Math 115 — Final Exam

December 14, 2012

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 10 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	12	
2	5	
3	4	
4	14	
5	10	
6	10	
7	11	
8	11	
9	11	
10	12	
Total	100	

1. [12 points] Let $f(x)$ and $g(x)$ be **increasing** continuous functions defined on the interval $[0, 10]$ with $f(0) = g(0) = 0$. Also suppose f is always concave down and g is always concave up. For each of the following statements, determine whether it is always true, sometimes true, or never true, and circle only one option. Explanations are not necessary and they will not be counted for credit.

a) $\int_0^{10} f(x)dx > \int_0^{10} g(x)dx.$

always **[sometimes]** **never**

b) $f'(10) < g'(10).$

always **[sometimes]** **never**

c) $g'(0) > g'(2).$

always **sometimes** **[never]**

d) $\int_0^{10} |f(x)|dx = \int_0^{10} f(x)dx.$

[always] **sometimes** **never**

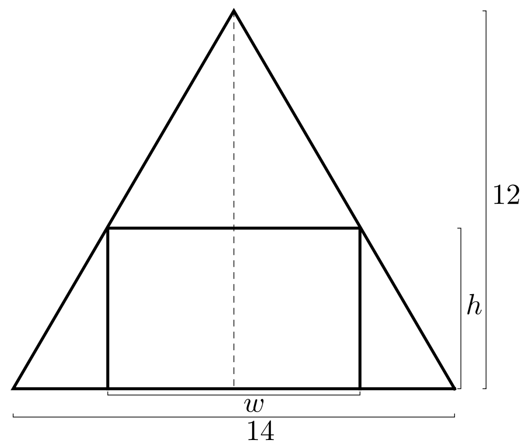
e) $\int_0^{10} f'(x)dx > 0.$

[always] **sometimes** **never**

f) If G is an antiderivative of g, then $G(10) > 0$.

always **[sometimes]** **never**

2. [5 points] Caleb has an attic apartment, and his bedroom has a triangular wall that is 14 feet wide and 12 feet tall at its tallest point. He wants to build a rectangular bookcase to put against the wall, as shown to the right. He is trying to maximize the area of the front of the bookcase.



a. [3 points] If the bookcase has width w and height h, write a formula relating w and h.

Solution: Exploiting properties of similar triangles, we get

$$\frac{w}{12-h} = \frac{14}{12},$$

so

$$w = \frac{7(12-h)}{6}.$$

b. [2 points] Using your answer from (a), find an expression for the area of the front of the bookcase in terms of the variable h.

Solution: Area $w \cdot h = \dfrac{7(12-h)h}{6}$.

3. [4 points] Suppose $g(x) = x^{2x}$. Write an explicit expression for $g'(5)$ using the limit definition of the derivative. Your expression should not contain the letter "g". Do not evaluate your expression.

Solution:

$$g'(5) = \lim_{h\to 0} \frac{g(5+h) - g(5)}{h} = \lim_{h\to 0} \frac{(5+h)^{2(5+h)} - 5^{10}}{h}.$$

4. [14 points] Consider the family of functions

$$y = x^{2b} + ax^b$$

where a and b are nonzero constants.

a. [4 points] Calculate y'.

Solution: $y' = 2bx^{2b-1} + abx^{b-1}$

b. [4 points] Calculate y''.

Solution: $y'' = 2b(2b-1)x^{2b-2} + ab(b-1)x^{b-2}$

c. [6 points] Find values of a and b so that the resulting function has an inflection point at $(x, y) = (1, -4)$. Justify that $(1, -4)$ is an inflection point of the function with the values of a and b that you found.

