{aligned}
$$

(b) We know that $P''(0) = 6, P'(0) = 3$. Since $P'(0) > 0$, i. and ii. are out. Since $P''(0) > 0$, iv. is out (it is concave down), so the correct answer is iii.

(c) Neither of the two graphs is consistent. In the first graph, when $t > t_0$, $P(t) < 4$, so $P'(t) = P(t)(4 - P(t)) > 0$, which contradicts the fact that $P(t)$ is decreasing when $t > t_0$.

In the second graph, when $t > t_0$, $P(t) > 4$, so $P'(t) = P(t)(4 - P(t)) < 0$. This contradicts the fact that $P(t)$ is increasing when $t > t_0$. We can also conclude that $P(t)$ can never reach 4.

(d) From (c), we know that $P(t)$ never reaches 4. An argument similar to that of (c) shows that $P(t)$ never reaches 0 as t decreases. So $0 < P(t) < 4$, and therefore $P'(t) = P(t)(4 - P(t)) > 0$; i.e. $P(t)$ is increasing and has no critical points. $P(t)$ has one inflection point, t_1, where $P(t_1) = 2$. When $t > t_1$, $P''(t) < 0$, so $P(t)$ is concave down. When $t < t_1$, $P''(t) > 0$, so $P(t)$ is concave up. Finally, $\lim_{t\to\infty} P(t) = 4$.



2. Show that $y = 3\cos 3t$ satisfies $\frac{d^2y}{dt^2} + 9y = 0$.

ANSWER:

If $y = 3\cos 3t$, then $\frac{dy}{dt} = -9\sin 3t$, and $\frac{d^2y}{dt^2} = -27\cos 3t$.

So $\frac{d^2y}{dt^2} + 9y = -27\cos 3t + 9(3\cos 3t) = 0$.

3. Find the value(s) of w for which $y = e^{wt}$ satisfies

$$\frac{d^2y}{dt^2} - 16y = 0$$

ANSWER:

If $y = e^{wt}$, $\frac{dy}{dt} = we^{wt}$ and $\frac{d^2y}{dt^2} = w^2e^{wt}$.

$$
\begin{aligned}
\text{So } \frac{d^2y}{dt^2} - 16y = w^2e^{wt} - 16e^{wt} &= 0 \\
e^{wt}(w^2 - 16) &= 0 \\
w &= \pm 4
\end{aligned}
$$

4. Pick out which functions are solutions to which differential equation. (Note: Functions may be solutions to more than one equation or to none; an equation may have more than one solution.)

(a) $y = \sin 2x + \cos 2x$

(b) $y = 2\cos x - \sin x$

(c) $y = e^{\frac{x}{2}}$

(d) $y = \frac{e^{-x}}{2}$

$I) \quad \frac{dy}{dx} = \frac{y}{2}$

$II) \quad \frac{dy}{dx} = -y$

$III) \quad \frac{d^2y}{dx^2} = -4y$

$IV) \quad \frac{d^2y}{dx^2} = -y$

$V) \quad \frac{d^2y}{dx^2} = y$

ANSWER:

(a) $y = \sin 2x + \cos 2x$

$\frac{dy}{dx} = 2\cos 2x - 2\sin 2x$

$\frac{d^2y}{dx^2} = -4\sin 2x - 4\cos 2x = -4y \ (III)$

(b) $y = 2\cos x - \sin x$

$\frac{dy}{dx} = -2\sin x - \cos x$

$\frac{d^2y}{dx^2} = -2\cos x + \sin x = -y \ (IV)$

(c) $y = e^{\frac{x}{2}}$

$\frac{dy}{dx} = \frac{1}{2}e^{\frac{x}{2}} = \frac{y}{2} \ (I)$

$\frac{d^2y}{dx^2} = \frac{1}{4}e^{\frac{x}{2}}$

(d) $y = \frac{e^{-x}}{2}$

$\frac{dy}{dx} = -\frac{e^{-x}}{2} = -y \ (IV)$

$\frac{d^2y}{dx^2} = \frac{e^{-x}}{2} = y \ (V)$

5. If $\frac{d^2S}{dt^2} = -9.8$, find S if the initial velocity is 15 m/sec upward and the initial position is 7 m above the ground.

ANSWER:

$$\frac{d^2S}{dt^2} = -9.8$$

Integrating gives us $\frac{dS}{dt} = -9.8t + C_1$
and integrating again gives $S = -4.9t^2 + C_1t + C_2$.

$$S = -4.9t^2 + 15t + 7$$

6. For what values of n (if any) is $y = e^{xn}$ a solution to the differential equation

$$-\frac{1}{3}y'' + y' + 6y = 0$$

ANSWER:

$y' = ne^{xn},\ y'' = n^2 e^{xn}$

$$-\frac{1}{3}y'' + y' + 6y = -\frac{1}{3}n^2 e^{nx} + ne^{nx} + 6e^{nx} = 0$$

$$e^{nx}\left(-\frac{1}{3}n^2 + n + 6\right) = 0$$

$$n = -3, 6$$

7. Is $y = e^x \sin x$ a solution to $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0$?

ANSWER:

$$\frac{dy}{dx} = e^x \cos x + e^x \sin x$$

$$\frac{d^2y}{dx^2} = 2e^x \cos x$$

$$\text{So } \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 2e^x \cos x - 2\left(e^x \cos x + e^x \sin x\right) + 2\left(e^x \sin x\right)$$

$$= \left(2e^x \cos x - 2e^x \cos x\right) + \left(2e^x \sin x - 2e^x \sin x\right)$$

$$= 0$$

So $y = e^x \sin x$ is a solution to $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0$.

Questions and Solutions for Section 11.2

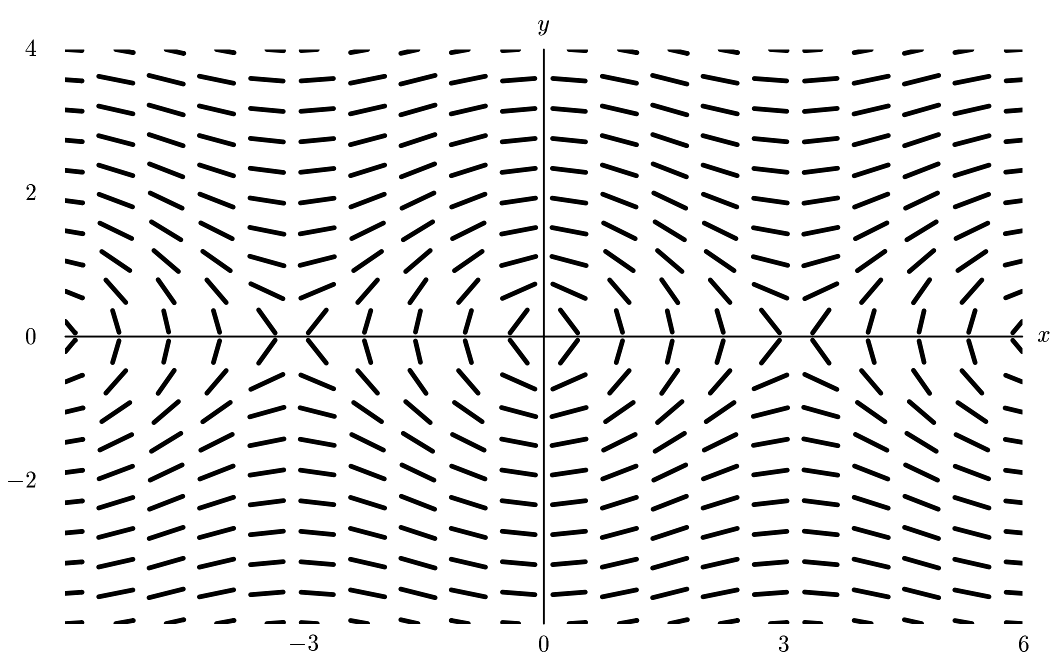
1. Note that the point $(0, 2)$ is on the graph of each of the following three equations:

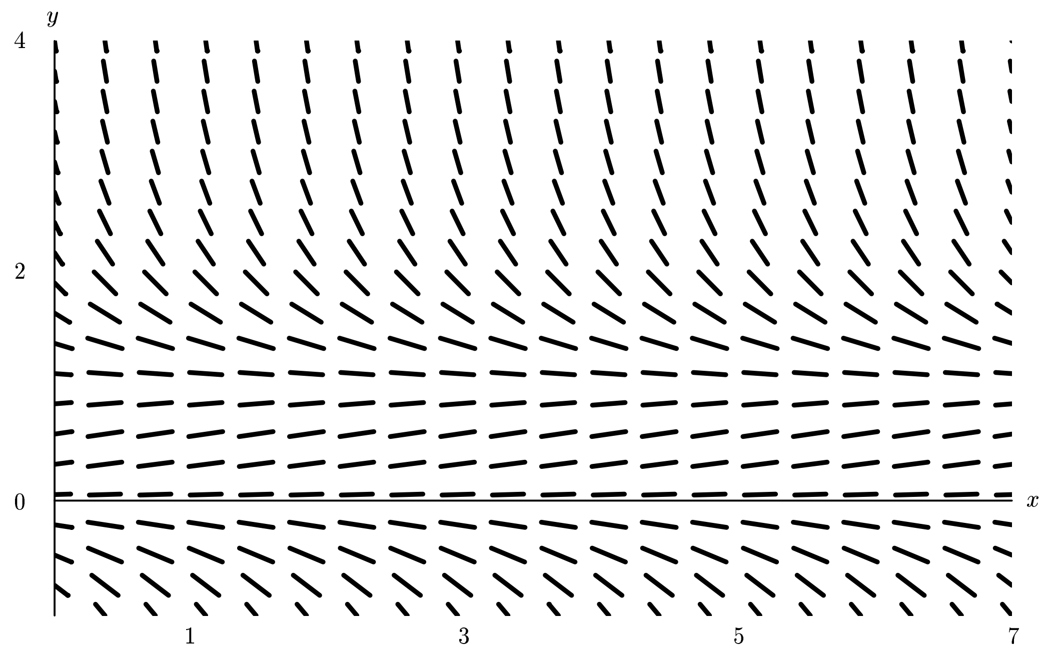
(a) $y^2 - 2\cos x = 2$

(b) $x \sin y + y = 2$

(c) $\ln|y/(1-y)| = 0.71x + \ln 2$

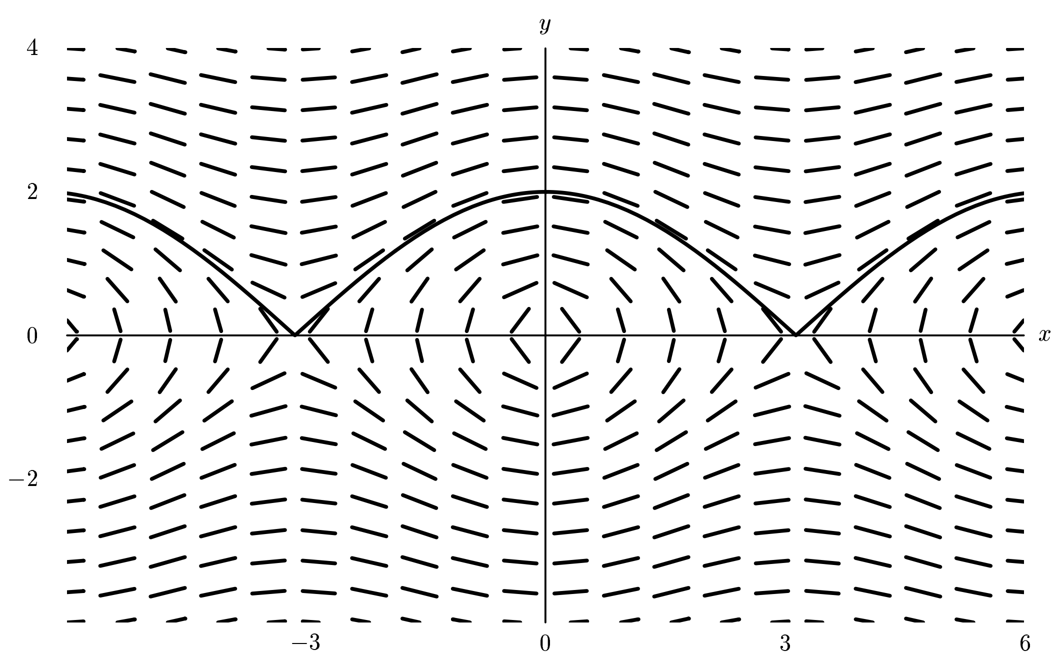
Following are slope fields for two of the three equations. Identify which two equations have these slope fields. Label each graph with the letter a, b, or c. and explain why you made each choice. On each graph, draw the curve described by the appropriate equation (a, b, or c) that goes through the point $(0, 2)$.



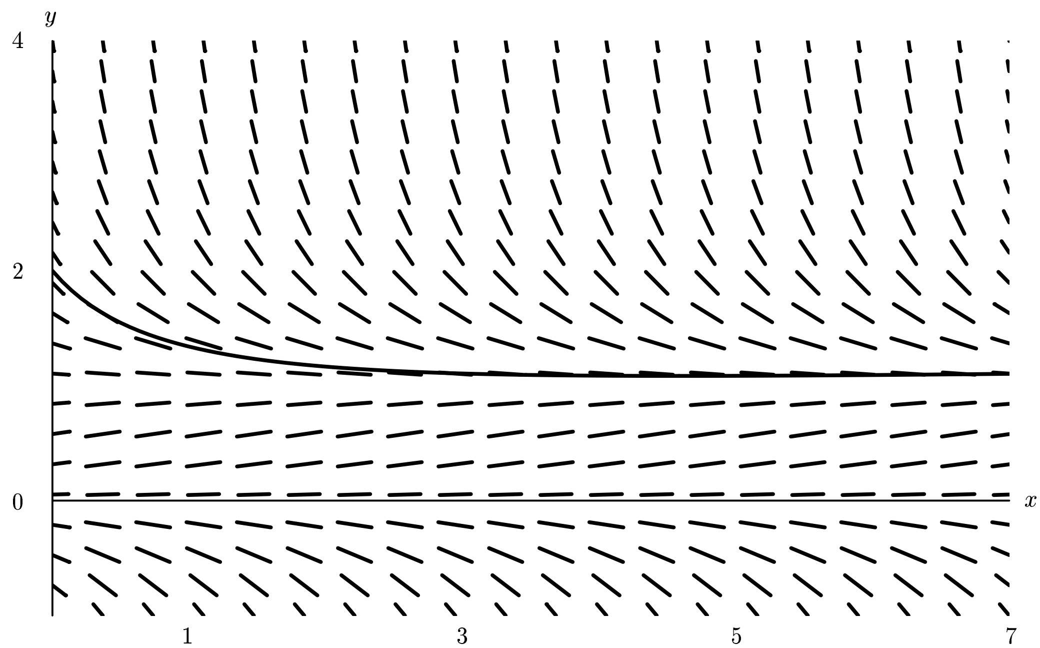


ANSWER:

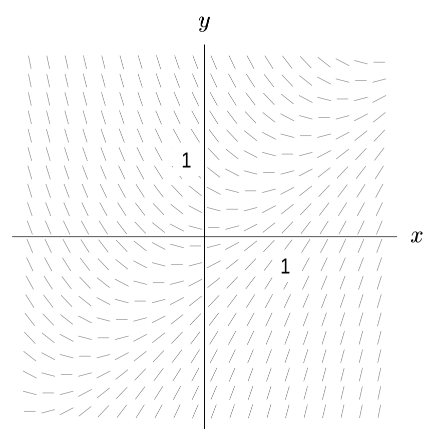
The first slope field seems to repeat every π units in the x-direction, so it corresponds to the slope field given by equation (a).



The second slope field does not depend on x, so it must correspond to equation (c).

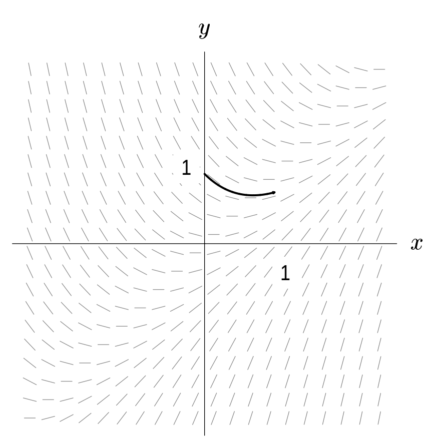


2. The slope field for the differential equation $\dfrac{dy}{dx} = x - y$ is shown below.

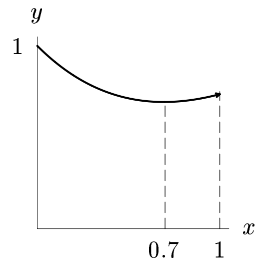


On the slope field, sketch the solution curve to the differential equation starting at $x = 0$, $y = 1$ and ending at $x = 1$. From your sketch, approximate the value of y when $x = 1$.

