Math 115 — Final Exam

December 15, 2011

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 14 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	14	
2	6	
3	16	
4	10	
5	8	
6	10	
7	14	
8	12	
9	10	
Total	100	

1. [14 points]
You are online playing the Facebook-based game, FarmVille, and you receive land with 5 stalks of corn on it. You decide that you would like to model the corn population on this patch of land using your calculus skills, so you recall that a good model for population growth is the logistic model

$$P(t) = \frac{L}{1 + Ae^{-kt}} \qquad L > 0, \quad A > 0, \quad k > 0.$$

a. [5 points] Using the ***limit definition of the derivative***, write an explicit expression for the derivative of the function $P(t)$ at $t = 1$. Do not evaluate this expression.

Solution:

$$P'(1) = \lim_{h \to 0} \frac{\frac{L}{1+Ae^{-k(1+h)}} - \frac{L}{1+Ae^{-k(1)}}}{h}$$

b. [5 points] Using the definition of the logistic model above, compute the following in terms of L, k, and A, showing your work or providing an explanation for each part:

i. [1 points] $\lim_{t \to \infty} P(t) = L$

Since the exponential piece is decreasing, it tends to 0 as $t \to \infty$. Then, the denominator goes to 1, and the limit of $P(t)$ as $t \to \infty$ is L.

ii. [1 points] $\lim_{t \to -\infty} P(t) = 0$

Since the exponential factor goes to infinity at $t \to -\infty$, the entire denominator goes to infinity, and the function tends to 0.

iii. [1 points] $P(0) = \dfrac{L}{1 + A}$

If we evaluate the function at 0, the exponential piece equals 1, so the denominator is just $1 + A$, giving the answer above.

iv. [2 points] $P'(0) = \dfrac{LAk}{(1 + A)^2}$

First, we see the derivative is $P'(t) = \dfrac{LAke^{-kt}}{(1 + Ae^{-kt})^2}$, so $P'(0) = \dfrac{LAk}{(1 + A)^2}$

c. [4 points] Your farmland satisfies the following conditions:

$$P(0) = 5, \; P'(0) = 1, \; \lim_{t \to \infty} P(t) = 100.$$

Based on your answers in part (b), compute the correct values of L, k, and A for the logistic equation modeling corn population on your land.

Solution: Since $\lim_{t\to\infty} P(t) = L$, the last piece of given information implies that $L = 100$. Since $P(0) = \dfrac{L}{1+A}$ and $L = 100$, we have from the given information

$$\frac{100}{1+A} = 5 \Rightarrow 20 = 1 + A \Rightarrow A = 19.$$

Finally, we use the last piece of information (combined with $L = 100$ and $A = 19$ to solve for k:

$$P'(0) = \frac{LAk}{(1+A)^2} = \frac{1900k}{400} = 1 \Rightarrow k = \frac{4}{19}$$

$L =$ 100 $A =$ 19 $k =$ $\frac{4}{19}$

2. [6 points] For each of the following statements, circle TRUE if the statement is always true and circle FALSE otherwise. Any ambiguous answers will be marked as incorrect.

a. [2 points] Suppose $A(t)$ and $B(t)$ are both everywhere differentiable functions which satisfy the equation $A^2 = e^B$ for all real numbers t. If, additionally, $\ln(2A(0)) = B(0)$, then $A'(0) = B'(0)$.