Solution: The point $(1, -4)$ must be on the graph of our function, so

$$-4 = 1^{2b} + a \cdot 1^b = 1 + a.$$

Thus $a = -5$.
$(1, -4)$ must be an inflection point, so we plug $x = 1$ and $a = -5$ into our second derivative and set it equal to zero:

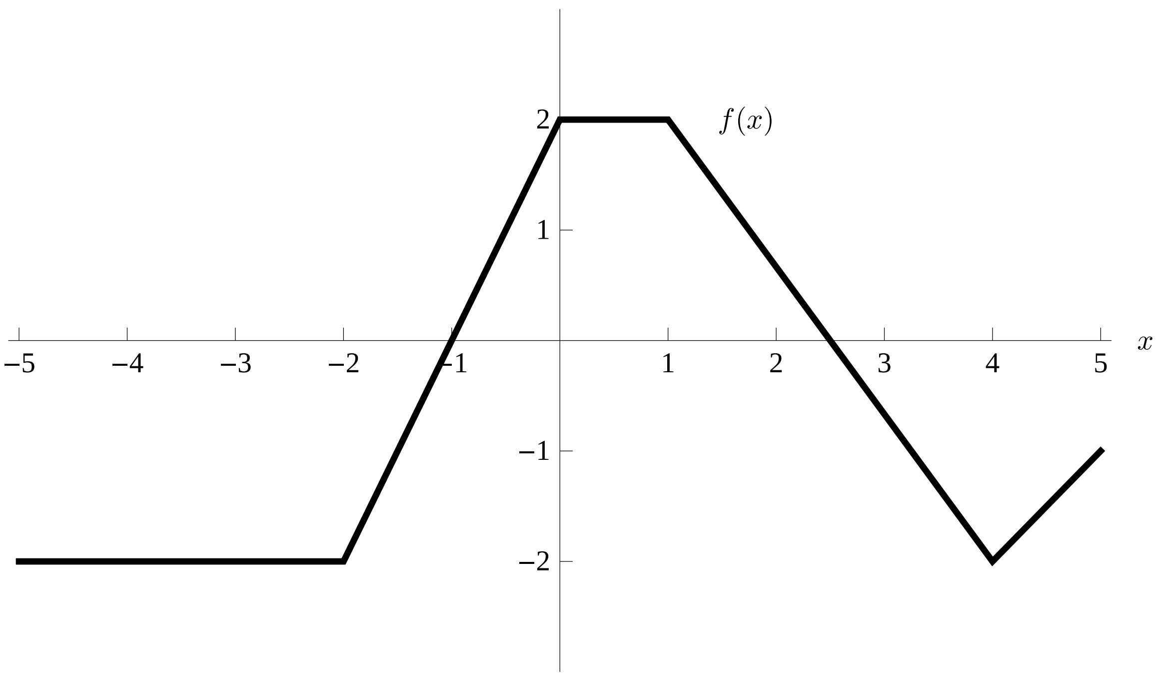
$$0 = 2b(2b-1) \cdot 1^{2b-2} - 5b(b-1) \cdot 1^{b-2} = 4b^2 - 2b - 5b^2 + 5b = -b^2 + 3b = -b(b-1).$$