ANSWER:

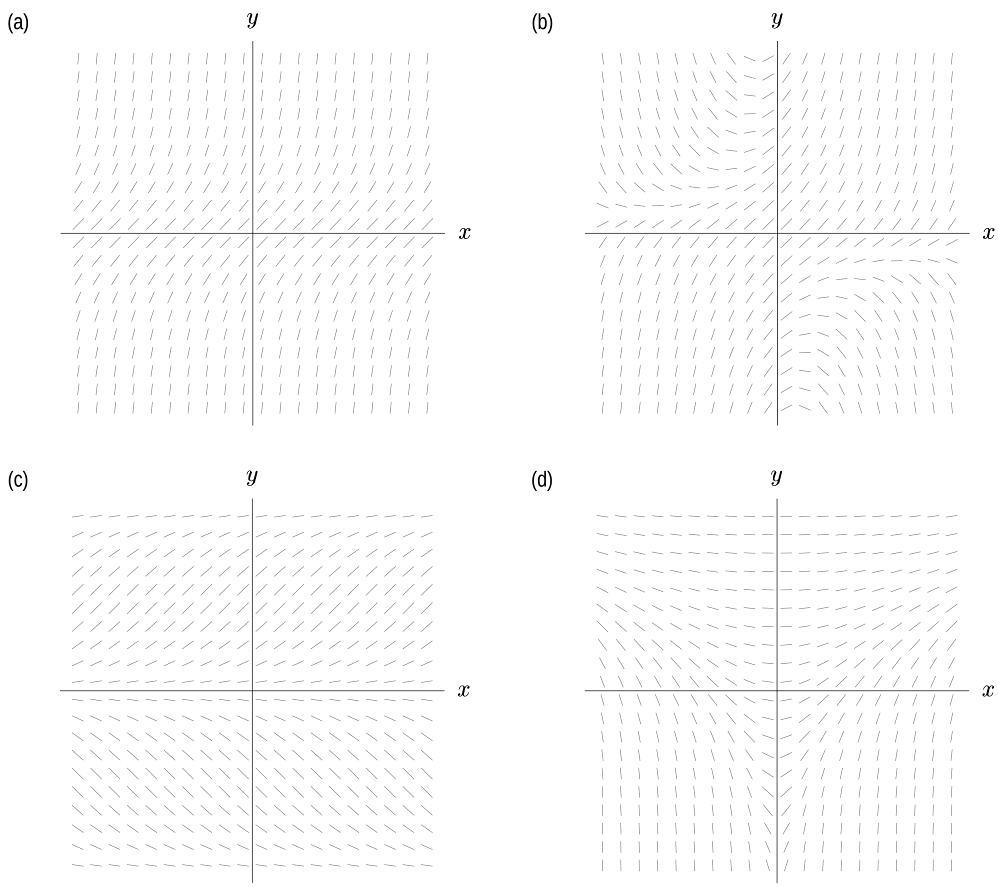


The sketch (shown above) looks like this:



Notice the solution curve starts increasing at about $x = 0.7$. At $x = 1$, it looks like $y \approx 0.75$.

3. Match the four direction fields (slope fields) with four of the differential equations. (One equation does not match!) No reasons are required.



(i) $y' = xy + 1$ corresponds to graph ____

(ii) $y' = \sin x$ corresponds to graph ____

(iii) $y' = xe^{-y}$ corresponds to graph ____

(iv) $y' = y^2 + 1$ corresponds to graph ____

(v) $y' = \sin y$ corresponds to graph ____

ANSWER:

(i) $y' = xy + 1$ corresponds to graph (b)
(ii) $y' = \sin x$ corresponds to none of the graphs
(iii) $y' = xe^{-y}$ corresponds to graph (d)
(iv) $y' = y^2 + 1$ corresponds to graph (a)
(v) $y' = \sin y$ corresponds to graph (c)

4. Match the slope fields to the equations. Explain.
[Note: One slope field will not have a corresponding equation.]

$$\text{(I) } \frac{dy}{dx} = (x-y)^2 \qquad \text{(II) } \frac{dy}{dx} = (x+y)^2 \qquad \text{(III) } \frac{dy}{dx} = x^2 - y^2$$

(a)

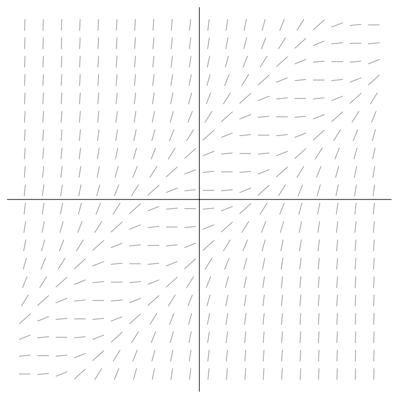


Figure 11.2.115

(b)

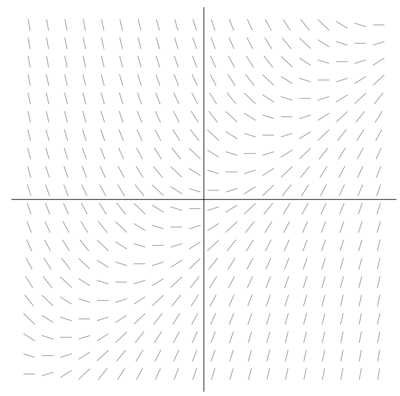


Figure 11.2.116

(c)

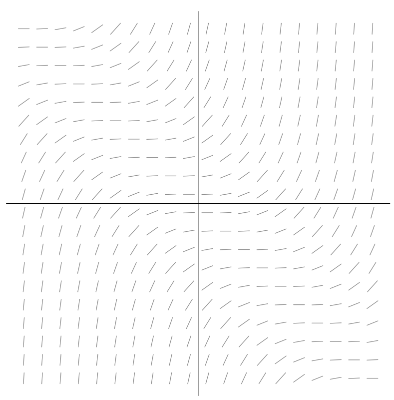


Figure 11.2.117

(d)

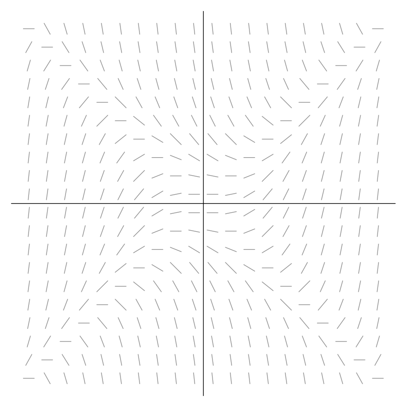


Figure 11.2.118

ANSWER:

(I) corresponds to (a).
Since $(x-y)^2 \geq 0$, the equation in (I) doesn't produce the slope field in (b) or in (d). Note that $(x-y)^2 = 0$ when $y = x$, i.e., along the line $y = x$, $\frac{dy}{dx} = 0$. This fits with (a).
(II) corresponds to (c).
Again, we can exclude (b) and (d) as $\frac{dy}{dx} = (x+y)^2 \geq 0$ for all x and y, so the slope must be positive everywhere. Since $\frac{dy}{dx} = (x+y)^2 = 0$ if and only if $y = -x$, we should find little horizontal line segments along the line $y = -x$ on the slope field. This occurs in (c), but not in (a), so (c) must be the answer.
(III) corresponds to (d).
We can exclude (a) and (c) because $\frac{dy}{dx}$ is negative at some points. Notice that $\frac{dy}{dx} = x^2 - y^2 = (x-y)(x+y)$, so $\frac{dy}{dx} = 0$ where $y = x$ or $y = -x$. The corresponding slope field should thus have horizontal line segments on the lines $y = x$ and $y = -x$. This is the case only in (d).

5. This problem concerns the differential equation

$$\frac{dy}{dx} = x - \frac{1}{2}y.$$

(a) Show that $y = 2x - 4$ is the unique solution of the equation that is a straight line. (Write $y = ax + b$ and show that a must equal 2 and b must equal -4.)

(b) Give a reasonable sketch of the direction field for the equation. Take account of your answer to part (a). (If you use the slope field program, try $-4 \leq x < 4$, $-4 \leq y \leq 4$ with x scale and y scale both 1.)

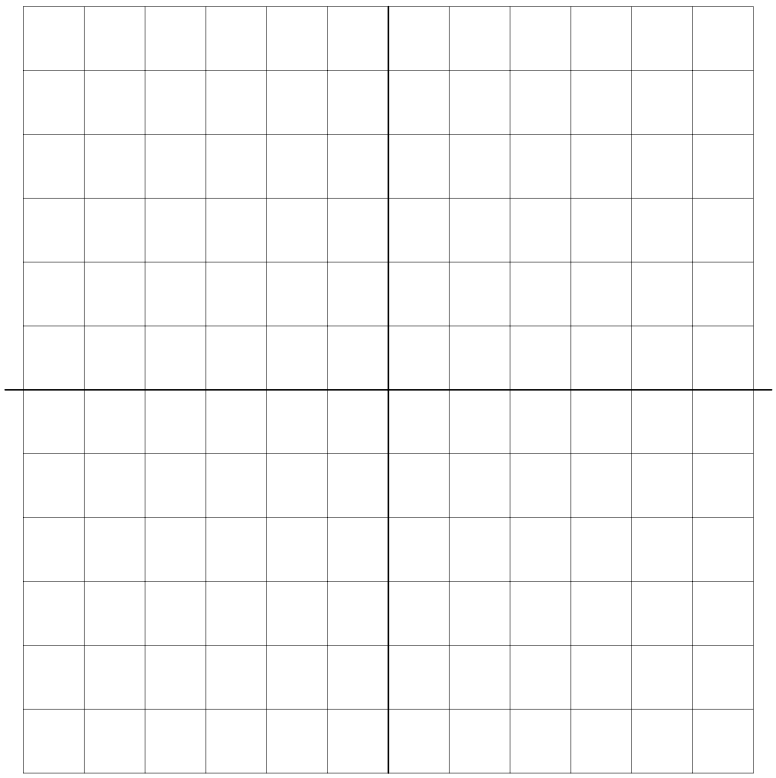


Figure 11.2.119

(c) Take a point (x_0, y_0) in the first quadrant which does not lie on the line in part (a). Can a solution curve through (x_0, y_0) cross the line? Why or why not?

(d) Show that

$$y = 2x - 4 + Ce^{-\frac{x}{2}}$$

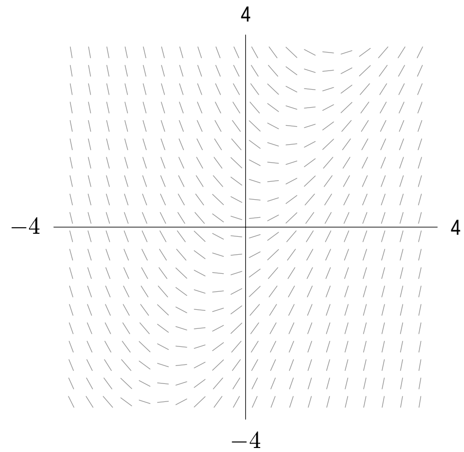
is a solution for *any* constant C.

(e) Find the solution passing through the point $(0, -2)$ and describe its qualitative behavior as $x \to \pm\infty$.

ANSWER:

(a) Suppose we substitute $y = ax + b$ into the right hand side of the equation to get $x - \frac{1}{2}(ax + b) = (1 - \frac{a}{2})x - \frac{b}{2}$. But we know that the left-hand side of the equations, $\frac{dy}{dx}$, must be a constant, namely a. This implies that the factor multiplying x in the right-hand side is zero, so $a = 2$. Then we are left with $a = -\frac{b}{2}$, so $b = -4$. Hence $y = 2x - 4$ is the sole linear solution.

(b)



(c) No. To find a solution curve for the differential equation, we only need to know one point it passes through (and then we can solve for the constants in this first order equation). If some two curves cross, then the point of intersection must define two different solution curves which is clearly not possible, so each point will yield one solution curve.

(d) For $y = 2x - 4 + Ce^{-\frac{x}{2}}$, $\frac{dy}{dx} = 2 - \frac{1}{2}Ce^{-\frac{x}{2}}$. On the other hand, $x - \frac{1}{2}y = x - (x - 2 + \frac{1}{2}Ce^{-\frac{x}{2}} = 2 - \frac{1}{2}Ce^{-\frac{x}{2}}$. Therefore, $y = 2x - 4 + Ce^{-\frac{x}{2}}$ is a solution of $\frac{dy}{dx} = x - \frac{1}{2}y$.

(e) For $x = 0, y = -2$, we have $-2 = -4 + C; C = 2$. So $y = 2x - 4 + 2e^{-\frac{x}{2}}$.
As $x \to \infty, y \to 2x - 4$. As $x \to -\infty, y \to \infty$ since $e^{-\frac{x}{2}} \to \infty$.

6. If a slope field for $\dfrac{dy}{dx}$ has constant slopes where x is constant, what do you know about $\dfrac{dy}{dx}$?
ANSWER:
We know that $\dfrac{dy}{dx}$ depends on x only.

7. If all the solutions curves for $\dfrac{dy}{dt}$ have $y = -3$ as a horizontal asymptote, what do you know about y?
ANSWER:
We know $\lim\limits_{t\to\infty} y = -3$.

Questions and Solutions for Section 11.3

1. Consider the differential equation

$$\frac{dy}{dx} = x^2 + y.$$

(a) Use Euler's method with two steps to approximate the value of y when $x = 2$ on the solution curve that passes through (1,3). Explain clearly what you are doing on a sketch. Your sketch should show the coordinates of all the points you have found.

(b) Are your approximate values of y an under- or over-estimate? Explain how you know.

ANSWER:

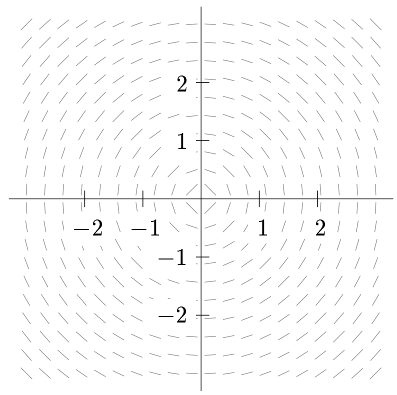
(a) (Use $\Delta x = 0.5$.)

x	y	slope	Δy
1	3	4	2
1.5	5	7.25	3.625
2	8.625		

$y = 8.625$

(b) From the slope field, we see that the curve is concave up, so we have an underestimate.
Alternatively, $y'' = \dfrac{d}{dx}(y') = \dfrac{d}{dx}(x^2 + y) = 2x + y' = 2x + x^2 + y > 0$.

2. The slope field for the differential equation $\frac{dy}{dx} = -\frac{x}{y}$ is shown below.



(a) Starting from the point $x = 0$, $y = 2$, use Euler's method with $N = 3$ subdivisions to approximate the value of y when $x = 1$.
(b) Sketch on the slope field where each step of Euler's method takes you. Based on your sketch, would you say Euler's method provides an underestimate or an overestimate of the true solution?
(c) Show that the equation $x^2 + y^2 = C$, where C is a constant, satisfies the differential equation, and find the value of C for the solution passing through the starting point $x = 0$, $y = 2$.

ANSWER:

(a) $\Delta x = \frac{1}{N} = \frac{1}{3}$

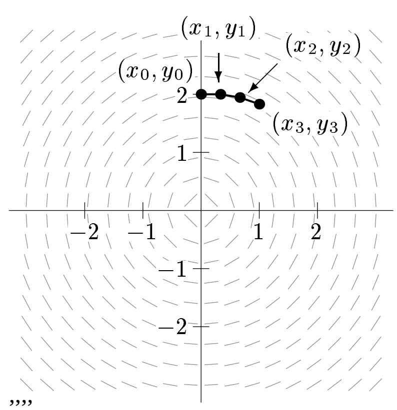
At $(0, 2)$, $\frac{dy}{dx} = -\frac{0}{2} = 0$. Therefore, $y_1 = y_0 + \frac{dy}{dx}\Delta x = 2 + 0\left(\frac{1}{3}\right) = 2$.

At $\left(\frac{1}{3}, 2\right)$, $\frac{dy}{dx} = -\frac{\frac{1}{3}}{2} = -\frac{1}{6}$. Therefore, $y_2 = y_1 + \frac{dy}{dx}\Delta x = 2 - \frac{1}{6}\left(\frac{1}{3}\right) = \frac{35}{18} \approx 1.944$.

At $\left(\frac{2}{3}, \frac{35}{18}\right)$, $\frac{dy}{dx} = -\frac{\frac{2}{3}}{\frac{35}{18}} = -\frac{12}{35}$. Therefore, $y_3 = y_2 + \frac{dy}{dx}\Delta x = \frac{35}{18} - \frac{12}{35}\left(\frac{1}{3}\right) = \frac{1153}{630} \approx 1.8302$.

Since $y(1) \approx y_3$, $y(1) \approx 1.8302$.

(b) See the slope field below. The solution curve starting at $(0, 2)$ is concave down for $0 \le x \le 1$. Therefore, Euler's method produces an <u>overestimate</u> of the solution, which should also be apparent from the sketch on the slope field.



,,,,

(c) $y = \sqrt{C - x^2} \Rightarrow \frac{dy}{dx} = \frac{-2x}{2\sqrt{C - x^2}} = -\frac{x}{\sqrt{C - x^2}} = -\frac{x}{y}$

Therefore, the equation $x^2 + y^2 = C$ satisfies the differential equation. Since the particular solution of interest passes through $(0, 2)$, plug that point into the equation to find $C = 4$.

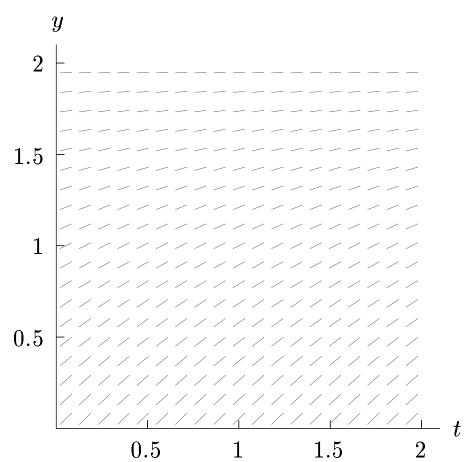
As an aside, notice this means that the true solution to the differential equation passing through $(0, 2)$ is the circle $x^2 + y^2 = 4$. Thus, the true value of y when $x = 1$ is $y = \sqrt{4 - 1^2} = \sqrt{3} = 1.7321$. The result from part (b) that Euler's method provides an overestimate of the solution is, therefore, correct.