[True] False

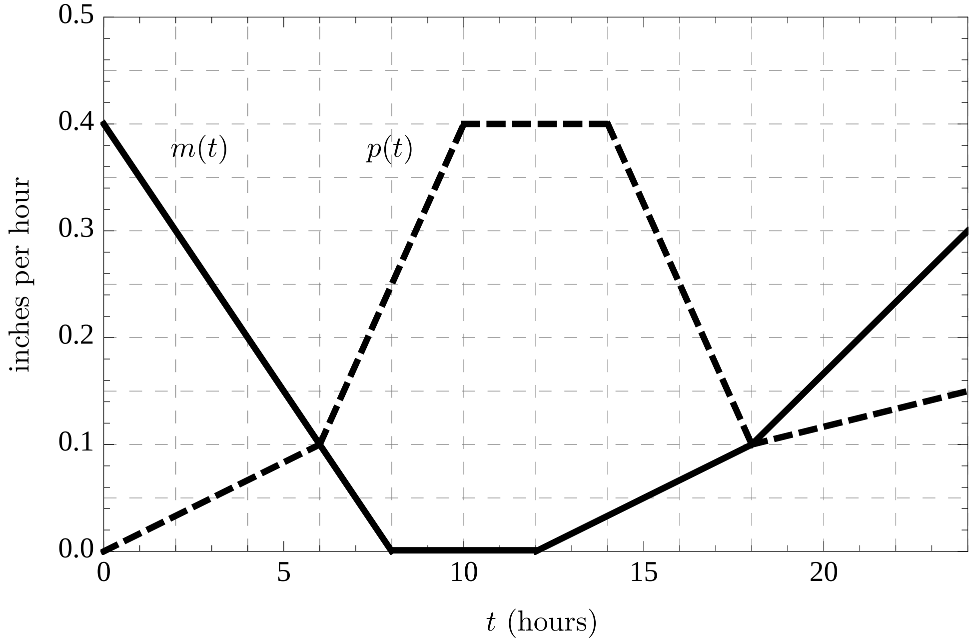
b. [2 points] If $f(x)$ is an everywhere continuous function and $\int_1^b f(t-b)dt = c$ for some real numbers b and c, then $\int_0^{1-b} 5f(t)dt = -5c$.

[True] False

c. [2 points] If $r(y)$ is a twice differentiable function whose first derivative is continuous, decreasing, and negative for all real numbers y, then $r(y)$ is concave up.

True [False]

3. [16 points] Suppose the graph below shows the rate of snow melt and snowfall on Mount Arvon, the highest peak in Michigan (at a towering 1970 ft), during a day (24 hour period) in April of last year. The function $m(t)$ (the solid curve) is the rate of snow melt, in inches per hour, t hours after the beginning of the day. The function $p(t)$ (the dashed curve) is the snowfall rate in inches per hour t hours after beginning of the day. There were 18 inches of snow on the ground at the beginning of the day.



a. [2 points] Over what time period(s) was the snowfall rate greater than the snow melt rate?

Solution: The snowfall rate was greater than the snow melt rate between hours 6 and 18 when the snowfall (dotted) curve is above the snow melt (solid) curve.

b. [2 points] When was the amount of snow on Mount Arvon increasing the fastest? When was it decreasing the fastest?

Solution: The amount of snow was increasing the fastest between hours 10 and 12. The amount of snow was decreasing the fastest at the very beginning of the day ($t = 0$).

c. [3 points] When was the amount of snow on Mount Arvon the greatest? Explain.

Solution: The amount of snow was increasing between $t = 6$ and $t = 18$ and decreasing at all other times. This means there should be the most snow at $t = 18$ (when the amount of snow stopped increasing) or at $t = 0$ (before snow started melting). The area between the curves represents the increase ($p(t) > m(t)$) or decrease ($p(t) > m(t)$) in snow over a given period of time. By inspection of the graph, there was much more of an increase between $t = 6$ and $t = 18$ than there was a decrease between $t = 0$ and $t = 6$, so there must have been the most snow at the end of the 18th hour ($t = 18$).