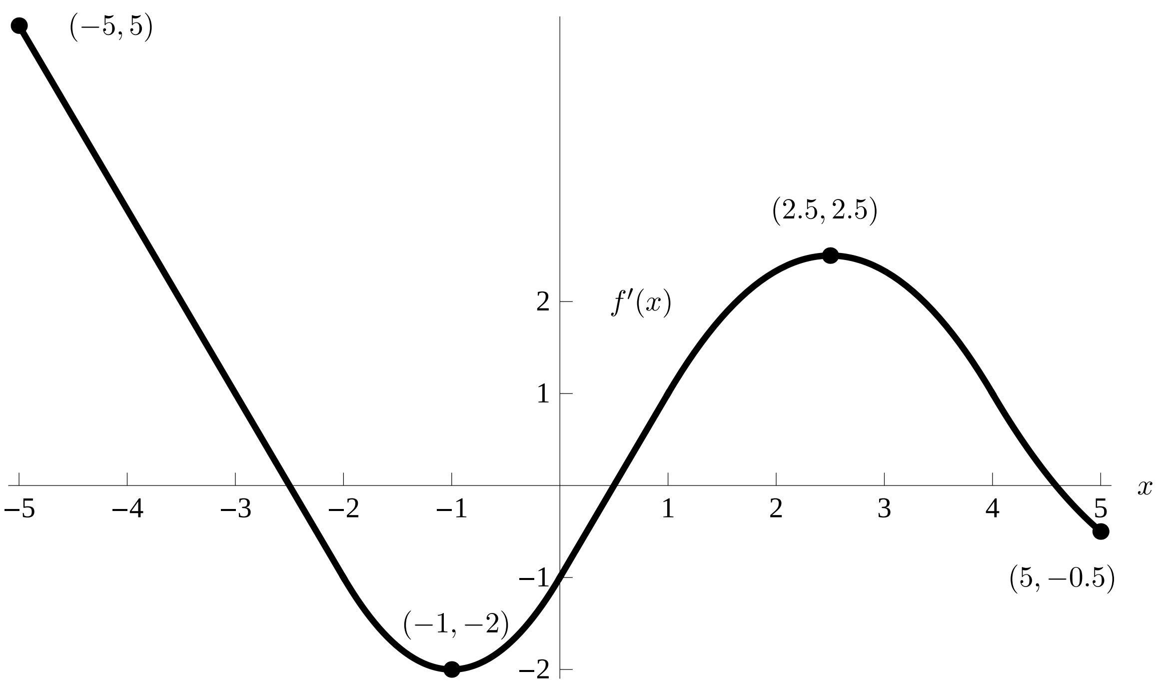
So $b = 0$ or 3, but b is nonzero so $b = 3$.
To check that $(1, -4)$ is indeed and inflection point, we check that y'' changes sign at $x = 1$. Plugging in the values of a and b we found, we get

$$y'' = 6(5)x^4 - 5(3)(2)x^2 = 30x^4 - 30x^2 = 30x^2(x^2 - 1).$$

The term $30x^2$ is always positive and $x^2 - 1$ changes sign from negative to positive at $x = 1$. Therefore, $(1, -4)$ is an inflection point of the function.

5. [10 points] The graph of a piecewise linear function $f(x)$ is shown below. On the axes provided, sketch a well-labeled graph of an antiderivative $F(x)$ of $f(x)$ satisfying $F(0) = -1$. Be sure to make the concavity of F clear and to label the y-coordinates of the local minima and maxima of F and the y-coordinates of F at $x = 5$ and $x = -5$.





6. [10 points] The company Canco manufactures and sells aluminum cans. The table below gives information about marginal cost (MC), in dollars, for Canco at various production levels q, in cans. The marginal revenue is constant and equal to \$0.05.

q	0	10000	20000	30000	40000	50000	60000
MC	\$0.05	\$0.04	\$0.03	\$0.02	\$0.06	\$0.08	\$0.11

a. [3 points] Assuming the marginal cost is either only increasing or only decreasing between each pair of consecutive table entries, give the smallest interval with endpoints from the table which must contain the production level that maximizes the company's profit. You do not need to justify your answer.

Solution: $30000 < q < 40000$

b. [2 points] Find a formula for $R(q)$, the revenue from selling q cans.

Solution: $MR = R'(q) = 0.05$, so $R(q) = 0.05q + C$. $R(0) = 0$, so $C = 0$, which means $R(q) = 0.05q$.

c. [5 points] Using a right sum, approximate the cost for the company to make the first 50,000 cans. Assume the fixed costs for the company are zero. You must write all the terms in the sum to receive credit.

Solution: Cost to make first 50,000 cans $\approx 10000(0.04+0.03+0.02+0.06+0.08) = \$2300$

7. [11 points] Consider the continuous function

$$f(x) = \begin{cases} x \cdot 2^{-x} & 1 \le x < 3, \\ \frac{1}{2-x} + \frac{11}{8} & 3 \le x \le 5. \end{cases}$$

Note that the domain of f is $[1, 5]$.

a. [7 points] Find the x-values of the critical points of f.