3. In a number of applications, differential equations of the following form appear:

$$\frac{dy}{dt} = k(A - y)$$

For a particular problem, let's say $k = 0.5$, and $A = 2$. You are given that $y = 1$ when $t = 0$, and you are interesting in finding the value of y when $t = 1$.

(a) The slope field for the differential equation is shown below. Sketch the solution to the differential equation starting from $y = 1$, $t = 0$.



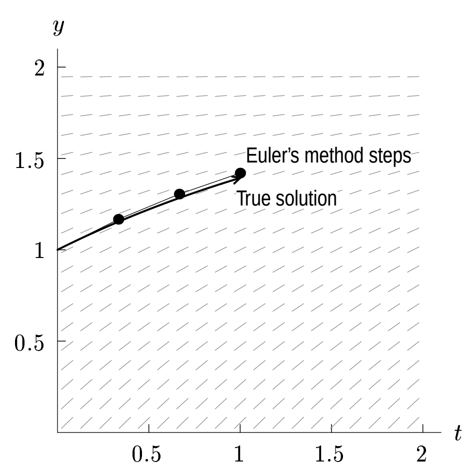
(b) From your sketch in part (a), estimate the value of y when $t = 1$. What happens to y as $t \to \infty$?

(c) Use Euler's method with $N = 3$ subdivisions of the interval $0 \leq t \leq 1$ to find an approximation of y when $t = 1$, if you start from $y = 1$ at $t = 0$.

(d) Sketch on the slope field where each Euler's method step takes you. Is the Euler method approximation of $y(1)$ an overestimate or an underestimate, and why?

ANSWER:

(a)



(b) The solution $y(1)$ looks to be approximately 1.4.
As $t \to \infty$, $y \to 2$.

(c) $\Delta t = \frac{1}{3}$. $y_0 = 1$. $y_0' = 0.5(2 - 1) = 0.5$

$$y_1 = y_0 + y_0'\Delta t = 1 + 0.5\left(\frac{1}{3}\right) = 1.16 \qquad y_1' = 0.5(2 - 1.16) = 0.416$$
$$y_2 = y_1 + y_1'\Delta t = 1.16 + 0.416\left(\frac{1}{3}\right) = 1.305 \qquad y_2' = 0.5(2 - 1.305) = 0.3472$$
$$y(1) \approx y_3 = y_2 + y_2'\Delta t = 1.305 + 0.3472\left(\frac{1}{3}\right) = 1.42\ldots$$

The Euler's method estimate is $y(1) \approx 1.42$.

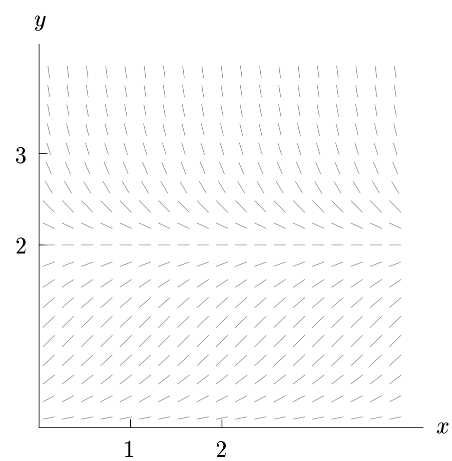
(d) See the slope field above. From the sketch of the solution in part (a), y appears to be concave down, which means that Euler's method yields an overestimate.
The true solution to the differential equation with initial condition $y = 1$ when $t = 0$ is $y(t) = 2 - e^{-0.5t}$. Thus, the exact solution for part (c) is $y(1) = 2 - e^{-0.5} = 1.39346934\ldots$.

4. As we shall see in the study of population growth with limited resources, differential equations such as the following appear:

$$\frac{dy}{dx} = y(2 - y)$$

Given that when $x = 0$, $y = 3$, you will find the value of y when $x = 1$.

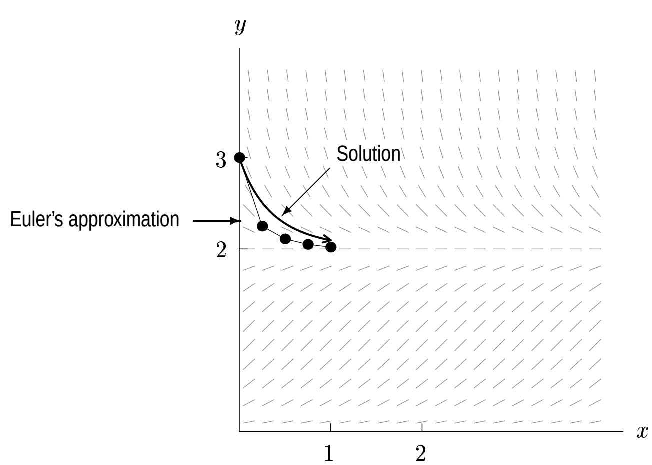
(a) The slope field for the differential equation is shown below. Sketch the solution to the differential equation passing through $(0, 3)$.



(b) From your sketch, estimate the value of y when $x = 1$. Also, what happens to y as $x \to \infty$?
(c) Use Euler's method with $N = 4$ subdivisions of the interval $0 \le x \le 1$ to find an approximation of y when$x = 1$, if you start from the point $(0, 3)$.
(d) Sketch on the slope field where the Euler's method steps take you. In particular, is the Euler method approximation of $y(1)$ an overestimate or an underestimate, and why?

ANSWER:

(a)



(b) $y(1) \approx 2.2$
As $x \to \infty$, $y \to 2$ asymptotically.

(c)

x	y	$\Delta y = \frac{dy}{dx}\Delta x = y(2-y)(0.25)$
0	3	$3(2-3)(0.25) = -0.75$
0.25	2.25	$2.25(2-2.25)(0.25) = -0.14$
0.5	2.11	$2.11(2-2.11)(0.25) = -0.06$
0.75	2.05	$2.05(2-2.05)(0.25) = -0.03$
1.0	2.02	

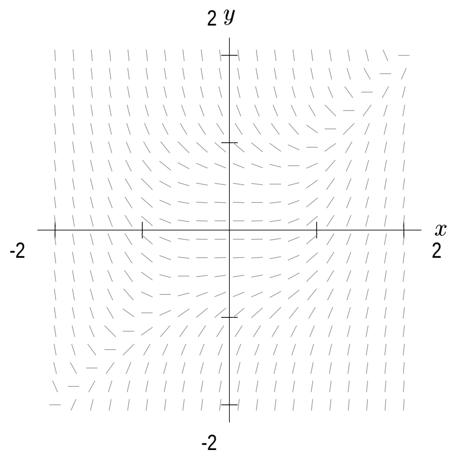
$y(1) \approx 2.02$ by Euler's method, $N = 4$

True Answer: $y(1) = \dfrac{2}{1 - \frac{1}{3}e^{-2}} = 2.09$

(d) Since the graph of $y(x)$ appears to be concave up over the interval, the Euler approximation is an underestimate.

5. For the differential equation represented by the slope field below, sketch the solution curve with $y(0) = 0$.

(a)



(b) On the same slope field, use $\Delta x = 0.5$ to sketch, as accurately as you can, two steps of Euler's approximation to this solution curve. (End at $x = 1$.)

ANSWER:

(a)

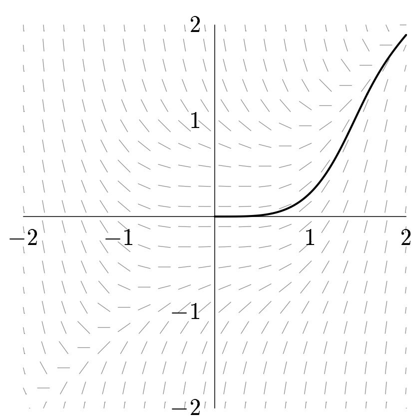


Figure 11.3.120: Solution curve using slope field (for part (a))

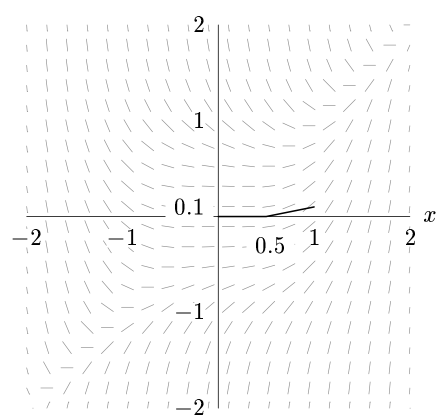


Figure 11.3.121: Euler's method with 2 divisions (for part (b))

(b) Using Euler's method with 2 divisions, we get $y(0.5) = 0.5(0) = 0$. And $y(1) = 0 + 0.5\left(\left.\dfrac{dy}{dx}\right|_{(0,0.5)}\right)$. The slope field at $(0.5, 0)$ is not very steep—it looks like it is approximately 0.2. So $y(1) \approx 0.5(0.2) = 0.1$. Since the solution curve is concave up, the Euler's method approximation must lie below it.

6. Consider the solution with $y(0) = 0$ to the differential equation

$$\frac{dy}{dx} = \frac{4}{1+x^2}$$

(a) What is the exact value of $y(1)$?
(b) If you use Euler's method with 1 million steps to approximate your answer to part (a), will your approximation be an over- or underestimate? Give a reason for your answer.
(c) Use Euler's method with 2 steps to approximate your answer to part (a).

ANSWER:

(a) Solving explicitly for y gives $y = 4\arctan x + C$. Since $y = 0$ when $x = 0$, we have $C = 0$ so $y = 4\arctan x$. Thus $y(1) = 4\arctan 1 = 4(\pi/4) = \pi$. So $y(1) = \pi$.
(b) A graph of y is shown in Figure 11.3.122.

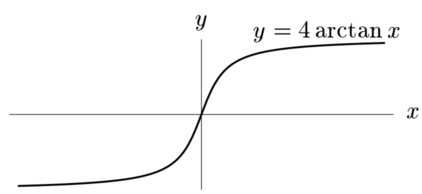


Figure 11.3.122

As we can see from the graph, the solution curve y is concave down, so Euler's method gives an overestimate.
(c) Take $\Delta x = 0.5$. Then $y(0.5) \approx y(0) + y'(0)\cdot\Delta x = 0 + 0.5(4) = 2$. Similarly, $y(1) \approx y(0.5) + y'(0.5)\cdot\Delta x \approx 2 + 0.5[4/(1+0.5)^2] = 2 + 0.5(3.2) = 3.6$. So $y(1) \approx 3.6$.

Questions and Solutions for Section 11.4

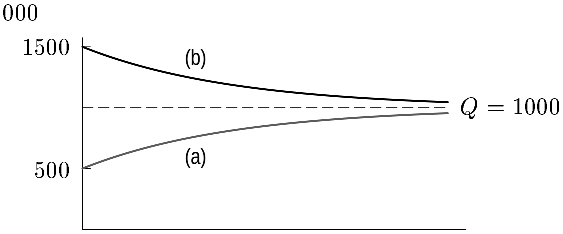
1. Consider the differential equation

$$\frac{dQ}{dt} = 300 - 0.3Q.$$

(a) Solve the differential equation subject to $Q(0) = 500$.
(b) Solve the differential equation subject to $Q(0) = 1500$.
(c) Sketch both solutions on the axes below. Give the coordinates of any intercepts and the equations of any asymptotes.

ANSWER:

(a) $\dfrac{dQ}{300 - 0.3Q} = dt \Rightarrow Q(t) = ce^{-0.3t} + 1000$
Using $Q(0) = 500$, we get $c = -500$.
$Q(t) = -500e^{-0.3t} + 1000$
(b) $Q(t) = 500e^{-.03t} + 1000$
(c)



2. Solve the following differential equations with the given initial conditions:

   (a) $\frac{dy}{dx} = \frac{x}{y} \quad y = 0$ when $x = 1$

   (b) $\frac{dy}{dt} = y^2 + 1 \quad y = 1$ when $t = 0$

   ANSWER:

   (a) $\int y\,dy = \int x\,dx$

   $\frac{y^2}{2} = \frac{x^2}{2} + C$

   $0 = \frac{1}{2} + C$

   so $C = -\frac{1}{2}$

   $y^2 = x^2 - 1$

   (b) $\int \frac{dy}{y^2+1} = \int dt$

   $\arctan y = t + C$

   $\arctan 1 = 0 + C$

   so $C = \arctan 1 = \frac{\pi}{4}$

   $\arctan y = t + \frac{\pi}{4}$

   $y = \tan\left(t + \frac{\pi}{4}\right)$

3. Find the solutions to the following differential equations with the given initial conditions. Solve for y as a function of x, and solve for all constants.

   (a) $\frac{dy}{dx} = \frac{1}{\sqrt{xy}} \quad y = 4$ when $x = 0$

   (b) $\frac{dy}{dx} = \sqrt{4-y^2} \quad y = 1$ when $x = \frac{\pi}{6}$

   ANSWER:

   (a) $\frac{dy}{dx} = \frac{1}{\sqrt{xy}} \Rightarrow \sqrt{y}dy = \frac{1}{\sqrt{x}}dx \Rightarrow \int \sqrt{y}dy = \int \frac{1}{\sqrt{x}}dx \Rightarrow \frac{2}{3}y^{\frac{3}{2}} = 2\sqrt{x} + C$

   Let $x = 0$, $y = 4$ to find C: $\frac{2}{3}(4)^{\frac{3}{2}} = 2\sqrt{0} + C \Rightarrow C = \frac{16}{3}$

   $\frac{2}{3}y^{\frac{3}{2}} = 2\sqrt{x} + \frac{16}{3} \Rightarrow y^{\frac{3}{2}} = 3\sqrt{x} + 8 \Rightarrow y = (3\sqrt{x}+8)^{\frac{2}{3}}$

   (b) $\frac{dy}{dx} = \sqrt{4-y^2} \Rightarrow \frac{dy}{\sqrt{4-y^2}} = dx \Rightarrow \int \frac{dy}{\sqrt{4-y^2}} = \int dx \Rightarrow \arcsin\frac{y}{2} = x + C$

   Let $x = \frac{\pi}{6}$, $y = 1$ to find C: $\arcsin\frac{1}{2} = \frac{\pi}{6} + C \Rightarrow \frac{\pi}{6} = \frac{\pi}{6} + C \Rightarrow C = 0$

   $\arcsin\frac{y}{2} = x$

   $\frac{y}{2} = \sin x$

   $y = 2\sin x$

4. Find the solution to the differential equation $y' = \frac{5}{1+y}$, satisfying $y(0) = 2$.

   ANSWER:

   $y' = \frac{5}{1+y}$ is the same as $\frac{dy}{dx} = \frac{5}{1+y}$, so

   $$\int (1+y)\,dy = \int 5\,dx,$$

   which gives $y + \frac{y^2}{2} = 5x + C$. But $y(0) = 2$, so $C = 4$ and

   $$y + \frac{y^2}{2} = 5x + 4.$$

5. Find the solutions to the following differential equations with the given initial conditions. Solve for y as a function of x, and solve for all constants.

   (a) $\dfrac{dy}{dx} = xy \quad y = 5$ when $x = 0$

   (b) $\dfrac{dy}{dx} = x\sec y \quad y = \dfrac{\pi}{6}$ when $x = 1$

   ANSWER:

   (a) $\dfrac{dy}{dx} = xy \qquad$ initial condition: $(0, 5)$

   $$\Rightarrow \frac{dy}{y} = xdx \Rightarrow \int \frac{dy}{y} = \int xdx \Rightarrow \ln|y| = \frac{x^2}{2} + C_1$$
   $$\Rightarrow e^{\ln|y|} = e^{\frac{x^2}{2}+C_1} \Rightarrow y = e^{\frac{x^2}{2}} \cdot e^{C_1} \Rightarrow y = C_2 e^{\frac{x^2}{2}}$$

   Plug in $(0, 5)$: $5 = C_2 e^0 \Rightarrow C_2 = 5 \Rightarrow y = 5e^{\frac{x^2}{2}}$

   (b) $\dfrac{dy}{dx} = x\sec y \qquad$ initial condition $\left(1, \dfrac{\pi}{6}\right)$

   $$\Rightarrow \frac{dy}{\sec y} = xdx \Rightarrow \int \cos y\, dy = \int xdx \Rightarrow \sin y = \frac{x^2}{2} + C_1$$

   Plug in $\left(1, \dfrac{\pi}{6}\right)$: $\sin\dfrac{\pi}{6} = \dfrac{1}{2} + C_1 \Rightarrow \dfrac{1}{2} = \dfrac{1}{2} + C_1 \Rightarrow C_1 = 0$

   $$\Rightarrow \sin y = \frac{x^2}{2} \Rightarrow y = \sin^{-1}\left(\frac{x^2}{2}\right) \quad \text{or} \quad y = \arcsin\left(\frac{x^2}{2}\right)$$

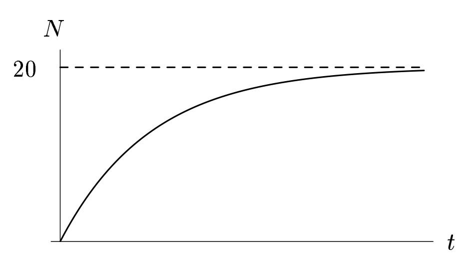
6. Solve $\dfrac{dN}{dt} = 4 - 0.2N$, with $N(0) = 0$, and sketch the solution for $t \geq 0$. Label any intercepts or asymptotes clearly.