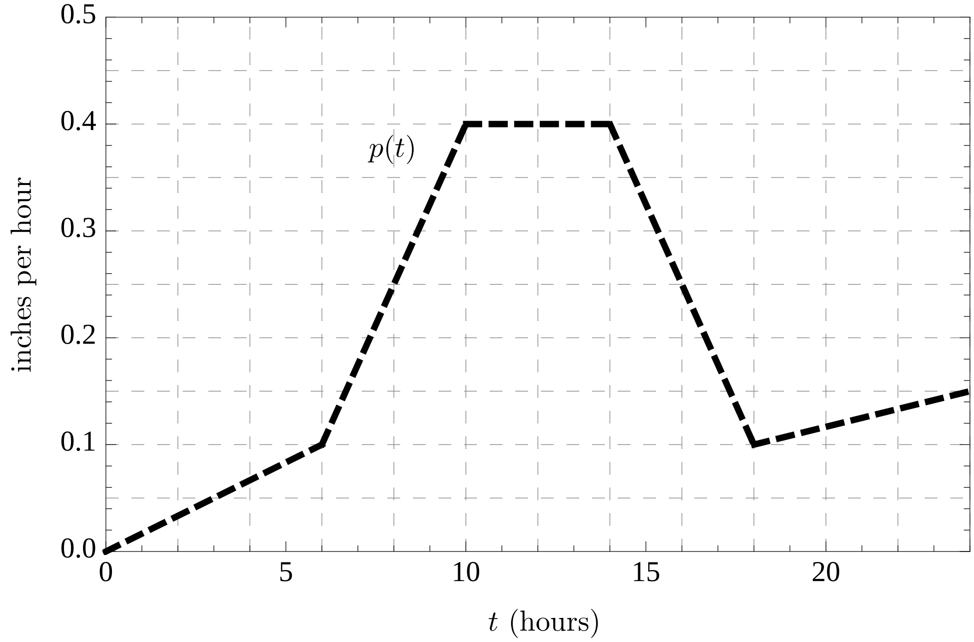
d. [3 points] How much snow was there on Mount Arvon at the end of the day (at $t = 24$)? Show work.

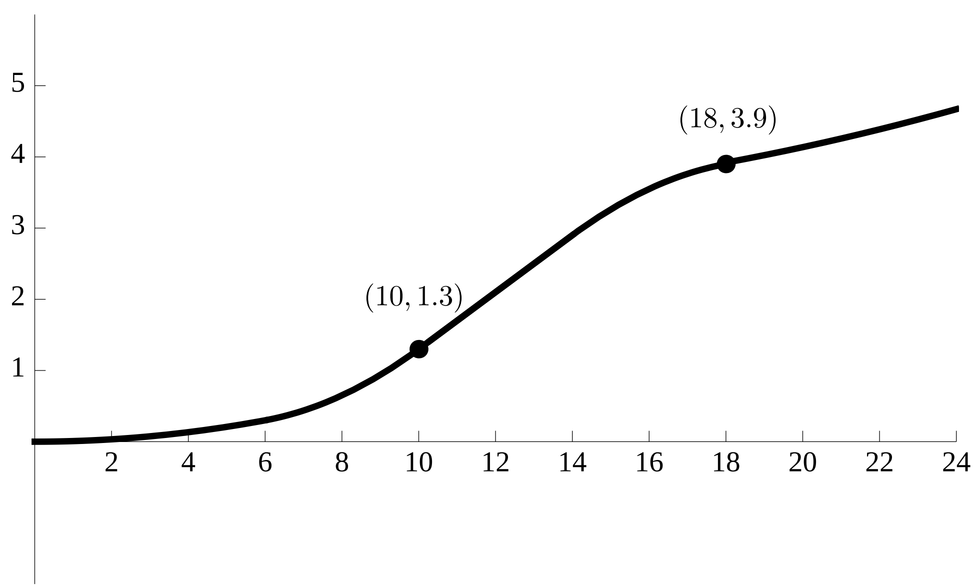
Solution: If A is the area between $m(t)$ and $p(t)$ from $t = 0$ to $t = 6$, B is the between $m(t)$ and $p(t)$ from $t = 6$ to $t = 18$ and C is the area between $m(t)$ and $p(t)$ between $t = 18$ and $t = 24$. Each "box" counts for 0.1 inches of snow. The amount of snow at the end of the day will be

$$18 + 0.1(-A + B - C) = 18 + 0.1(-12 + 32 - 4.5) = 19.55 \text{ inches.}$$

3. (continued)

e. [6 points] The graph of $p(t)$ is repeated below. On the empty set of axes, sketch a well-labeled graph of $P(t)$, an antiderivative of $p(t)$ satisfying $P(0) = 0$. Label and give the coordinates of the points on the graph of $P(t)$ at $t = 10$ and $t = 18$.





