Math 115 - An Example of Classifying Critical Points

Problem: Find the x-coordinates of all local maxima and local minima of the function $f(x)$ defined below. You must use calculus to find and justify your answers.

$$f(x) = x^5 - 10x^3 - 8$$

One possible complete solution to this problem is shown below.

To find local extrema, we first find the critical points of f.

$f'(x) = 5x^4 - 30x^2 = 5x^2(x^2-6)$

So $f'(x) = 0$ when $5x^2(x^2-6) = 0$

$5x^2 = 0$ or $x^2 - 6 = 0$

$x = 0$ or $x = \pm\sqrt{6}$

Note that f is differentiable everywhere, so these are the only critical pts.

We apply the first derivative test to classify them.

$f'(x) = 5x^2(x^2-6)$

$$* \left\{ \begin{array}{l} \text{Note: } 5x^2 > 0 \text{ for all nonzero } x \\ \text{For } x < -\sqrt{6},\ x^2 - 6 > 0, \text{ so } f'(x) = (\text{positive \#})(\text{positive \#}) > 0 \\ \text{For } x > \sqrt{6},\ x^2 - 6 > 0, \text{ so } f'(x) = (\text{positive \#})(\text{positive \#}) > 0 \\ \text{For } -\sqrt{6} < x < 0,\ x^2 - 6 < 0, \text{ so } f'(x) = (\text{positive \#})\cdot(\text{negative \#}) < 0 \\ \text{For } 0 < x < \sqrt{6},\ x^2 - 6 < 0, \text{ so } f'(x) = (\text{positive \#})\cdot(\text{negative \#}) < 0 \end{array} \right.$$

This is summarized on the right:

$f'>0$	$f'=0$	$f'<0$	$f'=0$	$f'<0$	$f'=0$	$f'>0$	x
	$-\sqrt{6}$		0		$\sqrt{6}$		

By the first derivative test, we thus find that f has a

- local maximum at $x = -\sqrt{6}$
- local minimum at $x = \sqrt{6}$

(critical point but no local extremum at $x=0$)

$*$ An alternative justification:

The sign of f' can only change at $x = -\sqrt{6}$, $x = 0$, and $x = \sqrt{6}$.

So, we choose a test point within each interval

$f'(-3) = 5(-3)^2((-3)^2-6) = 5(9)(3) = 145 > 0$

$f'(-1) = 5(-1)^2((-1)^2-6) = 5(-5) = -25 < 0$

$f'(1) = 5(1)^2(1^2-6) = 5(-5) = -25 < 0$

$f'(3) = 5(3)^2(3^2-6) = 5(9)(3) = 145 > 0$

$f'>0$	$f'=0$	$f'<0$	$f'=0$	$f'<0$	$f'=0$	$f'>0$	x
	$-\sqrt{6}$		0		$\sqrt{6}$		

Under the time pressure of an exam, appropriate justification might look something like this.

$f'(x) = 5x^4 - 30x^2 = 5x^2(x^2-6)$

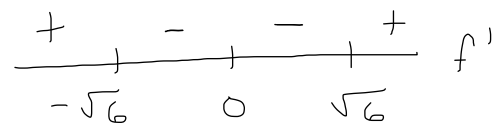
$\underline{f'(x)=0}$	$\underline{f'(x) \text{ DNE}}$
$5x^2(x^2-6)=0$	No such x
$x=0,\ x=\pm\sqrt{6}$	

$x < -\sqrt{6}$: $f'(x) = + \cdot + = +$

$-\sqrt{6} < x < 0$: $f'(x) = + \cdot - = -$

$0 < x < \sqrt{6}$: $f'(x) = + \cdot - = -$

$\sqrt{6} < x$: $f'(x) = + \cdot + = +$



By the first derivative test, $\boxed{\begin{array}{l} f \text{ has a local max at } x=-\sqrt{6} \\ \text{local min at } x=\sqrt{6} \\ \text{(critical pt but no local extremum at } x=0) \end{array}}$

A few key features of this solution:

$f'(x) = 5x^4 - 30x^2 = 5x^2(x^2-6)$

Find critical points algebraically.

$\underline{f'(x)=0}$	$\underline{f'(x) \text{ DNE}}$
$5x^2(x^2-6)=0$	No such x
$x=0,\ x=\pm\sqrt{6}$	

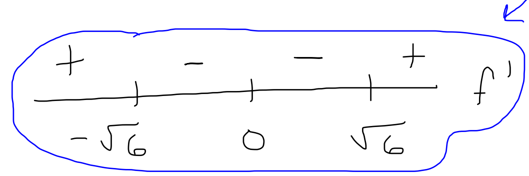
Clear labelling (including f')

$x < -\sqrt{6}$: $f'(x) = + \cdot + = +$

$-\sqrt{6} < x < 0$: $f'(x) = + \cdot - = -$

$0 < x < \sqrt{6}$: $f'(x) = + \cdot - = -$

$\sqrt{6} < x$: $f'(x) = + \cdot + = +$



Indicate the reasoning for the eventual "+" on the number line.

By the first derivative test, $\boxed{\begin{array}{l} f \text{ has a local max at } x=-\sqrt{6} \\ \text{local min at } x=\sqrt{6} \\ \text{(critical pt but no local extremum at } x=0) \end{array}}$

Clear final answer

* Alternative *

Under the time pressure of an exam, appropriate justification might look something like this.

Find critical points algebraically

$f'(x) = 5x^4 - 30x^2 = 5x^2(x^2-6)$

$f'(x) = 0$

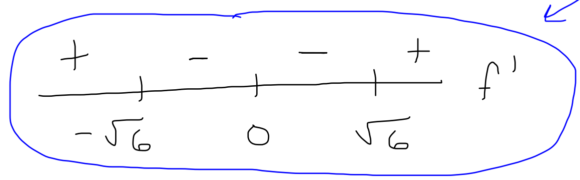
$5x^2(x^2-6) = 0$

$x = 0, \; x = \pm\sqrt{6}$

$f'(x)$ DNE

No such x

Clear labelling (including f')



$f'(-3) = 145$

$f'(-1) = -25$

$f'(1) = -25$

$f'(3) = 145$

Indicate the reasoning for the eventual "+" on the number line

Note:

"$f'(3) = 145$"

or "$f'(3) = + \cdot + = +$"

is okay, but

"$f'(3) > 0$" is not sufficient

By the first derivative test,

Clear final answer

f has a local max at $x = -\sqrt{6}$

local min at $x = \sqrt{6}$

(critical pt but no local extremum at $x = 0$)