   ANSWER:

$$\begin{aligned}
\frac{dN}{dt} &= 4 - 0.2N \\
\int \frac{dN}{4 - 0.2N} &= \int dt \\
\frac{1}{-0.2}\int \frac{dN}{(N-20)} &= \int dt \\
\ln|N - 20| &= -0.2t + C \\
N - 20 &= C' e^{-0.2t}
\end{aligned}$$

but $N(0) = 0$, so $C' = -20$:

$$\begin{aligned}
N &= 20 - 20e^{-0.2t} \\
&= 20(1 - e^{-0.2t})
\end{aligned}$$



7. This problem concerns the differential equation

$$\frac{dy}{dx} = \frac{2x}{y}$$

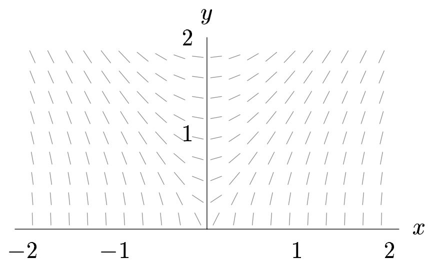
   Note: You can do part (d) without parts (a)–(c).

   (a) Sketch the slope field for the differential equation at the 14 points in the first and second quadrants $-2 \leq x \leq 2$, $0 \leq y \leq 2$ (x and y are integers—don't include $(0, 0)$).

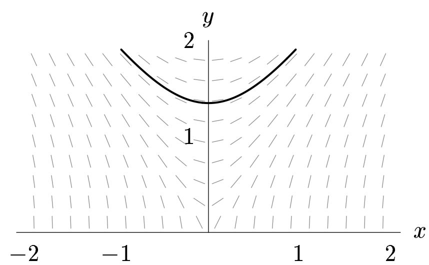
(b) Sketch in a solution curve passing through $(1, 2)$. Do you expect it to show any symmetry? Explain.
(c) Without plotting any more slopes in the third and fourth quadrants, describe what the slope field must look like there, in terms of the slope field in the first and the second quadrants.
(d) Find an equation for the solution to the differential equation $\frac{dy}{dx} = \frac{2x}{y}$ passing through $(1, 2)$. Is it consistent with your answer to part (b)?

ANSWER:

(a)



(b)



The curve is symmetric with respect to the y-axis because the slope at (x, y), which is $\frac{2x}{y}$, is exactly the opposite of the slope at $(-x, y)$, which is $-\frac{2x}{y}$.

(c) The slope field in the third and fourth quadrants can be obtained by reflecting the slope field in the first and second quadrants with respect to the x-axis.

(d)

$$\frac{dy}{dx} = \frac{2x}{y}$$
$$\int y\,dy = \int 2x\,dx$$
$$y^2 = 2x^2 + C.$$

Since the curve passes through $(1, 2)$, $2^2 = 2 + C$, and so $C = 2$. Hence the equation of the curve is $y^2 = 2x^2 + 2$. This curve is indeed symmetric with respect to the y-axis, and so our answer is consistent with part (b).

8. Find a formula for the solution to each of the following differential equations.

(a)

$$\frac{dy}{dx} = \frac{\cos^2 y}{x} \quad \text{with} \quad y(1) = \frac{\pi}{4}$$

(b)

$$xy' - (2 + x)y = 0 \quad \text{with} \quad y(1) = 1, \quad y(x) \geq 0 \text{ for all } x$$

ANSWER:

(a) By manipulating the differential equation, we have

$$\frac{dy}{dx} = \frac{\cos^2 y}{x}$$

$$\begin{aligned}\frac{dy}{\cos^2 y} &= \frac{dx}{x}\\ \int \sec^2 y dy &= \int \frac{1}{x} dx\\ \tan y &= \ln|x| + C\\ y &= \tan^{-1}(\ln|x| + C).\end{aligned}$$

Since $y(1) = \tan^{-1}(C) = \frac{\pi}{4}$, we get that $C = 1$, so

$$y(x) = \tan^{-1}(\ln|x| + 1).$$

(b) By manipulating the differential equation, we have

$$\begin{aligned}y' &= \left(\frac{2+x}{x}\right) y\\ \frac{dy}{dx} &= \left(\frac{2+x}{x}\right) y\\ \frac{dy}{y} &= \left(\frac{2+x}{x}\right) dx\\ \int \frac{dy}{y} &= \int \left(\frac{2}{x} + 1\right) dx\\ \ln|y| &= 2\ln|x| + x + C.\end{aligned}$$

Since $y(x) \geq 0$ for all x, $|y| = y$ so we can write $\ln y = 2\ln|x| + x + C$ or

$$\begin{aligned}y &= e^c \cdot e^{2\ln|x|+x} = e^c \cdot e^{\ln|x|^2} \cdot e^x\\ &= Ax^2 e^x, \quad \text{if we let } e^c = A.\end{aligned}$$

Using $y(1) = 1$ we get $y(1) = A \cdot 1 \cdot e^1 = 1$, or $A = \frac{1}{e}$, so

$$y(x) = \frac{1}{e} \cdot x^2 e^x = x^2 e^{x-1}.$$

9. Solve the following differential equations. Show your work.

(a) $\dfrac{dP}{dt} + 0.1P = 50$

(b) $\dfrac{dy}{dt} = \dfrac{\sin t}{y^2}$

(c) $\dfrac{dy}{dx} = e^{y-x}$ with $y(\ln 2) = -\ln 2$

ANSWER:

(a) We manipulate the differential equation as follows:

$$\begin{aligned}\frac{dP}{dt} &= -0.1(P - 500)\\ \frac{dP}{P - 500} &= -0.1dt\\ \int \frac{dP}{P - 500} &= \int -0.1dt\\ \ln|P - 500| &= -0.1t + C\\ P &= 500 + Ae^{-0.1t}.\end{aligned}$$

(b) We manipulate the differential equation as follows:

$$\begin{aligned}y^2 dy &= \sin t dt\\ \int y^2 dy &= \int \sin t dt\\ \frac{y^3}{3} &= -\cos t + C\\ y &= \sqrt[3]{A - 3\cos t}.\end{aligned}$$

(c) We manipulate the differential equation as follows:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{e^y}{e^x} \\
e^{-y}dy &= e^{-x}dx \\
\int e^{-y}dy &= \int e^{-x}dx \\
-e^{-y} &= -e^{-x} + C \\
e^{\ln 2} &= -e^{-\ln 2} + C, \quad \text{since when } x = \ln 2,\, y = -\ln 2 \\
-2 &= -\frac{1}{2} + C, \quad \text{so } C = -\tfrac{3}{2} \\
e^{-y} &= e^{-x} + \frac{3}{2} \\
y &= -\ln\left(e^{-x} + \frac{3}{2}\right).
\end{aligned}$$

10. Find the general solution to each of the following differential equations:

(a) $\dfrac{dy}{dt} = a - by$

(b) $\dfrac{dH}{dt} = -3H + tH$

ANSWER:

(a) We can manipulate the differential equation as follows:

$$\begin{aligned}
\frac{dy}{dt} &= -b\left(y - \frac{a}{b}\right) \\
\frac{dy}{y - \frac{a}{b}} &= -bdt \\
\int \frac{dy}{y - \frac{a}{b}} &= \int -bdt \\
\ln\left|y - \frac{a}{b}\right| &= -bt + K \\
y - \frac{a}{b} &= Ce^{-bt} \\
y &= \frac{a}{b} + Ce^{-bt}.
\end{aligned}$$

(b) Similarly,

$$\begin{aligned}
\frac{dH}{dt} &= (t-3)H \\
\frac{dH}{H} &= (t-3)dt \\
\int \frac{dH}{H} &= \int (t-3)dt \\
\ln|H| &= \frac{t^2}{2} - 3t + K \\
H &= Ce^{t^2/2-3t}.
\end{aligned}$$

11. (a) Find a formula for the solution to the differential equation

$$\frac{dy}{dx} = y\cos x \quad y(0) = 4.$$

(b) Find the exact minimum and maximum values of the function $y(x)$ you found in part (a).

ANSWER:

(a) Separating variables gives

$$\frac{dy}{dx} = y\cos x$$

$$\frac{dy}{y} = \cos x\, dx$$
$$\int \frac{dy}{dy} = \int \cos x\, dx$$
$$\ln|y| = \sin x + C$$
$$|y| = e^C e^{\sin x}$$
$$y = \pm e^C e^{\sin x}$$
$$y = Ae^{\sin x}$$

Since $y(0) = A = 4$, we get $y = 4e^{\sin x}$.

(b) Since e^t is an *increasing function*, it will reach its minimum at the minimal value of its input and it will reach its maximum at the maximal value of its input. So the minimum is reached when $\sin x = -1$ and hence

$$\text{Min} = 4e^{-1} = \frac{4}{e}.$$

Similarly, the maximum is reached when $\sin x = 1$ and so

$$\text{Max} = 4e.$$

12. Find the general solution to each of the following differential equations.

(a) $\dfrac{d^2y}{dx^2} = x^2 - 3x + 2$

(b) $\dfrac{d^2y}{dx^2} = \cos x$

ANSWER:

(a)

$$\frac{dy}{dx} = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + c_1$$
$$y = \frac{1}{12}x^4 - \frac{1}{2}x^3 + x^2 + c_1x + c_2.$$

(b)

$$\frac{dy}{dx} = \sin x + c_1$$
$$y = -\cos x + c_1x + c_2.$$

Questions and Solutions for Section 11.5

1. Consider the Hakosalo residence in Oulu, Finland. Assume that heat is lost from the house only through windows and the rate of change of temperature in °F/h is proportional to the difference in temperature between the outside and the inside. The constant of proportionality is $\frac{1}{29}$. Assume that it is 10°F outside constantly. On a Thursday at noon the temperature inside the house was 65°F and the heat was turned off until 5 pm.

(a) Write a differential equation which reflects the rate of change of the temperature in the house between noon and 5 pm.

(b) Find the temperature in the house at 5 pm. (You may do this analytically or using your calculator to get a rough estimate.)

(c) At 5 pm the heat is turned on. The heater generates an amount of energy that would raise the inside temperature by 2°F per hour if there were no heat loss. Write a differential equation that reflects what happens to the inside temperature after the heat is turned on.

(d) If the heat is left on indefinitely, what temperature will the inside of the house approach?

ANSWER:

(a) If $y(t)$ = temperature at time t when t is measured in hours since the heat was turned off, then $\dfrac{dy}{dt} = -\dfrac{1}{29}(y - 10), \qquad 0 \le t \le 5$

(b) Separate variables to get

$$\begin{aligned}\int \frac{dy}{y-10} &= -\frac{1}{29}\int dt \\ \ln|y-10| &= -\frac{1}{29}t + C \\ y - 10 &= C'e^{-\frac{1}{29}t} \\ y &= C'e^{-\frac{1}{29}t} + 10\end{aligned}$$

If the heat is turned off at noon, when $t = 0$, $y = 65$. Then we have $65 = C' + 10, C' = 55$, so $y = 55e^{-\frac{1}{29}t} + 10$. Therefore, at 5 p.m, the temperature is

$$y = 55e^{-\frac{5}{29}} + 10 \approx 56.29^\circ\text{F}.$$

(c) $\dfrac{dy}{dt} = 2 - \dfrac{1}{29}(y - 10) = -\dfrac{1}{29}y + \dfrac{68}{29}, t > 5.$

(d) We are looking for an equilibrium solution, i.e. where $\frac{dy}{dt} = 0$. This occurs when $y = 68$. It is a stable equilibrium, since for $y > 68$, $\frac{dy}{dt} < 0$, and for $y < 68$, $\frac{dy}{dt} > 0$. In the long run 68° F must be the temperature of the house.

2. Suppose there is a new kind of savings certificate that starts out paying 3% annual interest and increases the interest rate by 1% each additional year that the money is left on deposit. (Assume that interest is compounded continuously and that the interest rate increases continuously.)

(a) Write a differential equation for $\frac{dB}{dt}$, where $B(t)$ is the balance at time t.
(b) Solve the equation that you found in part (a), assuming an initial deposit of $\$1000$.
(c) When $t = 7$ years, the interest rate will have risen to 10%. Would it have been better to have invested $\$1000$ at a *fixed* interest rate of 5% for 7 years than to use the variable rate savings certificate described in part (a)? Explain your answer.

ANSWER:

(a) $\dfrac{dB}{dt} = B(0.03 + 0.01t)$

(b) $\displaystyle\int \frac{dB}{B} = \int (0.03 + 0.01t)\,dt$ which gives $\ln|B| = 0.03t + 0.005t^2 + C$, so $B = B_0e^{0.03t+0.005t^2}$. Since $B_0 = \$1000$, we have $B = 1000e^{0.03t+0.005t^2}$.

(c) With the rate increasing 0.01 every year, we would have $1000e^{0.21+0.245} = 1000e^{0.455}$ dollars at the end of 7 years. With a constant rate of 0.5, we would have $1000e^{0.05\cdot 7} = 1000e^{0.35}$ dollars at the end of 7 years which is smaller.

3. The population of aphids on a rose plant increases at a rate proportional to the number present. In 3 days the population grew from 800 to 1400.

(a) Write down a differential equation for the population of aphids at time t in days, where $t = 0$ is the day when there were 800 aphids.
(b) How long does it take for the population to get 10 times as large?
(c) What was the population on the day before there were 800?

ANSWER:

(a) If $P(t)$ is the aphid population at time t, then we know that

$$\begin{aligned}\frac{dP}{dt} &= kP \\ \int \frac{dP}{P} &= \int k\,dt \\ \ln|P| &= kt + C \\ P &= e^{kt+C} \\ &= P_0e^{kt}\end{aligned}$$

Since there are 800 aphids at time $t = 0$, we have $P = 800e^{kt}$.
At $t = 3$, $P = 1400$, so

$$1400 = 800e^{3k}$$

$$\ln\frac{7}{4} = 3k$$
$$k = \frac{1}{3}\ln\frac{7}{4}$$
$$\text{So,}\quad P = 800e^{\left(\frac{1}{3}\ln\frac{7}{4}\right)t}$$

(b) For P to become 10 times as large,

$$8000 = 800e^{\left(\frac{1}{3}\ln\frac{7}{4}\right)t}$$
$$\ln 10 = \left(\frac{1}{3}\ln\frac{7}{4}\right)t$$
$$t = \frac{3\ln 10}{\ln\frac{7}{4}} \approx 12.34 \text{ days.}$$

(c) On the day before there were 800 aphids, $t = -1$, so

$$P = 800e^{-\frac{1}{3}\ln\frac{7}{4}}$$
$$= 800\left(\frac{7}{4}\right)^{-\frac{1}{3}}$$
$$\approx 664 \text{ aphids.}$$

4. Newton's Law of Cooling states that the rate of change of temperature of an object is proportional to the difference between the temperature of the object and the temperature of the surrounding air.

A detective discovers a corpse in an abandoned building, and finds its temperature to be 27°C. An hour later its temperature is 21°C. Assume that the air temperature is 8°C, that normal body temperature is 37°C, and that Newton's Law of Cooling applies to the corpse.

(a) Write a differential equation satisfied by the temperature, H, of the corpse at time t. Measure t from the moment the corpse is discovered.
(b) Solve the differential equation.
(c) How long has the corpse been dead at the moment it is discovered?

ANSWER:

(a) $\dfrac{dH}{dt} = k(H-8)$ where k will be negative.

(b) $\dfrac{dH}{H-8} = k\,dt \Rightarrow H = Ce^{kt} + 8$
Since $H(0) = 27$, $27 = Ce^{\circ} + 8$, so $C = 19$.
Since $H(1) = 21$, $21 = 19e^{k(1)} + 8$ so $k = \ln\left(\frac{13}{19}\right) \approx -0.379$.
so $H = 19e^{-.379t} + 8$.

(c) We want $H(T) = 37$, so
$T = -\dfrac{\ln\left(\frac{29}{19}\right)}{.379} \approx 1.1$ hours or 1 hour 7 minutes before the body was found.

5. Cesium 137 (Cs^{137}) is a short–lived radioactive isotope. It decays at a rate proportional to the amount of itself present and has a half-life of 30 years (i.e., the amount of Cs^{137} remaining t years after A_0 millicuries of the radioactive isotope is released is given by $A_0e^{-\left(\frac{\ln 2}{30}\right)t}$. We will abbreviate millicuries by mCi).

As a result of its operations, a nuclear power plant releases Cs^{137} at a rate of 0.1 mCi per year. The plant began its operations in 1980, which we will designate as $t = 0$. Assume there is no other source of this particular isotope.

(a) Write an integral which gives the total amount of Cs^{137} T years after. (Note: The rest of the problem does not depend on correctly answering part (a).)
(b) Write a differential equation whose solution is $R(t)$, the amount (in mCi) of Cs^{137} in t years. (We are assuming $R(0) = 0$).
(c) After 20 years, approximately how much Cs^{137} will there be?
(d) In the long run, how much Cs^{137} will there be?
(e) Since Cs^{137} poses a great health risk, the government says that the maximum amount of Cs^{137} acceptable in the surrounding environment is 1 mCi (spread over the surroundings). What is the maximum rate at which the station can release the isotope and still be in compliance with the regulations?

ANSWER:

(a) $\displaystyle\int_0^T 0.1e^{\left(-\frac{\ln 2}{30}\right)t}\,dt.$

(b) $\frac{dR}{dt} = 0.1 - \frac{\ln 2}{30}R$

(c) Solving the equation in (b), we get:

$$\begin{aligned} \frac{dR}{0.1 - \frac{\ln 2}{30}R} &= dt; \\ -\frac{30}{\ln 2}\ln\left|0.1 - \frac{\ln 2}{30}R\right| &= t + C; \\ R &= -\frac{30}{\ln 2}(e^{-\frac{\ln 2}{30}t+D} - 0.1); \\ R &= \frac{30}{\ln 2}(0.1 - ke^{-\frac{\ln 2}{30}t}). \end{aligned}$$

Now, $R(0) = 0$, so $k = 0.1$, and we have $R = \frac{3}{\ln 2}(1 - e^{-\frac{\ln 2}{30}t})$.

Hence $R(20) = \frac{3}{\ln 2}(1 - e^{-\frac{2\ln 2}{3}}) \approx 1.60$ mCi.

