Math 115 — Second Midterm

November 13, 2012

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 9 pages including this cover. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	12	
2	15	
3	8	
4	13	
5	14	
6	12	
7	13	
8	13	
Total	100	

1. [12 points] The following questions relate to the implicit curve $2x^2 + 4x - x^2y^2 + 3y^4 = -1$.

a. [6 points] Calculate $\frac{dy}{dx}$.

Solution: Differentiating both sides with respect to x, we get

$$4x + 4 - 2xy^2 - 2x^2y\,\frac{dy}{dx} + 12y^3\,\frac{dy}{dx} = 0.$$

Moving all terms with no $\frac{dy}{dx}$ to the other side and factoring out $\frac{dy}{dx}$ gives us

$$\frac{dy}{dx}(12y^3 - 2x^2y) = 2xy^2 - 4x - 4.$$

So

$$\frac{dy}{dx} = \frac{2xy^2 - 4x - 4}{12y^3 - 2x^2y} = \frac{xy^2 - 2x - 2}{6y^3 - x^2y}.$$

b. [2 points] Q is the only point on the curve that has a y-coordinate of 1. Find the x-coordinate of Q.

Solution: Plugging $y = 1$ into the equation for the curve gives us

$$2x^2 + 4x - x^2 + 3 = -1.$$

Moving all the terms to the left, we get

$$x^2 + 4x + 4 = 0.$$

This factors as $(x+2)^2 = 0$, so $x = -2$.

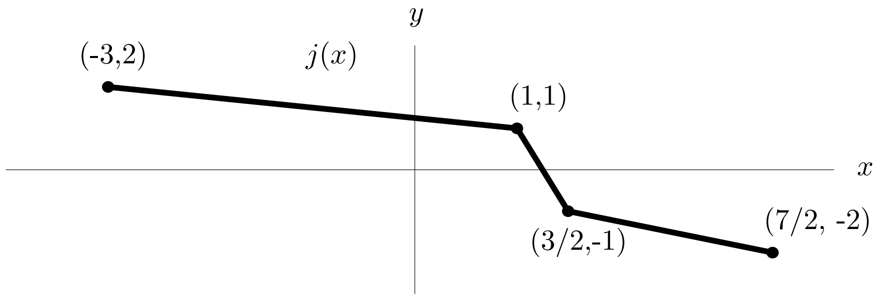
c. [4 points] Find the equation of the tangent line to the curve at Q.

Solution: To find the slope, we plug in $x = -2$ and $y = 1$ to $\frac{dy}{dx}$.

$$\text{slope} = \frac{-2 + 4 - 2}{6 - 4} = 0.$$

Thus, the tangent line is the horizontal line passing through Q, which has equation $y = 1$.

2. [15 points] The graph of a piecewise linear function $j(x)$ is given below. Use it to select the correct value of each derivative below. Circle only one answer for each part. Ambiguous marks will receive no credit.



a. [3 points] $\frac{d}{dx}[j(4\cos x)]$ at $x = \frac{\pi}{4}$.

(A) $-1/2$

(B) $\sqrt{2}$

(C) $-\sqrt{2}/2$

(D) $-\sqrt{2}$

b. [3 points] $\frac{d}{dx}[j(j(x))]$ at $x = 2$.

(A) $1/4$

(B) $1/8$

(C) $-1/4$

(D) $-1/8$

c. [3 points] $\frac{d}{dx}[2^{j(x)}]$ at $x = \frac{5}{4}$.

(A) $-2\ln 2$

(B) $-\frac{1}{2}\ln 2$

(C) $-4\ln 2$

(D) 0

d. [3 points] $\frac{d}{dx}[j^{-1}(x)]$ at $x = 0$.

(A) $1/4$

(B) $1/2$

(C) $-1/4$

(D) $-1/2$

e. [3 points] $j'(j(x))$ at $x = 3$.

