Math 115 — First Midterm

October 8, 2013

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ______________

1. **Do not open this exam until you are told to do so.**
2. This exam has 10 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	14	
2	10	
3	5	
4	7	
5	6	
6	11	
7	15	
8	12	
9	10	
10	10	
Total	100	

1. [14 points] Carla is trying to model the growth of the feet of her son, Taser, to predict what size boots she needs to buy him to last him through the winter. She has measured Taser's feet three times, once exactly nine months ago, once exactly three months ago, and once just today. Carla decides to measure t in months since she took her first measurement. Below is a table containing her measurements. Carla lost the record of her first measurement so the corresponding entry in the table is blank.

t (months)	0	6	9
foot length (inches)		6.4	7.2

a. [3 points] Write a linear function $L(t)$ modeling the length of Taser's feet t months after she took her first measurement.

Solution: The slope of $L(t)$ is $\dfrac{L(9)-L(6)}{9-6} = \dfrac{7.2-6.4}{9-6} = \dfrac{0.8}{3} \approx 0.267$. Using point-slope form gives

$$L(t) = \frac{0.8}{3}(t-6) + 6.4 = \frac{0.8}{3}t + 4.8$$

b. [5 points] Write a exponential function $E(t)$ modeling the length of Taser's feet t months after she took her first measurement.

Solution: An exponential function $E(t)$ will be of the form $E(t) = ab^t$. Plugging in the data points gives

$$6.4 = ab^6$$
$$7.2 = ab^9.$$

Dividing the second equation by the first gives $b^3 = \dfrac{7.2}{6.4}$ so $b = \left(\dfrac{7.2}{6.4}\right)^{1/3} \approx 1.040$. Using the first equation above, we have

$$6.4 = a\left(\left(\frac{7.2}{6.4}\right)^{1/3}\right)^6$$

so $a = \dfrac{6.4}{\left(\dfrac{7.2}{6.4}\right)^2} \approx 5.057$. Therefore our function is $E(t) = 5.057(1.040)^t$.

c. [2 points] According to the exponential model you found in **(b)**, what is the missing value in the table above?

Solution:

$$E(0) \approx 5.057$$

d. [4 points] Bob, the salesman at the shoe store, has a different model for foot growth. He gives Carla the formula

$$B(t) = \frac{50}{5 + 6e^{-t/8}}$$

for the length of Taser's feet t months since Carla took her first measurement. According to Bob's model, when will Taser's feet be 8 inches long? Give your answer in exact form with no decimals.

Solution: We need to solve $B(t) = 8$ for t.

$$\begin{aligned}
\frac{50}{5 + 6e^{-t/8}} &= 8 \\
\frac{50}{8} &= 5 + 6e^{-t/8} \\
\frac{\frac{50}{8} - 5}{6} &= e^{-t/8} \\
\ln \frac{\frac{50}{8} - 5}{6} &= -t/8 \\
-8 \ln \frac{\frac{50}{8} - 5}{6} &= t
\end{aligned}$$

2. [10 points] Louis owns a small soda company and is experimenting with new flavors. Let $b(p)$ model the number of thousands of bottles of bacon-flavored soda sold by his company per month if he charges p cents per bottle. You may assume $b(p)$ is differentiable and invertible.

a. [2 points] Give a practical interpretation of the statement $b^{-1}(8) = 150$.

Solution: In order to sell 8000 bottles of bacon-flavored soda per month, the company should charge 150 cents per bottle.

b. [3 points] Give a practical interpretation of the statement $(b^{-1})'(4) = -10$.

Solution: In order to increase the number of bottles sold per month from 4000 to 5000, the company should lower the price about 10 cents.
If the company is currently selling 4000 bottles per month, lowering the price by 10 cents will increase sales by about 1000 bottles per month.
(There are other possible answers.)

c. [3 points] Write an expression that is equal to the price (in cents) that the company would have to charge per bottle in order to sell twice as many bottles of bacon-flavored soda as it sells at a price of 125 cents per bottle.

Solution: $b^{-1}(2b(125))$

d. [2 points] Which of the following is a correct formula for a function $h(d)$ that gives the number of thousands of bottles sold per month at a price of d dollars per bottle? (Circle your answer.)