(d) From (c), $R = \frac{3}{\ln 2}(1 - e^{-\frac{\ln 2}{30}t})$. As $t \to \infty$, $e^{-\frac{\ln 2}{30}t} \to 0$, so $R \to \frac{3}{\ln 2} \approx 4.33$mCi.

(e) Take Rate $= r$, then $R = \frac{30}{\ln 2}(r - re^{-\frac{\ln 2}{30}t})$. As $t \to \infty, e^{-\frac{\ln 2}{30}t} \to 0$, so $R \to \frac{30}{\ln 2}r$. We set $1 = \frac{30}{\ln 2}r$, then $r = \frac{\ln 2}{30} \approx .023$ mCi/yr.

6. A bank account earns interest at a rate of 2% per year, compounded continuously. Money is deposited into the account in a continuous cash flow at a rate of \$500 per year.

(a) Write a differential equation describing the rate at which the balance $B(t)$ is changing. (t is time in years.)

(b) Solve the differential equation given an initial balance of \$1000.

(c) Find the amount of money in the bank account after 10 years, assuming an initial balance of \$1000.

ANSWER:

(a) The balance increases at a rate of \$500 per year from the cash flow and by $2\% \cdot B$ per year from interest, so we have

$$\frac{dB}{dt} = 0.02B + 500.$$

(b) We can manipulate the differential equation as follows:

$$\begin{aligned} \frac{dB}{dt} &= 0.02(B + 25{,}000) \\ \frac{dB}{B + 25{,}000} &= 0.02dt \\ \int \frac{dB}{B + 25{,}000} &= \int 0.02dt \\ \ln|B + 25{,}000| &= 0.02t + C \\ B + 25{,}000 &= Ae^{0.02t}. \end{aligned}$$

When $t = 0$, we have $B = 1000$, so $A = 26{,}000$, giving

$$B = 26000e^{0.02t} - 25000.$$

(c) Substituting $t = 10$ in the above equation gives $B = 26000e^{0.02(10)} - 25000 = \$6756.47$.

7. After a certain car is turned on, the engine block heats up according to the differential equation

$$\frac{dH}{dt} = -K(H - 100)$$

for K, a positive constant.

(a) If the engine block was 15° C when the car was started, solve the differential equation.

(b) If after running for 20 minutes the temperature is 90° C, find K.

ANSWER:

(a) By separating variables, we get

$$\frac{dH}{H - 100} = -K\,dt, \text{ so } \int \frac{dH}{H - 100} = \int -K\,dt.$$

So $\ln|H - 100| = -Kt + C$ and $H - 100 = Ae^{-Kt}$, where $A = \pm e^C$.

We have initial temperature of 15° C when $t = 0$. This gives $15 - 100 = A$. So $A = -85$ and $H = 100 - 85e^{-Kt}$.

(b) Using part (a), we get $90 = 100 - 85e^{-K(20)}$.
Solve for K and we get

$$e^{-20K} = \frac{-10}{-85}, \text{ so}$$
$$K = \frac{\ln\left(\frac{10}{85}\right)}{-20} \approx 0.107$$

Questions and Solutions for Section 11.6

1. A diligent student has a slow leak in her bike tire, but has been too busy studying for exams to fix it. Assume that the pressure in the tire decreases at a rate proportional to the difference between the atmospheric pressure (15 lbs.) and the tire pressure. Monday at 6:00 pm she pumped up the pressure to 85 lbs. By 6:00 pm Tuesday it was down to 75 lbs. How much longer can she wait to pump up the tire if she wants to keep the pressure at a minimum of 40 lbs.? (You may keep your answer in number of days from Monday at 6:00 pm if you like.)

   ANSWER:
   Counting in days from Monday at 6pm, we have

$$-\frac{dP}{dt} = k(15 - P), \text{ where } P \text{ is pressure and } k \text{ is constant;}$$
$$\int \frac{1}{P-15}\,dP = \int k\,dt;$$
$$\ln|P-15| = kt + C;$$
$$P = 15 + e^{kt+C}.$$

   When $t = 0, P = 85$. When $t = 1, P = 75$. Therefore, $85 = 15 + e^C$ and $C = \ln 70$. Now, $75 = 15 + e^k(70)$, and so $k = \ln\frac{6}{7} \approx -0.154$.

   To find when $P = 40$, solve $40 = 15 + (\frac{6}{7})^t \cdot 70$, yielding $(\frac{6}{7})^t = \frac{5}{14}$ and $t = \frac{\ln\frac{5}{14}}{\ln\frac{6}{7}} \approx 6.68$ days.

2. In trying to model the response to a stimulus, psychologists use the Weber Fechner Law. This law states that the rate of change of a response, r, with respect to a stimulus, s, is inversely proportional to the stimulus.

   (a) Model this law as a differential equation.
   (b) Solve this differential equation with the initial condition that $r(s_0) = r_0$ for some initial stimulus, s_0.

   ANSWER:

   (a) Let k be a positive constant. Then, the Weber Fechner Law states that

$$\frac{dr}{ds} = \frac{k}{s}.$$

   (b) To solve this equation, separate variables to obtain

$$dr = k \cdot \frac{ds}{s}$$

   and integrate

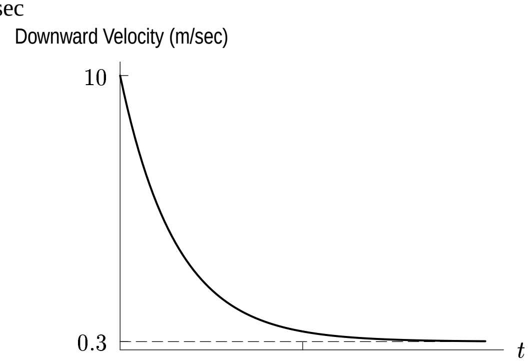
$$\int dr = k\int \frac{ds}{s}$$
$$r = k\ln|s| + C.$$

   Plugging in the initial condition gives $C = r_0 - k\ln|s_0|$. This means the solution is

$$r = k\ln|s| + r_0 - k\ln|s_0|$$
$$= r_0 + k\ln\left|\frac{s}{s_0}\right|.$$

3. The differential equation describing the motion of a 60-kg woman who has jumped into a swimming pool is given as follows:
$$a = \frac{dv}{dt} = g - \frac{k}{m}v$$
In this equation, $a = \dfrac{dv}{dt}$ on the left hand side is the woman's downward acceleration, v is the woman's downward velocity, $g =$ 9.8 m/sec^2 is the acceleration due to gravity, and $k =$ 1960 kg/sec is the woman's ballistic coefficient in water. This equation assumes that the woman does not attempt to swim to the surface and just allows herself to continue sinking.
   (a) In words, what does each term on the right hand side of the differential equation tell you?
   (b) What is the woman's terminal velocity in the water?
   (c) The woman enters the water with an initial downward velocity of 10 m/sec. Sketch a graph of her downward velocity as a function of time. (You do not need to solve the differential equation to do this.) Explain in words the motion of the woman.

   ANSWER:
   (a) $\dfrac{dv}{dt} = \underbrace{g}_{\text{Velocity increase due to pull of gravity}} - \underbrace{\frac{k}{m}v}_{\text{Velocity decrease due to drag of water}}$
   (b) $\dfrac{dv}{dt} = 0 \Rightarrow g - \dfrac{k}{m}v = 0 \Rightarrow V_t = \dfrac{mg}{k}$
$$V_t = \frac{(9.8 \text{ m/sec}^2)(60 \text{ kg})}{1960 \text{ kg/sec}} = 0.3 \text{ m/sec}$$
   (c)

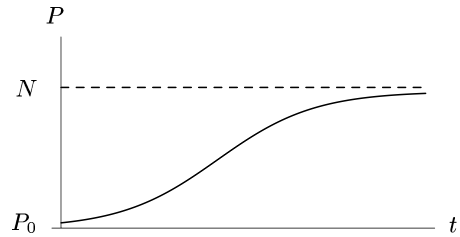


The velocity of the woman decreases exponentially from the moment she enters the water, leveling off to 0.3 m/s.

4. There is a theory that says the rate at which information spreads by word of mouth is proportional to the product of the number of people who have heard the information and the number who have not. Suppose the total population is N.
   (a) If $p = f(t)$ is the number of people who have the information, how many people do not have the information?
   (b) Write a differential equation that describes the rate, $\dfrac{dp}{dt}$, at which the information spreads by word of mouth.
   (c) Why does this theory make sense?
   (d) Sketch the graph of $p = f(t)$ as a function of time.

   ANSWER:
   (a) $N - p$ people do not have information.
   (b) $\dfrac{dp}{dt} = kp(N - p)$.
   (c) This theory makes sense for several reasons. The first is that we would expect the number of people who have the information to grow exponentially when p is small. That is, if p is small, then chances are that each person they come into contact with will not know the information. From the equation, we can see that $\dfrac{dp}{dt} = k(pN - p^2)$, so that when p is small, the term p^2 on the right-hand side will be small compared to the larger term pN. We also expect that at the very beginning and at the very end of the information spreading process, the rate of change of p will be zero. At the very beginning, information spreads slowly because very few people have it. At the end, it spreads slowly because there are few people left to hear it. Notice in the equation that when $p = 0$ or N (the beginning and end of the process), $\dfrac{dp}{dt} = 0$ as we suspected.

(d)



where P_0 is the number of people who have the information initially.

5. When a bacterial cell is suspended in a fluid, the concentration of a certain drug within the cell will change toward its concentration in the surrounding fluid at a rate proportional to the difference between the two concentrations.
   (a) Write the differential equation that expresses this relation. (Be sure to define the meaning of the literal quantities involved.)
   (b) Assume that the concentration in the surrounding fluid is held constant. What is the general solution of the equation in (a)? (This solution will necessarily contain some unknown constants.)
   (c) Suppose that a patient's blood is infected with these bacteria, which initially contain none of the drug. The patient is given enough of the drug to bring (and hold) its concentration in his blood to 0.0001. After two hours, the concentration within the bacterial cells is found to be 0.00004. Use this to evaluate the unknown constants in (b).
   (d) How long will it be before the concentration within the bacteria reaches 0.00008?

   ANSWER:

   (a) Let k equal the concentration of the drug in the cell, s represent the concentration of the drug in the fluid, and A be the constant of proportionality. Then, $\frac{dk}{dt} = (s-k)\cdot A$.
   (b) If we let s be constant,

$$\begin{aligned}\int \frac{dk}{s-k} &= \int A\,dt \text{ so}\\ -\ln(s-k) &= At + B\\ \text{and } s-k &= e^{-At-B}, \text{ so,}\\ k &= s - e^{-At-B}\end{aligned}$$

   Rewrite this as

$$k = s - e^{-B}e^{-At} = s - De^{-At},$$

   where $D = e^{-B}$.
   (c) Since $k(0) = 0$, $D = s$. We are given $s = 0.0001$, so $k = 0.0001 - 0.0001e^{-At}$. $k(2) = 0.00004$ implies $0.00004 = 0.0001 - 0.0001e^{-2A}$. Solving for A, $0.6 = e^{-2A}$ and $A = -\frac{1}{2}\ln(0.6) \approx 0.255$
   (d) Set $0.00008 = 0.0001\left(1 - e^{-0.255t}\right)$. Then

$$\begin{aligned}0.8 &= 1 - e^{-0.255t}\\ -0.2 &= -e^{-0.255t}\\ \frac{\ln(0.2)}{-0.255} &= t\\ t &\approx 6.312 \text{ hours.}\end{aligned}$$

6. A spherical raindrop evaporates at a rate proportional to its surface area. Note: you can do parts (e) and (f) WITHOUT doing parts (a)–(d).
   (a) If $V =$ volume of the raindrop and $S =$ surface area, write down a differential equation for $\frac{dV}{dt}$.
   (b) Your equation in (a) should include an unspecified constant k. What is the sign of k? Why?
   (c) Since $V = \frac{4}{3}\pi r^3$ for a sphere, write down an equation which relates $\frac{dV}{dt}$ for a sphere to r and $\frac{dr}{dt}$.
   (d) Since $S = 4\pi r^2$ for a sphere, you can write S in terms of r and $\frac{dV}{dt}$ in terms of r and $\frac{dr}{dt}$. Then the differential equation in part (a) becomes $\frac{dr}{dt} = k$. Show how this happens.
   (e) Solve the differential equation $\frac{dr}{dt} = k$, where k is a constant.
   (f) If it takes 5 minutes for a spherical raindrop to evaporate to $\frac{1}{8}$ of its original volume, how long will it take to completely evaporate?

ANSWER:

(a) $\dfrac{dV}{dt} = kS.$

(b) k is negative because the raindrop evaporates and its volume decreases.

(c) $\dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}.$

(d) Since $\dfrac{dV}{dt} = kS$, we have $4\pi r^2 \dfrac{dr}{dt} = k(4\pi r^2)$, which gives $\dfrac{dr}{dt} = k.$

(e) $dr = k\,dt$; $r = kt + r_0.$

(f)

$$V(t) = \frac{4\pi}{3}(kt + r_0)^3,$$

$$\text{So} \quad V(5) = \frac{4}{3}\pi(5k + r_0)^3.$$

$$\text{On the other hand,} \quad V(5) = \frac{1}{8}V(0) = \frac{1}{8}\cdot\frac{4}{3}\pi r_0^3.$$

$$\text{Thus,} \quad \frac{4}{3}\pi(5k + r_0)^3 = \frac{1}{8}\cdot\frac{4}{3}\pi r_0^3$$

$$5k + r_0 = \sqrt[3]{\frac{1}{8}}r_0 = \frac{1}{2}r_0$$

$$k = \frac{\left(\frac{1}{2} - 1\right) r_0}{5} = -\frac{1}{10}r_0.$$

Now, we want $V(t) = 0$ (i.e., the volume to be 0). Since $V(t) = \frac{4}{3}\pi(kt + r_0)^3$, we set $kt + r_0 = 0$. Then $t = -\dfrac{r_0}{k} = 10$ minutes.

7. A lake contains pollutants. A stream feeds clear mountain water into the lake at 2 gals/min. Polluted water is drained out of the lake at a rate of 2 gals/min by a second stream. If the volume of the lake if V gals and time, t, is measured in minutes, and if it is assumed that the pollutants are spread evenly through the lake at all times, then the differential equation for $Q(t)$, the quantity of pollutant in the lake at time t is (circle one):

(a) $\dfrac{dQ}{dt} = -2Q$

(b) $\dfrac{dQ}{dt} = -2t$

(c) $\dfrac{dQ}{dt} = -\dfrac{2Q}{V}$

(d) $\dfrac{dQ}{dt} = 2 - 2Q$

(e) $\dfrac{dQ}{dt} = 2 - \dfrac{2Q}{V}$

(f) $\dfrac{dQ}{dt} = 2V - Qt$

(g) $\dfrac{dQ}{dt} = 2Qt - V$

(h) $\dfrac{dQ}{dt} = Qe^{-2t}$

(i) $\dfrac{dQ}{dt} = \dfrac{Q}{V}e^{-2t}$

(j) $\dfrac{dQ}{dt} = \dfrac{2Q - 2V}{V}$

ANSWER:

The pollutants are leaving the lake and no new pollutants are entering, so the net rate is just the rate out. The total volume out is 2 gals/min of which only $2 \cdot (Q/V)$ is the pollutant (Volume · Concentration), so

$$\frac{dQ}{dt} = -2\frac{Q}{V},$$

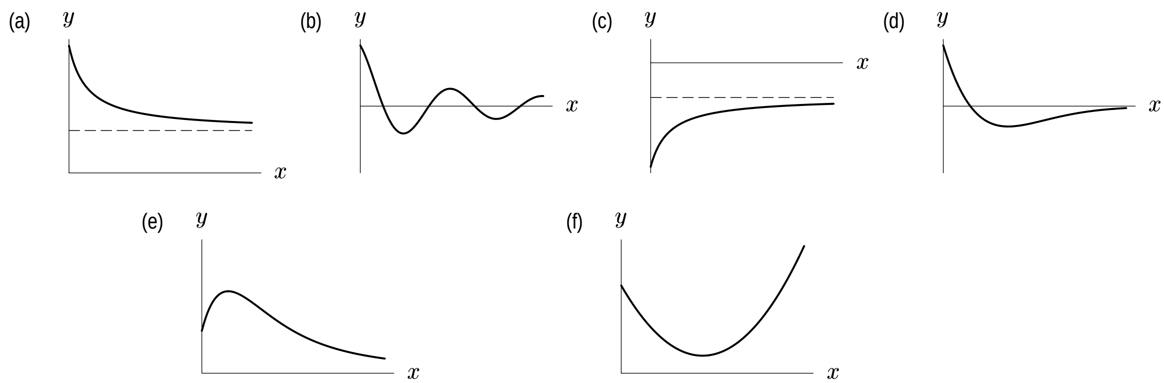
or choice (c).

Questions and Solutions for Section 11.7

1. Suppose y is a solution to the differential equation

$$\frac{dy}{dx} = f(y)$$

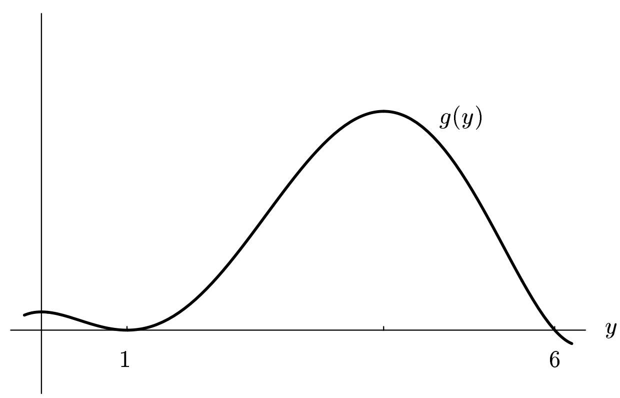
and that $f(y) \geq 0$ for all y. Which of the following could be a graph of y? (Circle one or more.)