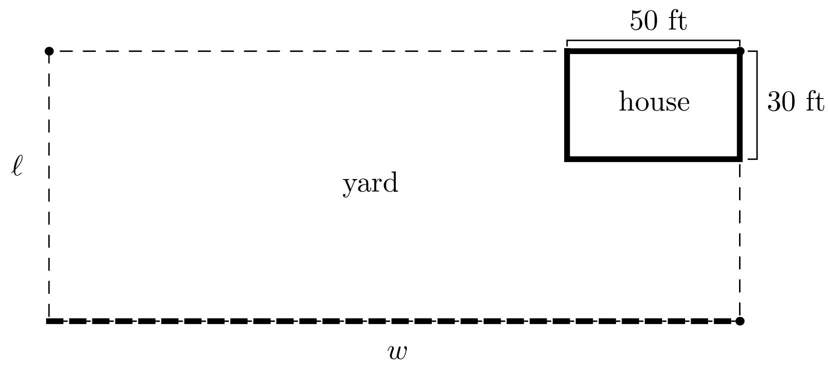
(A) $1/4$

(B) $1/8$

(C) $-1/4$

(D) $-1/2$

3. [8 points] Jason has a 50 ft by 30 ft house and wants to enclose his yard with a fence to keep his dogs in, as shown below. On the south side of his yard, he plans for the fence to be extra tall to shade his yard from the sun. Note that the fence does not extend around the sides of Jason's house. The extra tall fence (thick dashed line) costs \$15 per foot, and the rest of the fence (thin dashed line) costs \$5 per foot. Jason is going to spend \$4500 on his fence.



a. [5 points] ℓ is the length of the fenced in yard, and w is the width, as shown above. Write a formula for ℓ in terms of w. Your formula should not involve any other variables.

Solution: $4500 = 15w + 5(\ell + (w - 50) + (\ell - 30)) = 20w + 10\ell - 400$. So $\ell = 490 - 2w$.

b. [3 points] Write a formula for the total area A of the fenced yard (not including the house), in terms of w. Your answer should not include ℓ. (This is the equation Jason would use to find the values of w and ℓ maximizing the area he can enclose. You should **not** do the optimization in this case.)

Solution: $A = w\ell - 50 \cdot 30 = w\ell - 1500$. Substituting in $\ell = 490 - 2w$, this becomes $A = w(490 - 2w) - 1500$.

4. [13 points] Let $f(x) = e^{\sin\sqrt{x}}$. Let P be the point on the graph of f at which $x = 4\pi^2 (\approx 39.4784)$.

a. [3 points] Calculate $f'(x)$.

Solution:

$$f'(x) = \left(e^{\sin\sqrt{x}}\right)\left(\cos\sqrt{x}\right)\left(\frac{1}{2}x^{-1/2}\right) = \frac{e^{\sin\sqrt{x}}\cos\sqrt{x}}{2\sqrt{x}}$$

b. [4 points] Find an **exact** formula for the tangent line $L(x)$ to $f(x)$ at P. **Exact** means your answer should not involve any decimal approximations.

Solution:

$$\text{slope} = f'(4\pi^2) = \frac{e^{\sin(2\pi)}\cos(2\pi)}{2\cdot 2\pi} = \frac{1}{4\pi},$$

so $L(x) = \frac{x}{4\pi} + b$, where b is the vertical intercept. When $f(4\pi^2) = e^{sin(2\pi)} = 1$, so $1 = \frac{4\pi^2}{4\pi} + b$, which gives us $b = 1 - \pi$, so

$$L(x) = \frac{x}{4\pi} + 1 - \pi$$

.

c. [2 points] Use your formula for $L(x)$ to approximate $e^{\sin\sqrt{38}}$.

Solution:

$$e^{\sin\sqrt{38}} = f(38) \approx L(38) = \frac{38}{4\pi} + 1 - \pi \approx 0.8824.$$

d. [4 points] Recall that the error, $E(x)$, is the actual value of the function minus the value approximated by the tangent line. Given the fact that in this case $E(39) \approx 0.000613$ and $E(40) \approx 0.000719$, would you expect $f''(4\pi^2)$ to be positive or negative? Explain, without doing any calculations.

Solution: The errors are positive, which means that near P the tangent line lies below the curve, so the function is probably concave up at P. Since concave up corresponds to positive second derivative, we should expect the sign of $f''(4\pi^2)$ to be positive.

5. [14 points] The function f is has a continuous second derivative on the interval $10 \le x \le 19$. Some values of its derivative function f' are given in the table below.

x	10	11	12	13	14	15	16	17	18	19
$f'(x)$	-34	-3	-1	-2	-3	31	62	70	66	37

a. [4 points] f has exactly one inflection point on the interval $15 \le x \le 19$. Given the information provided, give the smallest x interval on which this inflection point is guaranteed to lie, making it clear whether your endpoints are included.

Solution: $16 < x < 18$ or $(16, 18)$.

b. [8 points] f has exactly four critical points, with x-values 11.2, 11.7, 12.6, and 14.2, respectively. Classify each point as a local minimum, a local maximum, or neither, given that f has either a local maximum or a local minimum at $x = 11.2$. For each point below, circle only one option.