Solution: To find the critical points, we first take the derivatives of the two functions and set them equal to zero.

$$\frac{d}{dx} x2^{-x} = 2^{-x} - x \ln 2 \cdot 2^{-x} = 2^{-x}(1 - x \ln 2) = 0.$$

2^{-x} is never 0 so $1 = x \ln 2$ so $x = \frac{1}{\ln 2} \approx 1.44$, which is between 1 and 3, so it is a critical point.

$$\frac{d}{dx}\left(\frac{1}{2-x} + \frac{11}{8}\right) = \frac{1}{(2-x)^2},$$

which is never 0, but undefined at 2, which is not between 3 and 5 and therefore not a critical point.

To check if there is a critical point at $x = 3$, we can either graph the function on a calculator and see that there is a sharp corner there, or we can check and see that the derivatives of the two functions are not equal there:

$$2^{-3}(1 - 3 \ln 2) \approx -0.13 \neq 1 = \frac{1}{(2-3)^2}.$$

Thus, the critical points are at $x = \frac{1}{\ln 2}$ and $x = 3$.

b. [4 points] Find the y-values of the global maximum and global minimum of f if they exist, or explain why they don't exist.

Solution: Since this is a closed interval, we can just test the critical points and endpoints.

$f(1) = 1 \cdot 2^{-1} = 0.5$

$f\left(\frac{1}{\ln 2}\right) = \frac{1}{\ln 2} \cdot 2^{-\frac{1}{\ln 2}} \approx 0.53$

$f(3) = 3 \cdot 2^{-3} = \frac{3}{8} = 0.375$

$f(5) = \frac{1}{2-5} + \frac{11}{8} = \frac{25}{24} \approx 1.042$

So the global maximum is $\frac{25}{24} \approx 1.042$ and the global minimum is $\frac{3}{8} = 0.375$.

8. [11 points] Let $W(t)$ be the temperature, in degrees Fahrenheit, of a cake t minutes after it is put in the oven. Assume $W(10) = 220$.

a. [3 points] Give a practical interpretation of the statement $\int_5^{10} W'(t)dt = 120$.

Solution: Between minutes five and ten in the oven, the cake's temperature increases by 120°F.

b. [3 points] Give a practical interpretation of the statement $\frac{1}{2}\int_3^5 W(t)dt = 80$.

Solution: Between minutes three and five in the oven, the cake's average temperature is 80°F.

c. [3 points] Write a single mathematical equation describing the following statement: The average temperature of the cake over the first five minutes in the oven is the same as its temperature after three minutes in the oven.

Solution:

$$\frac{1}{5}\int_0^5 W(t)dt = W(3)$$

d. [2 points] Assuming all of the above statements in (a)-(c) are true, what will the temperature of the cake be five minutes after it is put in the oven?

Solution: From (a), we deduce $W(10) - W(5) = 120$. So $220 - W(5) = 120$, which means $W(5) = 100$°F.

9. [11 points] A cube of ice is removed from the freezer and begins to melt. Let $\ell(t)$ be its side length, $V(t)$ its volume, and $S(t)$ its surface area, all dependent upon t, the number of minutes since it was removed from the freezer. The ice cube is melting (its volume is changing) at a rate proportional to its surface area. That is, $\frac{dV}{dt} = kS(t)$, for some number k. Initially the ice cube has a side length of 2 inches.

a. [4 points] Write V and S in terms of ℓ. Calculate the rate of change (with respect to time) of the side length of the ice cube in terms of ℓ and k.

Solution: $V = \ell^3$ and $S = 6\ell^2$.
We know, by the chain rule, that

$$\frac{dV}{dt} = \frac{dV}{d\ell}\frac{d\ell}{dt} = 3\ell^2\frac{d\ell}{dt}.$$

We are given that

$$\frac{dV}{dt} = kS(t) = k6\ell^2.$$

Setting these equal, we get

$$3\ell^2\frac{d\ell}{dt} = k6\ell^2.$$

So $\frac{d\ell}{dt} = 2k$ inches per minute.

b. [2 points] How fast is the **volume** of the ice cube changing immediately after it is removed from the freezer? Your answer will involve k.