ANSWER:

Since $f(y) \geq 0$ for all y, we know that $dy/dx \geq 0$ for all y so $y(x)$ is nondecreasing for all y. The only nondecreasing graph among the options given is choice (c).

2. Consider the differential equation $\dfrac{dy}{dt} = g(y)$. The graph of $g(y)$ is drawn below.



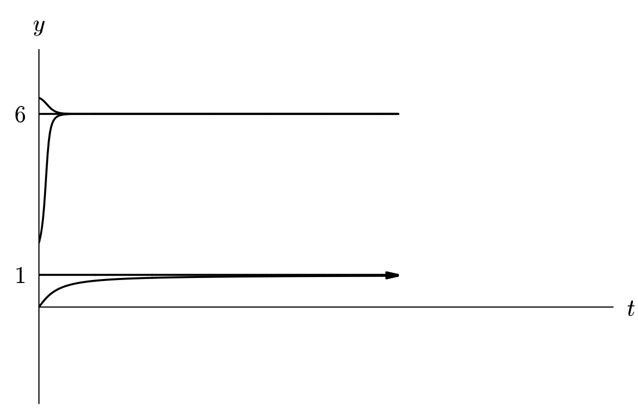
(a) What are the constant solutions to the differential equation $\dfrac{dy}{dt} = g(y)$? Which of them are stable?

(b) Sketch a graph of y as a function of t for representative initial values of y. You do not have to provide a scale for t.

ANSWER:

(a) The constant solutions to the differential equation $\dfrac{dy}{dt} = g(y)$ are $y = 1$ and $y = 6$ because for these values of y, $\dfrac{dy}{dt} = 0$, so y is constant. The only stable solution is $y = 6$. Although solutions with $y(0) < 1$ approach 1 in the long run, it is best not to call $y = 1$ a stable equilibrium since solutions with $y(0)$ slightly greater than 1 tend to veer away.

(b)



3. The rates of change of the populations for four different species are given by the differential equations (i), (ii), (iii), and (iv) below. (These species do not interact, and so the differential equations are all independent of one another.) P measures the population in thousands at time t months. Each species starts with 1000 members ($P = 1$) at time $t = 0$. Assume that each of these differential equations holds indefinitely.

(i) $\frac{dP}{dt} = 0.05$ (ii) $\frac{dP}{dt} = -0.05P$ (iii) $\frac{dP}{dt} = 0.05(1000 - P)$ (iv) $\frac{dP}{dt} = 0.0005P(1000 - P)$

(a) For each of the four species, find the size of the population P when the rate of change of the population is a maximum. (Remember $P(0) = 1$.)
(b) For each of the four species, find the size of the population P when the rate of change of the population is a minimum. (Remember $P(0) = 1$.)
(c) For each of the four species, find *all* equilibrium values of the population, and state whether each equilibrium value is stable or unstable.
(d) For each of the four species, sketch a graph of the population as a function of time given the initial condition $P(0) = 1$.

ANSWER:

(a) We want to know when the rate of change, $\frac{dP}{dt}$, is a max. To approach this problem, let $f(P) = \frac{dP}{dt}$ (the rate of change is given as a function of P). To find the max of $f(P) = \frac{dP}{dt}$, set $f'(P) = 0$, confirm whether this is a max, and find the P value for this max.

(i) $f(P) = 0.05$. The rate of change is constant ($f'(P) = 0$ for all P), so there is no maximum rate of change.

(ii) This equation describes exponential decay. Since we start with $P = 1$, $\frac{dP}{dt}$ will be negative for all time. No max.

(iii) $f(P) = 0.05(1000 - P) = 50 - 0.05P \quad f'(P) = -0.05$
$f'(P)$ can never equal zero, so we must check the (left) endpoint. The left endpoint is at $t = 0$, and $P(0) = 1$. Here $f(P) = \frac{dP}{dt} = 49.95$, and from there $\frac{dP}{dt}$ decreases to zero. This must be true, because $\frac{dP}{dt} > 0 \Rightarrow P$ is increasing $\Rightarrow \frac{dP}{dt}$ is decreasing. So $\frac{dP}{dt}$ has a max (of 49.95) where $P = 1$.

(iv) $f(P) = 0.0005P(1000 - P) = 0.5P - 0.0005P^2$
$f'(P) = 0.5 - 0.001P = 0 \Rightarrow 0.5 = 0.001P \Rightarrow P = \frac{0.5}{0.001} \Rightarrow P = 500$.
Check that it's a max: $f''(P) = -0.001 < 0 \Rightarrow$ max.

(b) (i) Again, since the rate of change $\frac{dP}{dt}$ is constant (meaning $P(t)$ is linear) there is no min.

(ii) (See a.ii.) $f(P) = -0.05$. At the left endpoint, where $P = 1$, we have $\frac{dP}{dt} = -0.05$ is a min. (From there $\frac{dP}{dt}$ increases to zero.)

(iii) $f'(P) = -0.05$ (see a.iii). The left endpoint, where $P = 1$, is a max of $\frac{dP}{dt} = 49.95$. From there, $\frac{dP}{dt}$ decreases to zero.
It never attains the value zero, but in some sense zero is a "min" since $\frac{dP}{dt}$ cannot be less than zero. $\frac{dP}{dt}$ approaches zero as P approaches 1000.
Answer: Either $\frac{dP}{dt}$ never attains a min, or it approaches a "min" of zero when $P = 1000$.

(iv) As seen in part (a), setting $f'(P) = 0$ yields a max of $\frac{dP}{dt}$, so we need to check endpoints. The (left) endpoint, where $p = 1$, is not a min. But $\frac{dP}{dt}$ decreases to zero as P approaches 1000. So the answer, as in part b.iii, is either no min, or a "min" at $P = 1000$.

(c) To find the equilibrium values, set $\frac{dP}{dt} = 0$. (No change in P.)

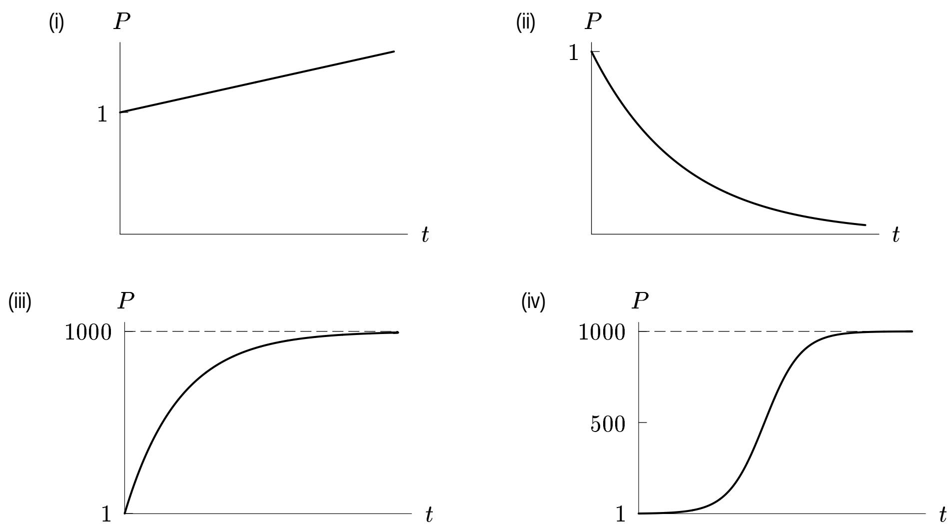
(i) No equilibrium, since $\frac{dP}{dt} = 0.05$.

(ii) $\dfrac{dP}{dt} = 0 = -0.05P \Rightarrow P = 0$. This is a stable equilibrium, because the population is dying off. Since $\dfrac{dP}{dt} < 0$, P always will decrease to zero, no matter where it starts.

(iii) $\dfrac{dP}{dt} = 0 = 0.05(1000 - P) \Rightarrow P = 1000$
This is a stable equilibrium, meaning the population will always approach 1000, no matter what size it is at $t = 0$. If $P < 1000$ at $t = 0$, then $\dfrac{dP}{dt} > 0$, and P increases until it levels off at 1000 as $\dfrac{dP}{dt}$ approaches 0. If $P(0) > 1000$, then $\dfrac{dP}{dt} < 0$, and P decreases until it levels off at 1000 as $\dfrac{dP}{dt}$ approaches zero.

(iv) $\dfrac{dP}{dt} = 0 = 0.0005P(1000 - P) \Rightarrow P = 0$ or $P = 1000$. $P = 0$ is an unstable equilibrium, because if the population starts out at any value other than zero, it will grow larger and move away from the value zero. (If it starts out greater than 1000, it will never be less than 1000, so it will never approach zero.)
$P = 1000$ is a stable equilibrium, because the size of the population always approaches 1000. If P starts out less than 1000, then $\dfrac{dP}{dt} > 0$, and P approaches 1000 as $\dfrac{dP}{dt}$ approaches 0. If P starts out greater than 1000, then $\dfrac{dP}{dt} < 0$, and P decreases to 1000 as $\dfrac{dP}{dt}$ approaches 0.

(d)



4. On January 1, 1879, records show that 500 of a fish called Atlantic striped bass were introduced into the San Francisco Bay. In 1899, the first year fishing for bass was allowed, 100,000 of these bass were caught, representing 10% of the population at the start of 1899. Owing to reproduction, at any time the bass population is growing at a rate proportional to the population at that moment.
   (a) Write a differential equation satisfied by $B(t)$, the number of Atlantic striped bass a time t, where t is in years since January 1, 1879 and $0 \le t < 20$.
   (b) Solve for $B(t)$, assuming $0 \le t < 20$.
   (c) Assume that when fishing starts in 1899, the rate at which bass are caught is proportional to the square of the population with constant of proportionality 10^{-7}. Write a differential equation satisfied by $B(t)$, for $t > 20$.
   (d) Assume that fishing practices from the start of 1899 are as described in part (c). What happens to the bass population in the long run?

   ANSWER:
   Let $B(t)$ be the bass population at time t.

   (a) We are told that for $0 \le t < 20$,
$$\frac{dB}{dt} = kB.$$
   Also
$$B(0) = 500, \quad \text{and} \quad B(20) = 10^6.$$

(b) So $B(t) = Ae^{kt}$, where A and k are constants. Since $B(0) = A = 500$, we know $A = 500$ so $B(t) = 500e^{kt}$. Since $B(20) = 500e^{20k} = 10^6$, we have $e^{20k} = 2000$, so $20k = \ln 2000$ or $k = \dfrac{\ln 2000}{20} \approx 0.38$. So

$$B(t) = 500e^{0.38t}.$$

(c) We know that $\frac{dB}{dt}$ = Rate of growth − Rate of fishing, or

$$\frac{dB}{dt} = 0.38B - 10^{-7}B^2.$$

(d) The equilibria occur when $\frac{dB}{dt} = 0$, or at $B = 0$ and $B = \dfrac{0.38}{10^{-7}} = 3.8 \cdot 10^6$. The equilibrium at $B = 3.8 \cdot 10^6$ is stable since $\frac{dB}{dt} > 0$ for $0 < B < 3.8 \cdot 10^6$ and $\frac{dB}{dt} < 0$ for $B > 3.8 \cdot 10^6$. So in the long run $B = 3.8 \cdot 10^6$.

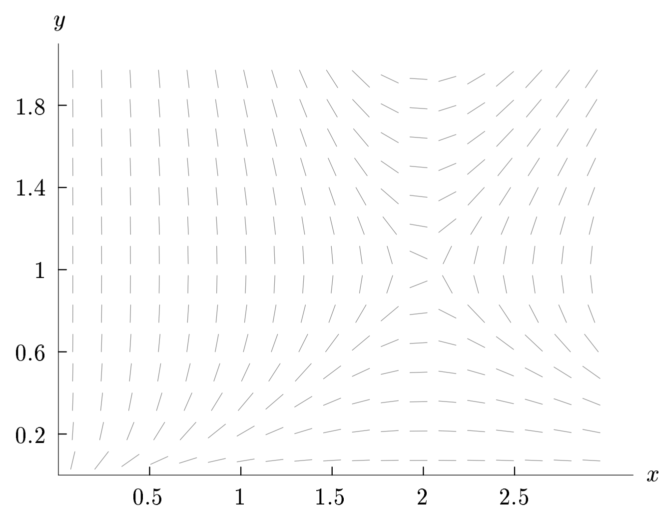
Questions and Solutions for Section 11.8

1. Suppose the equations

$$\frac{dy}{dt} = -4y + 2xy, \qquad \frac{dx}{dt} = -x + xy$$

describe the rates of growth of two interacting species, where x is the number of species A, measured in thousands, and y is the number of species B, measured in thousands.

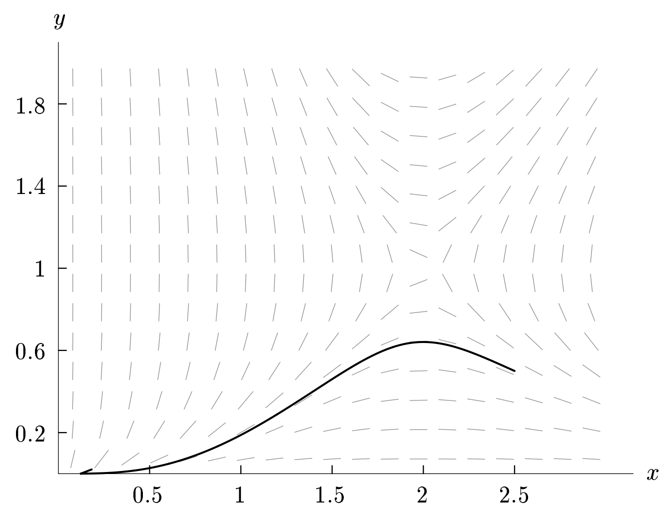
(a) Describe what happens to each species in the absence of the other.
(b) For each species, is the interaction with the other species favorable or unfavorable?
(c) Summarize in words the nature of the interaction between these two species.
(d) Determine all the equilibrium points in the xy-phase plane.
(e) The slope field in the xy-phase plane is shown below. Sketch the trajectory for initial conditions of $x = 2.5$, $y = 0.5$. (In other words, there are initially 2500 of species A and 500 of species B). Be sure to indicate with arrows which direction along the trajectory the populations will go in time, and show how you found this direction.



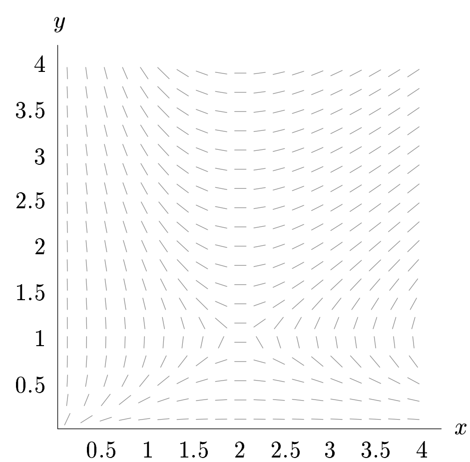
ANSWER:

(a) Both die out.
(b) Favorable to both.
(c) Each species would be lost without the other–symbiosis.
(d) $y' = -y(4 - 2x)$, $\qquad x' = -x(1 - y)$ so Equilibrium points: $(0, 0)$, $(2, 1)$
(e) At $(2.5, 0.5)$:
$$\frac{dy}{dt} = -4(0.5) + 2(2.5)(0.5) = -2 + 2.5 = 0.5 \text{ (positive)}.$$

$\frac{dx}{dt} = -2.5 + (2.5)(0.5) = -2.5 + 1.25 = -1.25$ (negative).
Therefore if the initial conditions are $(2.5, 0.5)$, species A is decreasing while species B is increasing.



2. Suppose the equations $\frac{dy}{dt} = 2y - xy$ and $\frac{dx}{dt} = x - xy$ describe the rates of growth of two interacting species, where x is the number of species A, measured in thousands, and y is the number of species B, measured in thousands.
   (a) In one sentence, summarize the nature of the interaction between these two species.
   (b) The slope field in the xy-phase plane is shown below. Sketch the trajectory for the initial conditions $x = 1$, $y = 2$. (In other words, there are initially 1000 of species A and 2000 of species B.) Be sure to indicate with arrows which direction along the trajectory the populations go in time, and show how you found this direction.

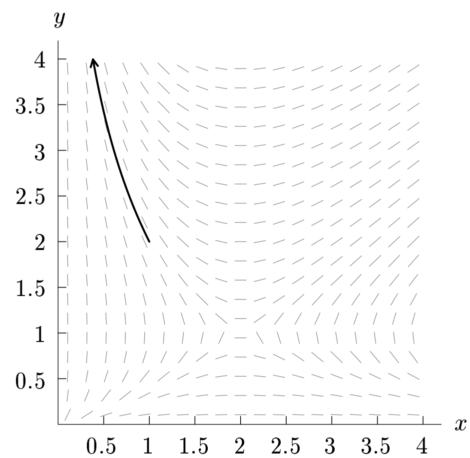


   (c) Describe the long-run behavior of the two populations with the initial conditions given in part (b).

   ANSWER:

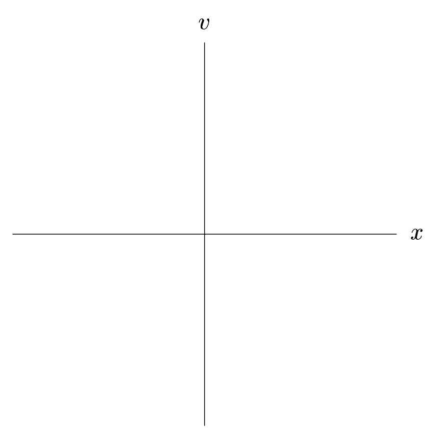
   (a) The two populations are competitors, since each would grow exponentially without the other; each suffers in the presence of the other.
   (b) At $(1, 2)$ we have $\frac{dy}{dt} = 2 > 0$ and $\frac{dx}{dt} = -1 < 0$. This means y is increasing and x is decreasing, so the direction of the trajectory must be upwards and to the left.