At $x = 11.2$, f has **a local maximum** **[a local minimum]**

At $x = 11.7$, f has **[a local maximum]** **a local minimum** **neither**

At $x = 12.6$, f has **a local maximum** **a local minimum** **[neither]**

At $x = 14.2$, f has **a local maximum** **[a local minimum]** **neither**

c. [2 points] Is there at least one inflection point on the interval $11 < x < 12$? (Circle one.)

[Yes] **No** **Not possible to determine**

6. [12 points] For **(a)-(d)** below, write a formula for a function that satisfies the given criteria on the given line below. You do not need to show any work. No credit will be given for anything but your final answer, which must be written on the given line.

a. [3 points] f has a critical point at $x = 0$, but has no local maximum or minimum at $x = 0$.

$f(x) =$ x^3

b. [3 points] g has domain $(-\infty, \infty)$ and is increasing and concave down for all x.

$g(x) =$ $-e^{-x}$

c. [3 points] $h''(0) = 0$, but h has no inflection point at $x = 0$.

$h(x) =$ x^4

d. [3 points] j has an inflection point at $x = 0$ that is not a critical point.

$j(x) =$ $\sin x$

7. [13 points] Consider the family of functions

$$y = ax^b \ln x$$

where a and b are nonzero constants.

a. [4 points] Calculate $\frac{dy}{dx}$ in terms of the constants a and b.

Solution: $\frac{dy}{dx} = abx^{b-1} \ln x + ax^b \left(\frac{1}{x}\right) = abx^{b-1} \ln x + ax^{b-1} = ax^{b-1}(b \ln x + 1)$.

b. [9 points] Find specific values of a and b so that the resulting function has a local maximum at the point $(e, 1)$. You must show that $(e, 1)$ is a local maximum to receive full credit.

Solution: The point $(e, 1)$ must be on the curve, so $1 = ae^b \ln e = ae^b$. So $a = e^{-b}$.
We also know $(e, 1)$ is a critical point, so

$$0 = ae^{b-1}(b \ln e + 1) = ae^{b-1}(b - 1).$$

$a \neq 0$ and $e^{b-1} \neq 0$ so $b + 1 = 0$, which means $b = -1$ and $a = e^{-(-1)} = e$.
To check that $(e, 1)$ is a local maximum, we use the second derivative test:
First, we plug in the values of a and b to our derivative and get $\frac{dy}{dx} = ex^{-2}(1 - \ln x)$. So we have

$$\frac{d^2y}{dx^2} = e(-2)x^{-3}(1 - \ln x) + ex^{-2}\left(\frac{-1}{x}\right).$$

Now we can plug in $x = e$ and get $-2e \cdot e^{-3}(1 - \ln e) - e \cdot e^{-3} = -e^{-2} < 0$. Thus, $(e, 1)$ is a local maximum.

8. [13 points] Two smokestacks d miles apart deposit soot on the ground between them. The concentration of the combined soot deposits on the line joining them, at a distance x from one stack, is given by

$$S = \frac{c}{x^2} + \frac{k}{(d-x)^2}$$

where c and k are positive constants which depend on the quantity of smoke each stack is emitting. If $k = 27c$, find the x-value of the point on the line joining the stacks where the concentration of the deposit is a minimum. Justify that the point you found is actually a global minimum.

Solution: First we plug in $k = 27c$ and get

$$S = \frac{c}{x^2} + \frac{27c}{(d-x)^2}.$$

Then we take the derivative and set it equal to zero to find the critical points on the domain $0 < x < d$:

$$S' = \frac{-2c}{x^3} + \frac{(-2)(27c)(-1)}{(d-x)^3} = \frac{-2c}{x^3} + \frac{2 \cdot 27c}{(d-x)^3} = 0$$

Now we solve for x:

$$\frac{2 \cdot 27c}{(d-x)^3} = \frac{2c}{x^3}$$

$$\frac{27}{(d-x)^3} = \frac{1}{x^3}$$

Taking the cube root of both sides gives $\frac{3}{d-x} = \frac{1}{x}$. So $3x = d - x$. Thus $x = \frac{d}{4}$.

Now our domain is $(0, d)$ and $x = \frac{d}{4}$ is our only critical point. As $x \to 0$, $S \to \infty$, and as $x \to d$, $S \to \infty$. Thus, our global minimum must be at our sole critical point at $x = \frac{d}{4}$.