$h(d) = 100b(d)$ $\qquad$ $h(d) = \dfrac{b(d)}{100}$ $\qquad$ $\boxed{h(d) = b(100d)}$ $\qquad$ $h(d) = b\left(\dfrac{d}{100}\right)$

3. [5 points] Use the limit definition of the derivative to write an explicit expression for $r'(3)$ where $r(t) = (t + 5)^{2t}$. Do not simplify or evaluate the limit. Your answer should not include the letter r.

Solution:

$$r'(3) = \lim_{h\to 0} \frac{(3 + h + 5)^{2(3+h)} - (3 + 5)^{2(3)}}{h}$$

4. [7 points] After the success of his new bacon-flavored soda, Louis wants to try making a flavor that customers will find more refreshing in the hot summer months. Louis notices daily sales of his new spearmint soda vary seasonally. Sales reach a high of $300 around August 1 and a low of $120 around February 1. Suppose that daily sales of the soda (in dollars) can be modeled by a sinusoidal function $S(t)$ where t is the time in months since January 1. Note that August 1 is seven months after January 1. You do not need to show work for this problem.

a. [2 points] What are the period and amplitude of the function $S(t)$?

Period = **12**

Amplitude = **90**

b. [5 points] Write a formula for the function $S(t)$.

Solution:

$$S(t) = -90\cos\left(\frac{2\pi}{12}(t-1)\right)+210 = 90\sin\left(\frac{2\pi}{12}(t-4)\right)+210 = 90\cos\left(\frac{2\pi}{12}(t-7)\right)+210$$

(There are other possible answers.)

5. [6 points] For which value(s) of a is the following function continuous? Show all of your work.

$$f(x) = \begin{cases} \dfrac{x^2-9}{x-3} & \text{for } x < 3 \\ ax^2+2x+15 & \text{for } x \geq 3 \end{cases}$$

Solution: The function $f(x)$ is continuous everywhere except possibly at $x = 3$.
First find the left-hand limit at $x = 3$:

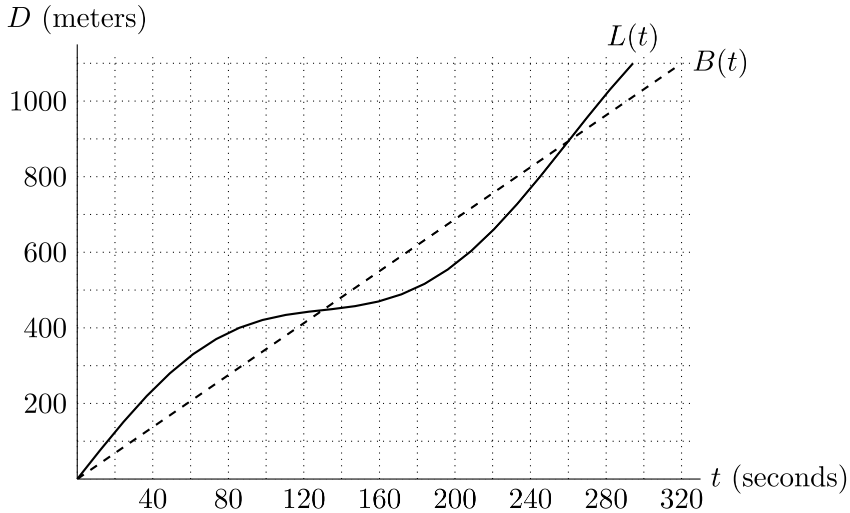
$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{x^2-9}{x-3} = \lim_{x\to 3^-} \frac{(x+3)(x-3)}{x-3} = \lim_{x\to 3^-} x+3 = 6$$

Then find the right-hand limit at $x = 3$:

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} ax^2+2x+15 = 9a+21$$

We need the left and right limits at 3 to be equal. So we solve $6 = 9a+21$ to get $a = -\dfrac{5}{3}$.
The function with $a = -\dfrac{5}{3}$ is continuous at $x = 3$ because $\lim_{x\to 3} f(x) = 6$ and $f(3) = 6$.