(c) From the trajectory we see that y grows (in fact, exponentially) and x dies out.



3. On a fine spring day you stand in the Square and throw a bean bag high into the air and catch it. Sketch a trajectory which reflects the bean bag's trip. Label the points A, B and C on the trajectory where

A = the point corresponding to the instant the bag is tossed;
B = the bag reaches its highest altitude;
C = the point corresponding to the instant you catch the bag;
x = the altitude of the bean bag;
v = the velocity of the bag.

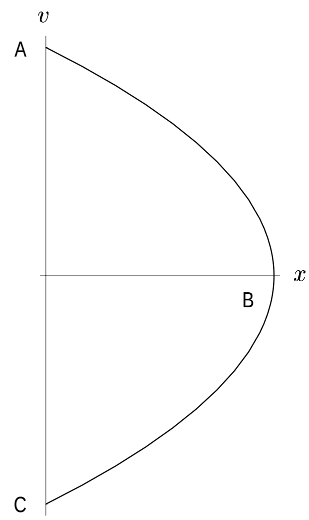


ANSWER:

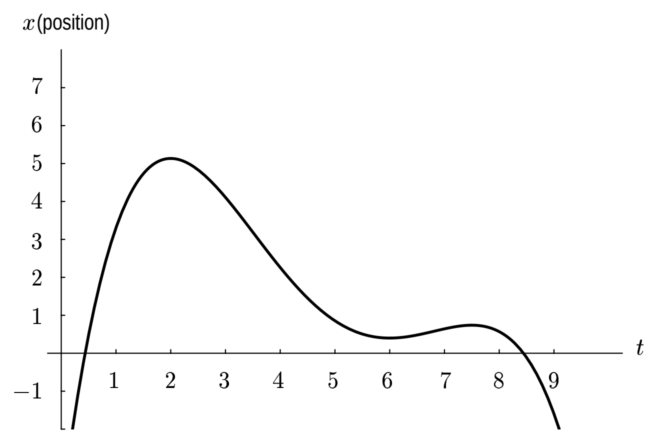
We take the acceleration to be constant at $-32\text{ft}/\text{sec}^2$. Then $v = -32t + v_0$ and $x = -16t^2 + v_0 t$ (we take $v(0) = v_0$ and $x(0) = 0$). Hence, $t = \frac{v_0 - v}{32}$ and

$$x = -16(\frac{v_0 - v}{32})^2 + v_0\frac{v_0 - v}{32} = -\frac{1}{64}v^2 + \frac{1}{64}v_0^2.$$

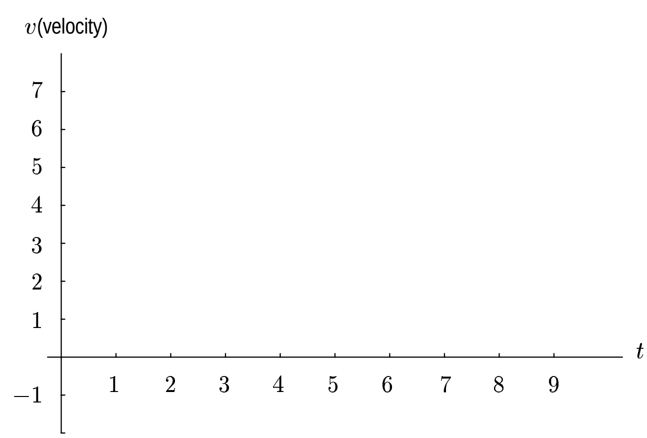
(Remember that v_0, the initial velocity, is a constant.) The graph is shown below.



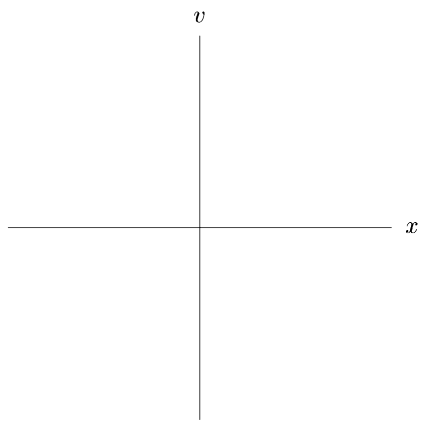
4. Below is a graph of position versus time.



(a) Sketch a rough graph of v versus t.

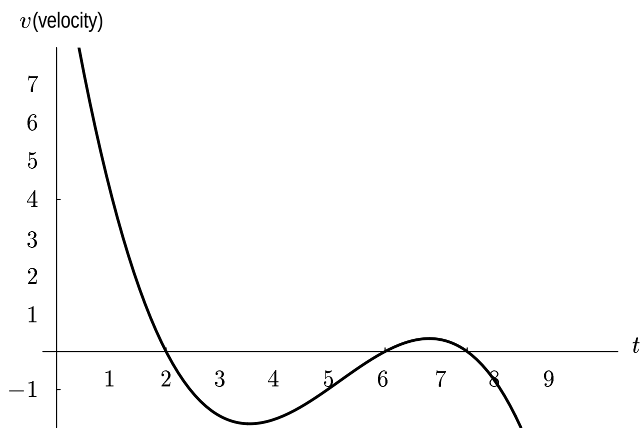


(b) Graph the corresponding trajectory in the xv-plane.



ANSWER:

(a)



(b) Taking values of x and v at the same time t, we get the following set of (x, v) ordered pairs:

$t = 1/2$	$(0, 8)$
$t = 1$	$(3, 4)$
$t = 2$	$(5, 0)$
$t = 3$	$(4, -1.5)$
$t = 4$	$(2.5, -1.5)$
$t = 5$	$(1, -1)$
$t = 5.5$	$(1/2, -1/2)$
$t = 6$	$(1/3, 0)$
$t = 6.5$	$(1/2, 1/3)$
$t = 7$	$(2/3, 1/2)$
$t = 7.5$	$(4/3, 0)$
$t = 8$	$(1/2, -1)$
$t = 8.5$	$(0, -2)$

Plotting these gives the following picture:

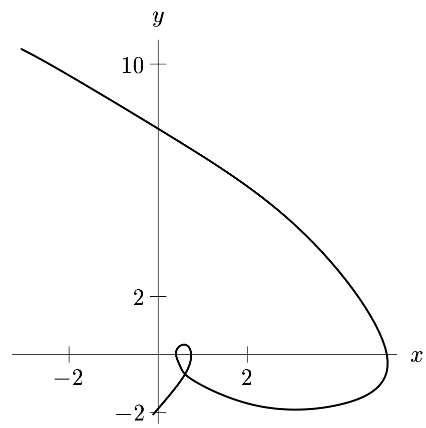


Figure 11.8.123

5. A fatal infectious disease is introduced into a growing population. Let S denote the number of susceptible people at time t and let I denote the number of infected people at time t. Suppose that, in the absence of the disease, the susceptible population grows at a rate proportional to itself, with constant of proportionality 0.3. People in the infected group die at a rate proportional to the infected population with constant of proportionality 0.2. The rate at which people get infected is proportional to the product of the number of susceptibles and the number of infecteds, with constant of proportionality 0.001.
   (a) Write a system of differential equations satisfied by S and I.
   (b) Find the equilibrium points for this system.
   (c) For each of the initial conditions below, cricle what happens as t starts to increase. (No reasons need be given.)
      (i) $S = 400$, $I = 100$
         - S increases, I increases.
         - S increases, I decreases.
         - S decreases, I increases.
         - S decreases, I increases.
         - None of the above
      (ii) $S = 400$, $I = 400$
         - S increases, I increases.
         - S increases, I decreases.
         - S decreases, I increases.
         - S decreases, I decreases.
         - None of the above
      (iii) $S = 200$, $I = 300$
         - S increases, I increases.
         - S increases, I decreases.
         - S decreases, I increases.
         - S decreases, I decreases.
         - None of the above
      (iv) $S = 100$, $I = 100$
         - S increases, I increases.
         - S increases, I decreases.
         - S decreases, I increases.
         - S decreases, I decreases.
         - None of the above

ANSWER:

(a) The system of differential equations is

$$\frac{dS}{dt} = 0.3S - 0.001SI,$$
$$\frac{dI}{dt} = 0.001SI - 0.2I.$$

(b) In equilibrium, $\frac{dS}{dt} = \frac{dI}{dt} = 0$, i.e.

$$\frac{dS}{dt} = -0.001S(I - 300) = 0,$$
$$\frac{dI}{dt} = 0.001I(S - 200) = 0.$$

If we solve these two equations we find that the equilibria are at $(0, 0)$ and $(200, 300)$.

(c) Figure 11.8.124 shows how the nullclines around $(200, 300)$ divide first quadrant into four regions.

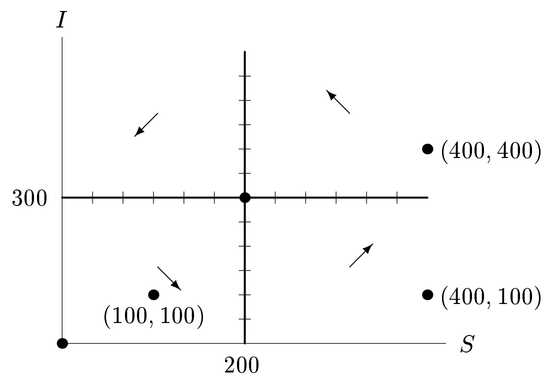


Figure 11.8.124

Figure 11.8.124 also shows the directions of arbitrary trajectories in each of these regions; for example, a trajectory in the upper-right region moves northwest. Using this figure, we can determine how S and I change in each of the four situations:

(i) S and I both increase.
(ii) S decreases; I increases.
(iii) None of the above.
(iv) S increases; I decreases.

Questions and Solutions for Section 11.9

1. Which system of equations could have the phase plane diagram of Figure 11.9.125?

(a) $\frac{dx}{dt} = -5x + y \qquad \frac{dy}{dt} = x - 5y$

(b) $\frac{dx}{dt} = x - 5y \qquad \frac{dy}{dt} = 5x - y$

(c) $\frac{dx}{dt} = -x + 5y \qquad \frac{dy}{dt} = -5x - y$

(d) $\frac{dx}{dt} = x - 5y \qquad \frac{dy}{dt} = -5x - y$

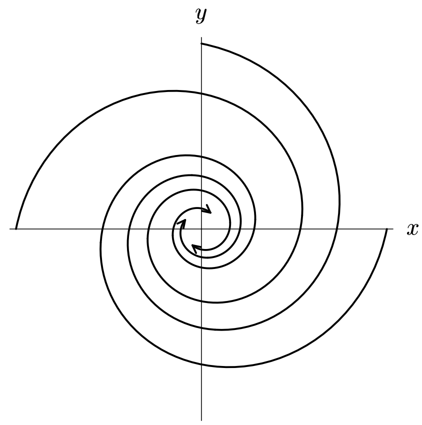


Figure 11.9.125: Phase Plane

ANSWER:

On the positive x axis, all trajectories satisfy the conditions $\frac{dx}{dt} < 0$, $\frac{dy}{dt} < 0$. If we substitute $y = 0$ and some positive x-value into the equation, we see that only (c) can satisfy the above condition.

2. Let x be the number of reptiles, y be the number of mammals, and z be the number of plants on the island of Komodo, all measured in thousands (e.g., $x = 50$ means 50,000 reptiles). The following differential equations give the rates of growth of reptiles, mammals, and plants on the island:

$$\frac{dx}{dt} = -0.2x - 0.04xy + 0.0008xz$$

$$\frac{dy}{dt} = -0.1y + 0.01xy$$

$$\frac{dz}{dt} = 2z - 0.002z^2 - 0.1xz$$

(a) Describe what happens to each class if the other two were not present. Use the terms we learned in class, such as "exponential growth," to describe the type of population growth each class experiences in the absence of the other two.

(i) Reptiles:

(ii) Mammals:

(iii) Plants:

(b) Say that initially there are 100,000 plants (i.e., $z = 100$) on Komodo, and there are no reptiles or mammals.

(i) Would the plant population increase or decrease initially?

(ii) Describe what would happen to the plant population in the long run. (If the plant population tends toward a particular value, then give that value.)

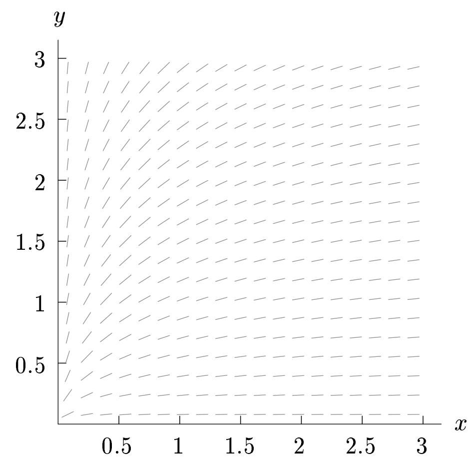
(iii) At what plant population would the number of plants be increasing the fastest?

(c) Who is eating whom on Komodo? Describe the nature of the interaction between each class.

(d) Find the equilibrium populations of reptiles, mammals, and plants (again assuming that none of the populations is zero).

Now assume that there are no plants ($z = 0$), but the reptile and mammal populations are non-zero.

(e) The slope field of the xy phase plane is shown below. Sketch the trajectory starting from initial populations of 2000 reptiles and 2000 mammals ($x = 2$ and $y = 2$). Be sure to indicate with arrows in which direction the populations move in time along the trajectory. Finally, describe what happens to the reptile and mammal populations in the long run. Why does this make sense?



ANSWER:

(a) (i) Reptiles: $\dfrac{dx}{dt} = -0.2x \Rightarrow$ exponential decay

(ii) Mammals: $\dfrac{dy}{dt} = -0.1y \Rightarrow$ exponential decay

(iii) Plants: $\dfrac{dz}{dt} = 2(z - 0.001z^2) \Rightarrow$ logistic growth

(b) $\Rightarrow$ Logistic growth, carrying capacity $z = 1000[2z(1 - 0.001z)] = 0 \Rightarrow z = 1000]$

(b) (i) Carrying capacity $= 1000 > P_0 = 100 \Rightarrow$ Plant population increases initially.

(ii) $\lim_{t\to\infty} z(t) = 1000$ (Population tends toward carrying capacity.)

(iii) $\max\left(\dfrac{dz}{dt}\right) = \dfrac{1}{2}$ (carrying capacity)$= 500$; here the second derivative is zero.

(c) $\dfrac{dy}{dt} \propto +0.01xy \Rightarrow$ Mammals eat reptiles.

$\dfrac{dx}{dt} \propto +0.008xz \Rightarrow$ Reptiles eat plants.

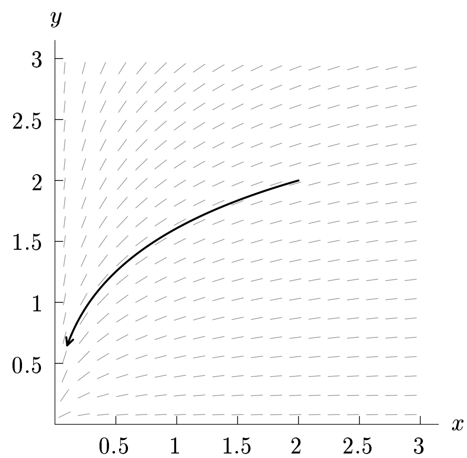
(d) (i) $x(-0.2 - 0.04y + 0.008z) = 0$
Set $\frac{dy}{dt} = y(-0.1 + 0.01x) = 0$ to get $x = 10$.

and $\frac{dz}{dt} = z(2 - 0.002z - 0.1x) = z(2 - 0.002z - 1) = z(1 - 0.002z) = 0$ to get $z = 500$.

Then $\frac{dx}{dt} = -0.2 - 0.04y + 0.0008(500) = 0$ when $y = 5$

Equilibrium populations are:
$x = 10$, i.e., 10,000 reptiles
$y = 5$, i.e., 5,000 mammals
$z = 500$, i.e., 500,000 plants.

(e)



Plug in $x = 2$, $y = 2$:

$$\frac{dx}{dt} = -0.2(2) - 0.04(2)(2) < 0$$

$$\frac{dy}{dt} = -0.1(2) + 0.01(2)(2) < 0$$

So x and y are $\downarrow$. Arrows go $\swarrow$.
Reptile and mammal populations $\to 0$.
Makes sense because plants are the ultimate food source for both!

3. The interaction of two populations $x(t)$ and $y(t)$ is modeled by the system

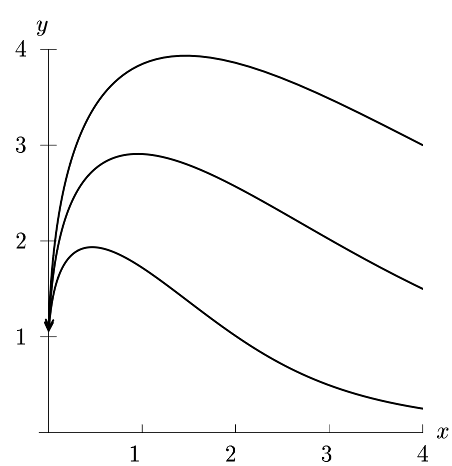
$$\frac{1}{x}\frac{dx}{dt} = 1 - x - ky, \qquad \frac{1}{y}\frac{dy}{dt} = 1 - y + kx,$$

where k is a positive constant.