4. [10 points] Your family farm has a small herd of dairy cows. You decide to track the daily milk production levels of your favorite cow, Bessie, over the course of several months. Below is your table of measurements from every two months of Bessie's milk production level $p(t)$ in liters per day:

day t	0	60	120	180	240	300	360
liters per day $p(t)$	18	24	18	6	0	6	18

a. [4 points] The function modeling Bessie's daily milk production $p(t)$ is sinusoidal. What is an equation for this function?

Solution: The period of this sinusoidal will be 360 (so $B = \frac{\pi}{180}$), based on the information in the table with an amplitude of 12 and a vertical shift of 12 as well. Since the maximum is at 60 days instead of at 0, we will need to shift the cosine function to match the data. Therefore, an equation for this function is $p(t) = 12\cos\left(\frac{\pi}{180}(x - 60)\right) + 12$. There are several other possible functions, including

$$p(t) = -12\sin\left(\frac{\pi}{180}(t - 150)\right) + 12$$

b. [3 points] Suppose you use a right-hand Riemann sum to approximate the total milk produced by Bessie between day 60 and day 240. Is this an overestimate, an underestimate, or can it not be determined? Explain your reasoning.

Solution: The key observation here is that between day 60 and day 240, the function $p(t)$ is decreasing. The total milk produced by Bessie between day 60 and 240 is $\int_{60}^{240} p(t)dt$. When we use a right-hand Riemann sum to approximate the area under a decreasing curve, it gives an underestimate. Therefore, the right-hand sum will give an underestimate of the milk produced by Bessie.

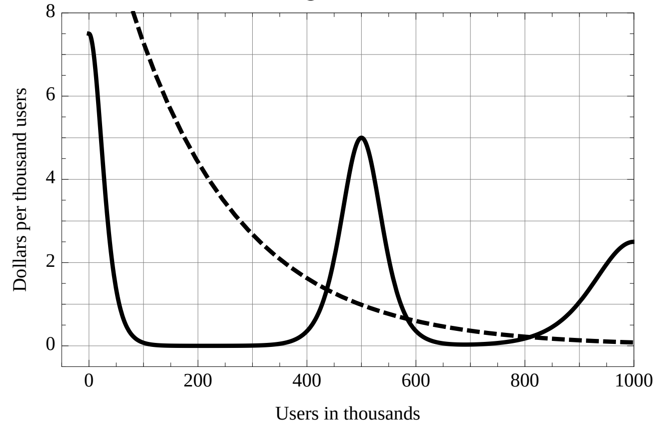
c. [3 points] How many measurements would you need to take between day 60 and day 240 to be sure your right-hand Riemann sum approximation is no more than 10 liters off from the exact amount of milk Bessie produces?

Solution: In order to figure out the error in the approximate of an integral by a Riemann sum, we use the formula $|LHS - RHS| = |f(a) - f(b)|\frac{b-a}{n}$. In this problem, we are only concerned with the interval $(60, 240)$, we want to find the value of n which makes this difference less than or equal to 10. If we use the table, we can plug in all the necessary values and get

$$\left|(24 - 0)\frac{240 - 60}{n}\right| \leq 10 \Rightarrow 24(180) \leq 10n \Rightarrow n \geq 432.$$

It will take at least 432 measurements between day 60 and day 240 in order to approximate the exact amount of milk Bessie produces over that period.

5. [8 points] The owners of a social networking site are concerned about their profit margins, so they develop the graphs for marginal revenue and marginal cost in terms of the number of users on the site. Use the graphs below to answer the following questions. The dashed graph is marginal revenue and the solid line is marginal cost.