6. [11 points] Link and Boots decided to have a race down a straight portion of Pauline Boulevard that is 1.1 kilometers long. Let $L(t)$ and $B(t)$ be Link's and Boots's respective distances from their starting point t seconds after the race began. A graph of $L(t)$ and $B(t)$ is shown below.



a. [1 point] Who won the race? (Circle your answer.)

$\boxed{\text{Link}}$ Boots

b. [2 points] Estimate the times at which Link and Boots were running at the same speed.

Solution: They are running the same speed when the two curves have the same slope. This occurs at about $t = 65$ and $t = 195$.

c. [3 points] Estimate Link's average velocity over the first 100 seconds of the race. Include units.

Solution:

$$\text{average velocity} = \frac{L(100) - L(0)}{100} \approx \frac{425 - 0}{100} = 4.25 \text{ meters/second}$$

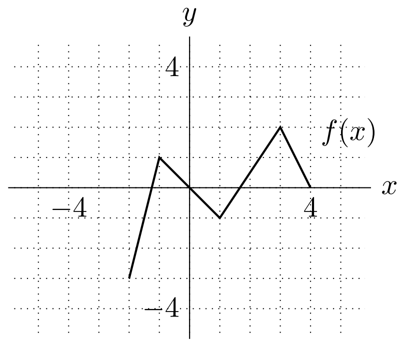
d. [3 points] Estimate Link's instantaneous velocity 40 seconds after the race began. Include units.

Solution: Estimate the slope of the tangent line to the graph of $L(t)$ at $t = 40$. The slope is about 5.1, which means his velocity is about 5.1 meters/second.

e. [2 points] 160 seconds after the race began, is Link's acceleration positive, negative, or equal to zero? (Circle your answer.)

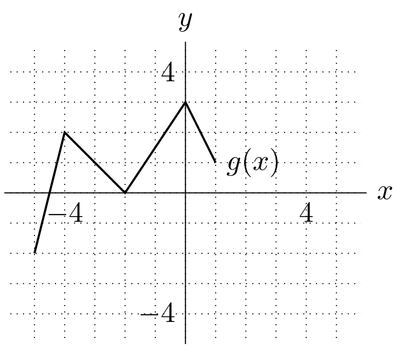
$\boxed{\text{positive}}$ negative zero

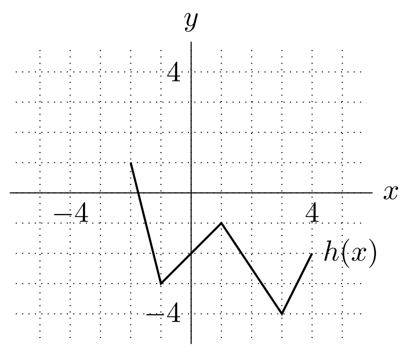
7. [15 points] The graph of a function $f(x)$ is shown below. The domain of $f(x)$ is $-2 \le x \le 4$.



You do not need to show work on this page.

a. [6 points] Each of the functions $g(x)$ and $h(x)$ shown below is a transformation of the function $f(x)$. Write a formula for each function in terms of $f(x)$.



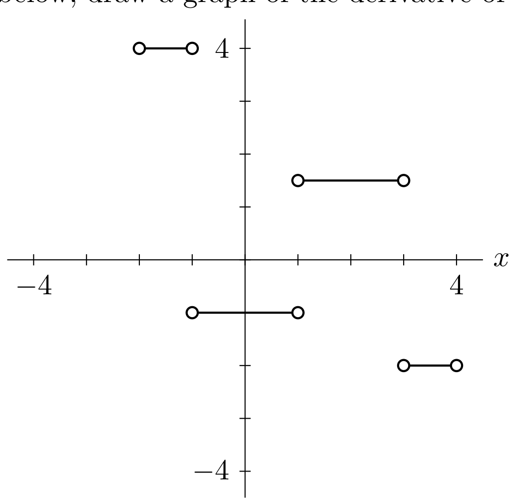


$g(x) =$ $f(x+3)+1$ $h(x) =$ $-f(x)-2$

b. [4 points] Determine the domain and range of the function $j(x) = -2f(x-6)+3$.

Domain: **4** $\le x \le$ **10** Range: **-1** $\le y \le$ **9**

c. [5 points] On the axes below, draw a graph of the derivative of $f(x)$.