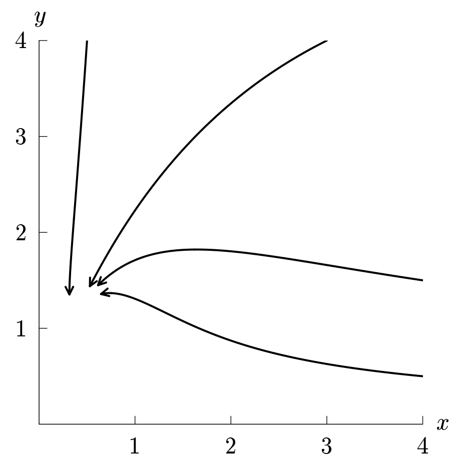
(a) What type of interaction is modeled here (Symbiosis, Predator—Prey, Competition)?
(b) Do the qualitative phase plane analysis for the case $k > 1$. For example, try $k = 2$. What happens in the long run?
(c) Do the qualitative phase plane analysis for the case $k < 1$. For example, try $k = \frac{1}{2}$. What happens in the long run?

ANSWER:

(a) We note that an increase in y causes a decrease in $\frac{dx}{dt}$ (due to the $-ky$ term) and an increase in x causes an increase in y (due to the $+kx$ term). Thus, it is Predator-Prey model, with y being the Predator and x being the Prey.
(b) For $k = 2$, we have $\frac{1}{x}\frac{dx}{dt} = 1 - x - 2y$ and $\frac{1}{y}\frac{dy}{dt} = 1 - y + 2x$. The phase plane diagram is below; one can see the trajectories spiraling toward $(0, 1)$. As for the equilibrium points, we have $x(1-x-2y) = 0$ and $y(1-y+2x) = 0$. Therefore, either $x = 0$, and then $y = 0$ or $y = 1$; or $y = 0$, and then $x = 0$ or $x = 1$; or $1 - x - 2y = 0$ and $1 - y + 2x = 0$, and then $x = \frac{1}{5}$, $y = \frac{3}{5}$. So we have $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(-\frac{1}{5}, \frac{3}{5})$ as our equilibrium points.



(c) For $k=\frac{1}{2}$, we have $\frac{1}{x}\frac{dx}{dt}=1-x-\frac{1}{2}y$ and $\frac{1}{y}\frac{dy}{dt}=1-y+\frac{1}{2}x$. The Phase Plane Diagram is below; one can see the trajectories spiraling toward $(\frac{2}{5},\frac{6}{5})$. As for the equilibrium points, we have $x(1-x-\frac{1}{2}y)=0$ and $y(1-y+\frac{1}{2}x)=0$. Therefore, either $x=0$, and then $y=0$ or $y=1$; or $y=0$, and then $x=0$ or $x=1$; or $1-x-\frac{1}{2}y=0$ and $1-y+\frac{1}{2}x=0$, and then $x=-\frac{2}{5}$, $y=\frac{6}{5}$. So we have $(0,0)$, $(1,0)$, $(0,1)$, and $(\frac{2}{5},\frac{6}{5})$ as our equilibrium points.



4. The acceleration of a moving object is given by

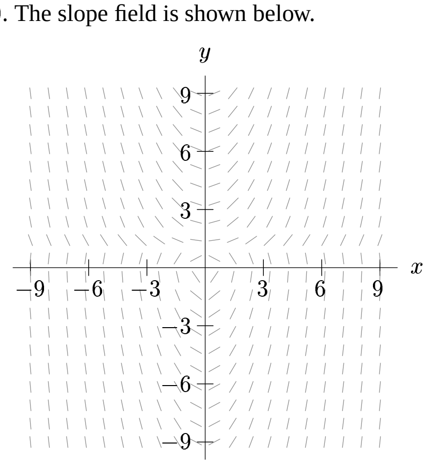
$$\frac{d^2x}{dt^2}=x\left(\frac{dx}{dt}-1\right)$$

where $x(t)$ is the position at time t.

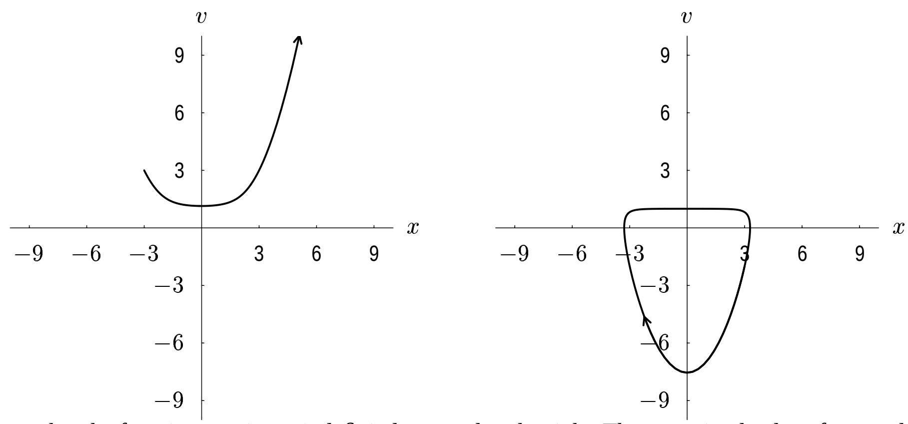
(a) Set up a system of first order differential equations for position x and velocity v.
(b) Do the qualitative phase plane analysis for this system, indicating clearly the null clines and constant solutions. (Consider positive and negative values for x and v.)
(c) With the help of your graphing calculator, sketch two trajectories: one for initial values $x(0)=-3$, $v(0)=3$ and the other for $x(0)=3, v(0)=-2$.
(d) Consider the trajectory with initial values $x(0)=-3$ and $v(0)=3$. Approximate the maximum and minimum values of x (if such values exist). Approximate the maximum and minimum values of v (if such values exist).
(e) Consider the trajectory with initial values $x(0)=3$ and $v(0)=-2$. Approximate the maximum and minimum values of x (if such values exist). Approximate the maximum and minimum values of v (if such values exist).
(f) Draw some general conclusions about what happens to $x(t)$ and $v(t)$ for different initial values.

ANSWER:

(a) We have $v=\frac{dx}{dt}$ and $\frac{dv}{dt}=x(v-1)$.
(b) To find points in the phase plane where slopes are vertical, we solve $\frac{dx}{dt}=0$, to obtain $v=0$. To find points with horizontal slopes, we solve $\frac{dv}{dt}=0$, to obtain $x=0$ or $v=1$. Therefore, slopes are vertical along the x-axis and horizontal along the v-axis and the line $v=1$. As for equilibrium points, we want both $\frac{dv}{dt}$ and $\frac{dx}{dt}$ equal to 0, which is only possible for $x=v=0$. The slope field is shown below.



(c)



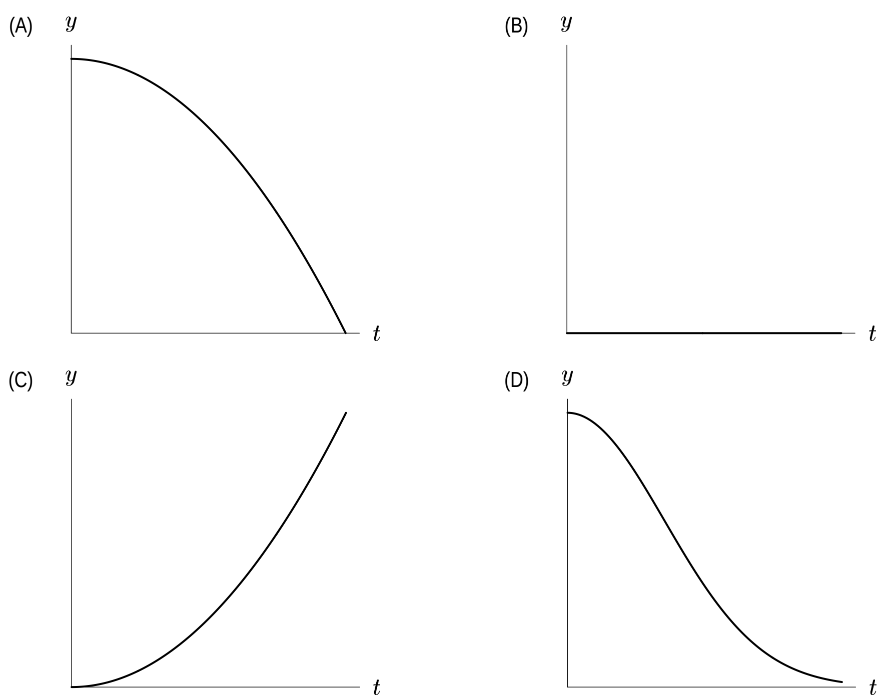
(d) We see that the function continues indefinitely up and to the right. Thus, maximal values for v and x do not exist. We also see from the graph that the minimal value for x is $x = -3$ (at the starting point) and the minimal value for v is $v \approx 1.14$.

(e) From the graph, we approximate the maximal value for x to be $x \approx 3.24$, the maximal value for v to be $v \approx 1.00$, the minimal value for x to be $x \approx -3.24$, and the minimal value for v to be $v \approx -7.60$

(f) If initially the velocity is greater than one and the position is positive, the acceleration will be positive forcing the velocity and position to increase further. Thus, an object in this state will never fall, and does in fact climb without bound. If the velocity is greater than one, but position is negative, then the object will at first slow down (while moving in the positive direction) and then speed up again. If, however, the velocity is ever less than one, then a cycle will arise in which velocity decreases until the position is negative at which point the object begins to fall more slowly. The object eventually changes direction and the cycle recurs.

Review Questions and Solutions for Chapter 11

1. Each graph below is a solution to at least one of the differential equations. Match them up in such a way that each graph is used only once. No work need be shown.



(a) $\dfrac{d^2y}{dt^2} = 2$ has solution graph________________________

(b) $\dfrac{dy}{dt} = y - 2$ has solution graph________________________

(c) $\dfrac{dy}{dt} = y(2 - y)$ has solution graph________________________

(d) $\dfrac{dy}{dt} = -yt$ has solution graph________________________

ANSWER:

(a) C. $y = t^2$.
(b) A. $y = -e^t + 1$, $y' < 0$ if $y < 2$.
(c) B. Zero solution, $y = 0$.
(d) D. $y = e^{-\frac{t^2}{2}}$

2. Discuss the solutions of the differential equation

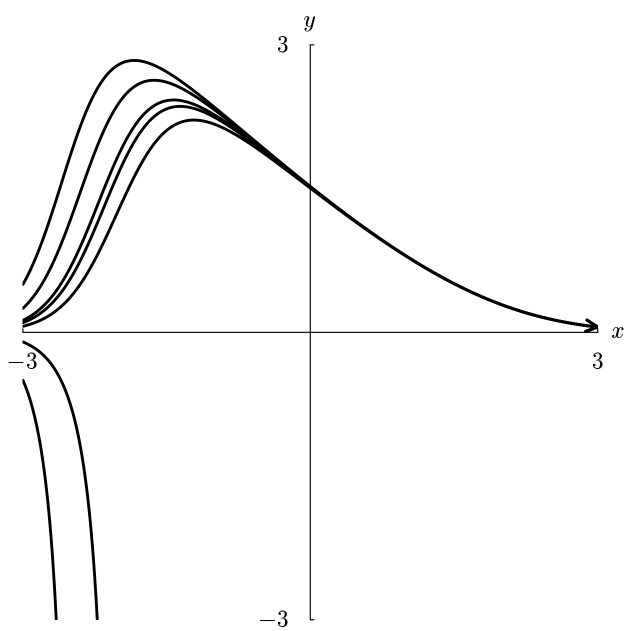
$$\frac{dy}{dx} = y(1 - x - y).$$

Your discussion should include at least the following points:

- Are there any constant solutions?
- A sketch of several solution curves.
- Explain why the solution function f for which $f(-1) = 1$ has a global maximum. What is the "long run" behavior of this function? (i.e., what can you say about $\lim\limits_{x\to\infty} f(x)$?)
- Many solution functions have global maxima. Where do these maximum points appear on the graph?
- A discussion that includes other significant information may receive extra credit. Although you may use your calculator to get the picture, try to base your arguments on information taken directly from the differential equation.

Note: Solutions starting from negative values of y are likely to grow so fast the the calculator will overflow, causing it to stop and give an error message.

ANSWER:



Solution Curves to $\frac{dy}{dx} = y(1 - x - y)$

The only constant solution is the curve $y = 0$. One can see this because if $y = c$,

$$\frac{dy}{dx} = c(1 - x - c) = 0.$$

But this holds for all x only if $c = 0$.

Solutions will have global maxima when $\frac{dy}{dx} = 0$. This happens when $y = 0$ or when $1 - x - y = 0$. The first cannot happen, since solution curves do not intersect and $y = 0$ is a solution. Therefore, critical points can only occur on the line $x + y = 1$. If y is positive, $\frac{dy}{dx}$ will be positive for $x > y - 1$ and negative for $x < y - 1$, so all critical points with positive y will be maxima. Moreover, if $y(x) > 0$ for some x, then y is decreasing for $x > 1$, so $y(x)$ approaches some limit greater than or equal to 0. Hence $y'(x) \to 0$ when $x \to \infty$. This means that

$$\frac{dy}{dx} = y(1 - x - y) \to 0$$

since $1 - x - y \to -\infty$, then $y \to 0$ (it's the other factor) as desired.

3. TRUE/FALSE questions. For each statement, write whether it is true or false and provide a short explanation or counterexample.

(a) $y = x^2 + x$ is a solution to $\frac{dy}{dx} = 2(y - x^2) + 1$.

(b) The solution of $\frac{dy}{dx} = x + 1$ passing through $(0, 1)$ is the same as the solution passing through $(0, 0)$, except it has been shifted one unit upward.

(c) The solutions of $\frac{dP}{dt} = kP(L - P)$ are always concave down.

ANSWER:

(a) TRUE. Substitute $y = x^2 + x$ in the left side of the equation $\dfrac{dy}{dx} = 2(y - x^2) + 1$.

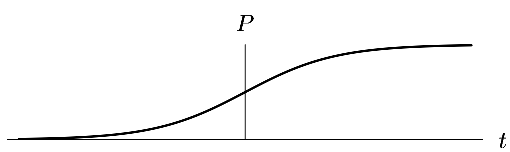
$$\text{Left side} = 2x + 1.$$

Now substitute in the right side:

$$\begin{aligned}\text{Right side} &= 2(x^2 + x - x^2) + 1 \\ &= 2x + 1.\end{aligned}$$

(b) TRUE. The slope field is independent of y, so solutions beginning at the same x-coordinate with different y-coordinates will be vertical translations of each other.

(c) FALSE. This is the logistic equation, some of whose solutions look like:



Note: A quick way to see that the statement is false is to note that when $k = 0$, $dP/dt = 0$, and thus $P = \text{const}$, which is not concave down.

4. Consider the differential equation

$$\frac{dP}{dt} = \sin P$$

(a) What are the equilibrium solutions with $-1 \le P \le 8$? Give exact answers.

(b) Sketch solution curves to this differential equation for $-1 \le P \le 8$. Include all the equilibrium solutions and the solutions going through the points $(0, 1)$, $(0, 3)$, $(0, 5)$, $(0, 7)$.

(c) What happens to the value of $P(t)$ as $t \to \infty$ if $P(0) = 1$?

(d) What happens to the value of $P(t)$ as $t \to \infty$ if $P(0) = 6$?

(e) What equilibrium solutions in $-1 \le P \le 8$ are stable and which are unstable?

(f) Find $\dfrac{d^2P}{dt^2}$ in terms of P.

(g) Use your answer to part (f) to decide at which of the following points the solution curve is concave up. (Circle your answer(s). No reasons need be given.)

$(1, 1)$ $(3, 3)$ $(5, 1)$ $(7, 3)$

ANSWER:

(a) $\frac{dP}{dt} = 0$ when $\sin P = 0$, i.e. if $P = k\pi$, so we have equilibria at $P = 0$, $P = \pi$, and $P = 2\pi$.

(b) For P in $(0, \pi)$, $\sin P$ is positive, so $\frac{dP}{dt}$ is positive, so P is increasing; likewise P is increasing for P in $(2\pi, 3\pi)$. By a similar argument, P is decreasing for P in $(\pi, 2\pi)$. Thus the graph of P will be as shown in Figure 11.10.126:

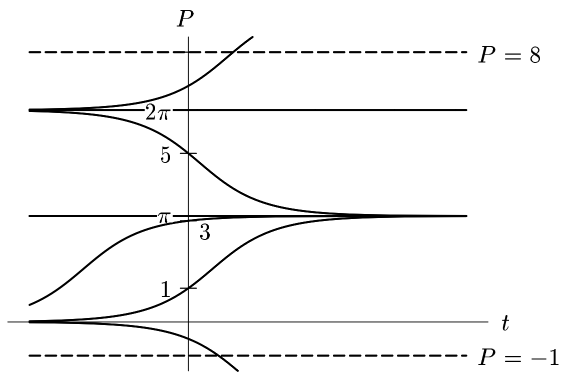


Figure 11.10.126

(c) Looking at Figure 11.10.126, we note that $P = 1$ is in $(0, \pi)$, so $P(t) \to \pi$ as $t \to \infty$.

(d) Looking at Figure 11.10.126, we note that $P = 6$ is in $(\pi, 2\pi)$ so $P(t) \to \pi$ as $t \to \infty$.

(e) $P = \pi$ is stable; $P = 0$ and $P = 2\pi$ are unstable.

(f) We differentiate the expression for $\frac{dP}{dt}$:

$$\begin{aligned}\frac{d^2P}{dt^2} &= \frac{d}{dt}\left(\frac{dP}{dt}\right)\\ &= \frac{d}{dt}(\sin P)\\ &= \cos P\left(\frac{dP}{dt}\right)\\ &= \cos P \sin P.\end{aligned}$$

(g) $(\cos P)(\sin P)$ is positive for $P = 1$ and negative for $P = 3$ so solution curves are concave up at $(1, 1)$ and $(5, 1)$.