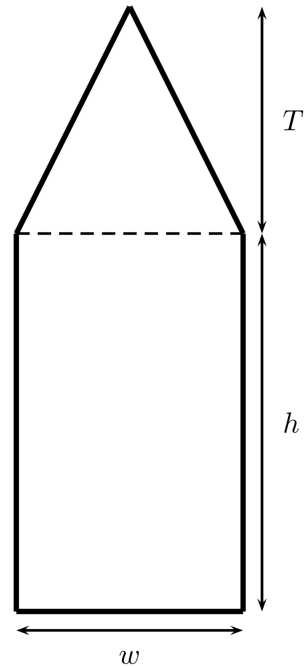
a. [3 points] At which approximate number(s) of users is marginal cost equal to marginal revenue?

Solution: At around 450, 575, and 825 thousand users, the two curves intersect, so marginal cost is equal to marginal revenue at those three points.

b. [5 points] Which of your answers from part (a) maximizes profit? Clearly justify your answer.

Solution: In order to maximize profit, we need to decide which of the points above could be local maxima. Since profit is revenue minus cost, we can find the sign of the derivative of profit by seeing where $MR - MC$ is positive and negative. Since $MR > MC$ for when the number of users is less than 450,000 and between 580,000 and 800,000 users, and $MR < MC$ everywhere else, 450 thousand users and 800 thousand users are local maxima of the profit function. In addition, since the area between the marginal cost and marginal revenue curves is greater between 450 and 580 (and counted negatively) than the area between 580 and 800, at 450 thousand users profit will be maximized.

6. [10 points] Consider a window the shape of which is a rectangle of height h surmounted by a triangle having a height T that is two times the width w of the rectangle (see the figure below which is not drawn to scale). If the total area of the window is 5 square feet, determine the dimensions of the window which minimize the perimeter.



Solution: From the statement of the problem, we have $T = 2w$ and adding the areas of the triangle and the rectangle, we have the total area of the window to be

$$A = wh + 0.5Tw = wh + w^2 = 5.$$

Solving for h, this gives $h = 5/w - w$. To calculate the perimeter of the window, we must calculate the length, ℓ, of the two sides of the triangle which lay on the perimeter. by the Pythagorean theorem, we get $\ell = \sqrt{17}w/2$. Now the perimeter of the window is

$$P = 2h + w + 2\ell = \frac{10}{w} - 2w + w + \sqrt{17}w = \frac{10}{w} - w + \sqrt{17}w.$$

Setting the derivative of P equal to zero, we have

$$P' = \frac{-10}{w^2} - 1 + \sqrt{17}.$$

Which means $w = \sqrt{\frac{10}{-1+\sqrt{17}}}$ is our only critical point. The second derivative of P is $20w^{-3}$ which is positive for all $w > 0$. This means P is concave up everywhere. Since we only have one critical point, it must be a global minimum. Therefore the dimensions which minimize the perimeter of the window are $w = \sqrt{\frac{10}{-1+\sqrt{17}}}$ feet, $T = 2\sqrt{\frac{10}{-1+\sqrt{17}}}$ feet, and $h = 5\sqrt{\frac{-1+\sqrt{17}}{10}} - \sqrt{\frac{10}{-1+\sqrt{17}}}$ feet.

7. [14 points] For positive A and B, the force between two atoms is a function of the distance, r, between them:

$$f(r) = -\frac{A}{r^2} + \frac{B}{r^3} \qquad r > 0.$$

a. [2 points] Find the zeroes of f (in terms of A and B).

Solution: Finding a common denominator for f, we have

$$f(r) = \frac{-Ar + B}{r^3}.$$

This means $f(r) = 0$ when the numerator is zero, so $r = \frac{B}{A}$ is the only zero of f.

b. [7 points] Find the coordinates of the critical points and inflection points of f in terms of A and B.