8. [12 points] In Ann Arbor, the average property value P, in dollars per square foot, can be modeled as a function of the distance x, in miles, you are away from the city center. This relationship can be written $P = g(x)$. Below is a table containing information about $g(x)$. Use the information in the table to answer the parts of this question.

x	0.1	0.2	0.3	0.4	0.5
$g(x)$	200	162	142	130	119
$g'(x)$	-401	-298	-160	-115	-118

a. [3 points] Estimate $g'(0.15)$ using only values of $g(x)$ from the table.

Solution:

$$g'(0.15) \approx \frac{g(0.2) - g(0.1)}{0.2 - 0.1} = \frac{162 - 200}{0.1} = -380$$

b. [3 points] Estimate $g''(0.45)$ using only values of $g'(x)$ from the table.

Solution:

$$g''(0.45) \approx \frac{g'(0.5) - g'(0.4)}{0.5 - 0.4} = \frac{-118 - (-115)}{0.1} = -30$$

c. [3 points] Assuming the concavity of $g(x)$ does not change on the interval $0.1 < x < 0.3$, do you expect $g(x)$ to be concave up, concave down, or neither over this interval? Explain.

I expect $g(x)$ to be ***concave up*** *because . . .*

Solution: $\ldots g'(x)$ is increasing on the interval $0.1 < x < 0.3$.
$\ldots g(x)$ decreases more between $x = 0.1$ and $x = 0.2$ than between $x = 0.2$ and $x = 0.3$.

d. [3 points] Write a sentence expressing the meaning of

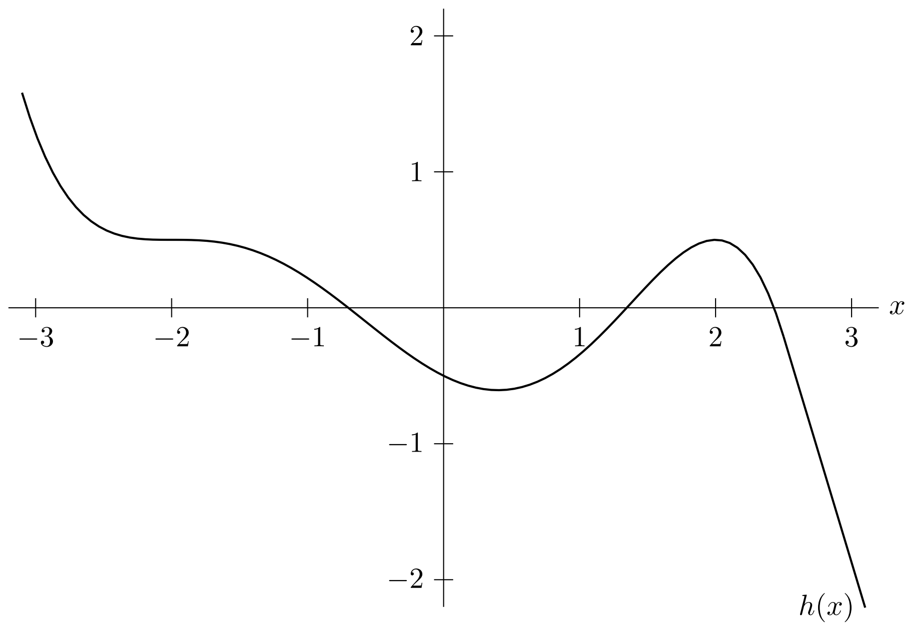
$$g'(0.3) = -160$$

which could be understood by someone who knows no calculus. The beginning of the sentence is given below.