Solution: The side length at time $t = 0$ is 2 inches, so
$\frac{dV}{dt}\big|_{t=0} = k \cdot 6 \cdot 2^2 = 24k$ in^3 per minute

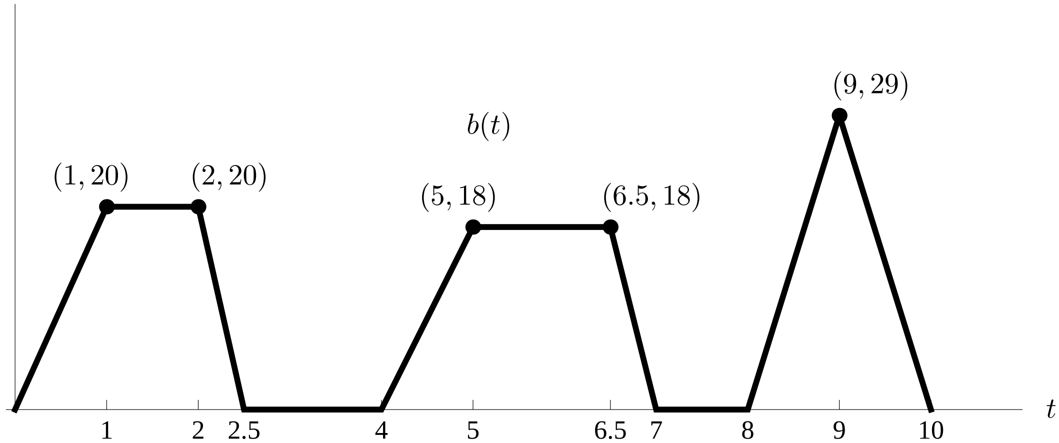
c. [2 points] What is the sign of k? Briefly explain.

Solution: From part (b) we see that k has the same sign as $\frac{dV}{dt}\big|_{t=0}$, which is negative, since the volume is decreasing. Thus, k must be negative as well.

d. [3 points] How long will it take the ice cube to melt completely? Your answer may involve k.

Solution: The side length is changing at a constant rate of $2k$ and starts at 2 inches, so it will take T minutes, where $2 + 2kT = 0$. So $T = -\frac{1}{k}$ minutes.

10. [12 points] Byron is blowing up a balloon. The rate at which he is blowing air into the balloon at time t is $b(t)$ cubic inches per second, graphed below. When $t = 0$, the balloon is empty.



a. [2 points] How much air has Byron blown into the balloon after 3 seconds?

Solution:
$$\int_0^3 b(t)dt = 10 + 20 + 5 = 35,$$
so after 3 seconds he has blown 35 cubic inches of air into the balloon.

After 3 seconds, the balloon springs a leak, and the air leaks out at a constant rate of r cubic inches per second.

b. [4 points] How much air is in the balloon 8 seconds after Byron started blowing it up? Your answer will involve r.

Solution:
$$\int_0^8 b(t)dt - 5r = 35 + 9 + 27 + 4.5 - 5r = 75.5 - 5r,$$
so after 8 seconds, there are $75.5 - 5r$ cubic inches of air in the balloon.

c. [3 points] Let $B(t)$ be the amount of air in the balloon after t seconds. Suppose $B(t)$ has a critical point at $t = 8.25$. Find r.

Solution: $B'(8.25) = 0$, which means $b(8.25) - r = 0$. We know $b(8.25) = \frac{29}{4}$, so $r = \frac{29}{4} = 7.25$ cubic inches per second.

d. [3 points] Is the critical point at $t = 8.25$ a local maximum, local minimum, or neither? Briefly explain.

Solution: At $t = 8.25$, $b(t)$ changes from being less than r to greater than r, so $B'(t)$ goes from negative to positive. Thus, by the first derivative test, the critical point is a local minimum.