Solution: Seeking critical points, we take the derivative of $f(r)$ and set it equal to zero

$$f'(r) = \frac{2A}{r^3} - \frac{3B}{r^4} = \frac{2Ar - 3B}{r^4} = 0.$$

Solving, we have that $r = \frac{3B}{2A}$ is our only critical point.
Now seeking inflection points, we take the second derivative of $f(r)$ and set it equal to zero.

$$f''(r) = -\frac{6A}{r^4} + \frac{12B}{r^5} = \frac{12B - 6Ar}{r^5} = 0.$$

Solving, we have that $r = \frac{2B}{A}$ is a candidate for an inflection point. Now we must test to see whether this is an inflection point. We compute $f''(\frac{B}{A}) = \frac{6B}{(B/A)^5} > 0$ since A and B are both positive, and also, $f''(\frac{3B}{A}) = \frac{-6B}{(3B/A)^5} < 0$ since A and B are both positive. This means f'' changes sign from positive to negative across the point $r = \frac{2B}{A}$, so it must be an inflection point.

c. [5 points] If f has a local minimum at $(1, -2)$ find the values of A and B. Using your values for A and B, justify that $(1, -2)$ is a local minimum.

Solution: We already know our only critical point is $r = \frac{3B}{2A}$. If f has a local minimum at $(1, -2)$, we must have that $1 = r = \frac{3B}{2A}$, so that $2A = 3B$. In addition, $-2 = f(1) = -A + B$. Solving these equations simultaneously, we have $A = 6$ and $B = 4$. We have already computed

$$f''(r) = \frac{12B - 6Ar}{r^5} = \frac{48 - 36r}{r^5}.$$

So $f''(1) = 48 - 36 = 12 > 0$ which means the critical point $(1, -2)$ is a local minimum since f is concave up at this point.

8. [12 points] Below is a table of values for the function $t(y)$ which gives the number of tweets per day, in millions, on the social media website Twitter, y years after January 1, 2007. For this problem assume $t(y)$ is an increasing function.

year y	0	1	2	3	4
millions of tweets per day $t(y)$	0.005	0.3	2.5	35	50

a. [4 points] Using the table, estimate the expression

$$365 \int_1^4 t(y)dy$$

using a left-hand Riemann sum. Please write all of the terms in the sum for full credit.

Solution: To estimate this integral, we will use a left hand sum with 3 subdivisions. Since $n = 3$, $\Delta y = (4 - 1)/3 = 1$. Therefore, the left hand sum is

$$365 \int_1^4 t(y)dy \approx 365(t(1) \cdot 1 + t(2) \cdot 1 + t(3) \cdot 1) = 365(37.8) = 13,797 \text{ million tweets}$$

b. [4 points] Give a practical interpretation of the expression $365 \int_1^4 t(y)dy$.

Solution: Since $t(y)$ gives tweets per day when you input a year, the units on the definite integral are (millions of tweets per day)(year). When we multiply by 365 days per year, we have the units of $365 \int_1^4 t(y)dy$ are millions of tweets. So the definite integral represents the total number tweets in millions that appeared on Twitter between January 1, 2008 and January 1, 2011.

c. [4 points] Suppose $T(y)$ is the total number of tweets, in millions, y years after January 1, 2007. If $T(3) = 9797$, estimate the total number of tweets between January 1, 2007 and January 1, 2011. Indicate what method you use to obtain your estimate and be sure to show your work.

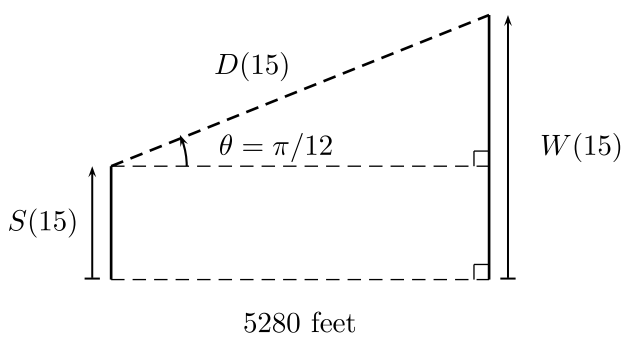
Solution: By the fundamental theorem of calculus, we know that

$$T(y) - T(3) = 365 \int_3^y t(w)dw.$$

In order to find $T(4)$, we need to compute the definite integral $\int_3^4 t(w)dw$. From the table above, we can use a left hand sum to compute $365 \int_3^4 t(w)dw = 365(35 \cdot 1) = 12775$. From our formula, we get that $T(4) = 12775 + T(3) = 22572$ million tweets. If we use a right hand sum to compute the value of the integral, we get $365 \int_3^4 t(w)dw = 365(50 \cdot 1) = 18250$ for a total of $T(4) = 28047$ million tweets.