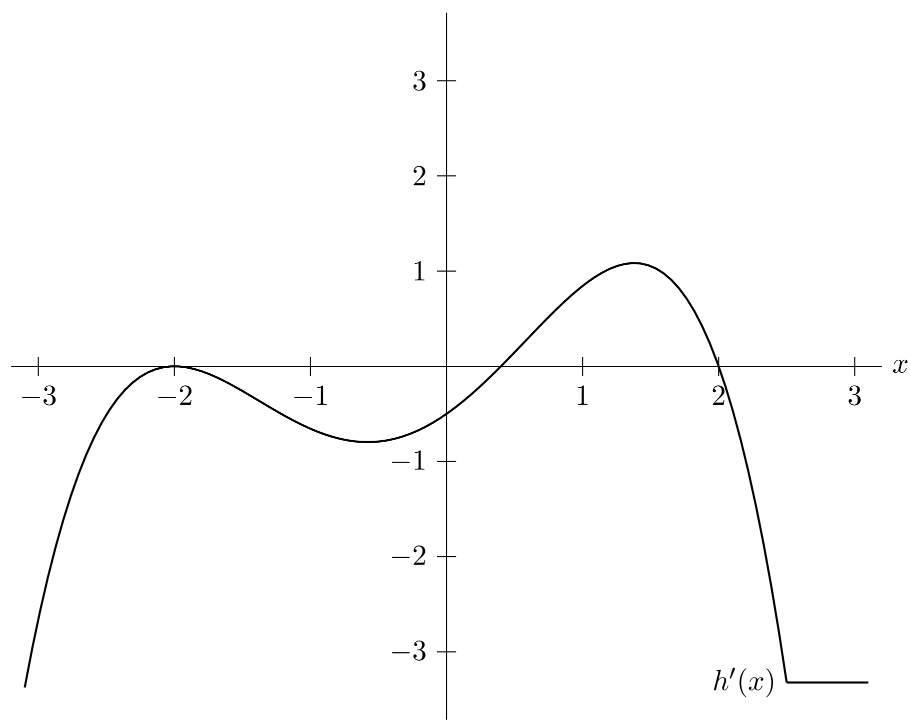
If I am 0.3 miles from the center of Ann Arbor looking at properties and I travel 0.05 miles toward the city center, . . .

Solution: . . . average property values will increase by about 8 dollars per square foot.

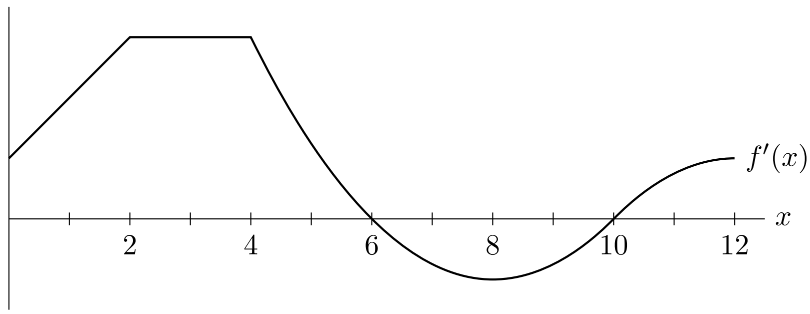
9. [10 points] Given below is the graph of a differentiable function $h(x)$ which is linear for $x > 2.5$. On the second set of axes, sketch a possible graph of $h'(x)$. Be sure your graph is drawn carefully.



Solution:



10. [10 points] The graph of $f'(x)$, the ***derivative*** of a function $f(x)$, is shown below.



For each of the following questions, circle ALL correct answers. You do not need to show work for this problem.

a. [2 points] On which of the following intervals is $f(x)$ increasing?

$\boxed{0 < x < 2}$ $\boxed{2 < x < 4}$ $\boxed{4 < x < 6}$ $6 < x < 8$ $8 < x < 10$ $\boxed{10 < x < 12}$

b. [2 points] On which of the following intervals is $f(x)$ concave down?

$0 < x < 2$ $2 < x < 4$ $\boxed{4 < x < 6}$ $\boxed{6 < x < 8}$ $8 < x < 10$ $10 < x < 12$

c. [2 points] On which of the following intervals is $f(x)$ linear?

$0 < x < 2$ $\boxed{2 < x < 4}$ $4 < x < 6$ $6 < x < 8$ $8 < x < 10$ $10 < x < 12$

d. [2 points] On which of the following intervals is $f''(x)$ increasing?

$0 < x < 2$ $2 < x < 4$ $\boxed{4 < x < 6}$ $\boxed{6 < x < 8}$ $\boxed{8 < x < 10}$ $10 < x < 12$

e. [2 points] Suppose $f(0) = -4$. Which of the following statements could be true?

$f(6) < -4$ $f(6) = -4$ $\boxed{f(6) > -4}$