9. [10 points] You are sitting on a ship traveling at a constant speed of 6 ft/sec, and you spot a white object due east moving parallel to the ship. After tracking the object through your telescope for 15 seconds, you identify it as the white whale. Let $W(t)$ denote the distance of the whale from its starting point in feet, and $S(t)$ denote the distance of the ship from its starting point in feet, with t the time in seconds since you saw the whale. You also note that at the end of the 15 seconds you were tracking the whale, you have turned your head $\pi/12$ radians north to keep it in your sights.

a. [1 point] If initially the creature is 5280 ft (1 mile) from the ship due east, use the angle you have turned your head to find the distance $D(t)$ in feet between the ship and the creature at 15 seconds. The figure below should help you visualize the situation. Recall that, for right triangles, $\cos(\theta)$ is the ratio of the adjacent side to the hypotenuse.



Solution: Since $\cos(\pi/12) = \frac{5280}{D(15)}$, we find that $D(15) = \frac{5280}{\cos(\pi/12)} \approx 5466.258$ft.

b. [2 points] Let $\theta(t)$ give the angle you've turned your head after t seconds of tracking the whale. Write an equation $D(t)$ for the distance between the ship and the whale at time t (Hint: your answer may involve $\theta(t)$).

Solution: From the previous part, we know that the distance between the ship and the whale is 5280 divided by the cosine of the angle you've turned your head. Since $\theta(t)$ gives how far you've turned your head, we can find the distance at any time t using the function $D(t) = \frac{5280}{\cos(\theta(t))}$.

c. [3 points] If at precisely 15 seconds, you are turning your head at a rate of .01 radians per second, what is the instantaneous rate of change of the distance between the ship and the whale?

Solution: Since $D(t) = \frac{5280}{\cos(\theta(t))}$, we take the derivative with respect to t on both sides to get

$$\frac{dD}{dt} = \frac{5280\sin(\theta(t))}{\cos^2(\theta(t))}\theta'(t).$$

Since $\theta(15) = \pi/12$ and we are given $\theta'(15) = .01$, we get

$$\left.\frac{dD}{dt}\right|_{t=15} = \frac{5280\sin(\pi/12)}{\cos^2(\pi/12)}(.01) \approx 14.6468\text{ft/sec}.$$

d. [4 points] What is the speed of the whale at $t = 15$ seconds? Hint: Use the Pythagorean theorem.

Solution: The right triangle in the figure above has hypotenuse $D(t)$ and sides with length $D(t)$ and $W(t) - S(t)$, so the Pythagorean theorem states

$$D(t)^2 = 5280^2 + (W(t) - S(t))^2.$$

If we take the t derivative of both sides, we get

$$2D(t)\frac{dD}{dt} = 2(W(t) - S(t))\left(\frac{dW}{dt} - \frac{dS}{dt}\right).$$

To find $\left.\frac{dW}{dt}\right|_{t=15}$, we will need $D(15) = 5466.258$ and $\left.\frac{dD}{dt}\right|_{t=15} = 14.6468$. We also need to find $W(15) - S(15)$, but since this is one side of the right triangle, we can use tangent to find this distance: $W(15) - S(15) = 5280\tan(\pi/12) \approx 1414.7717$. Finally, we also need to know $\frac{dS}{dt}$, but in the description of the problem it says that ship is traveling at a constant speed of 6 ft/sec. Plugging all of this information into our equation we have

$$2(5466.258)(14.6468) = 2(1414.7717)\left(\frac{dW}{dt} - 6\right) \Rightarrow 56.5909 = \left.\frac{dW}{dt}\right|_{t=15} - 6.$$

Therefore, $\left.\frac{dW}{dt}\right|_{t=15} \approx 62.5909$ ft/sec