Math 115 — Second Midterm

Nov 11, 2014

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
8. Include units in your answer where that is appropriate.
9. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones.
10. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	5	
2	11	
3	13	
4	12	
5	12	
6	11	
7	11	
8	7	
9	10	
10	8	
Total	100	

1. [5 points] Let $h(x)$ be a differentiable function such that $h'(x)$ is also differentiable everywhere. Suppose that $h(3) = 9$, $h'(3) = 2$, and $h''(x) > 0$ for all real numbers x.

a. [2 points] Let $L(x)$ be the local linearization of $h(x)$ at $x = 3$. Find a formula for $L(x)$.

Solution: The graph of $L(x)$ is the tangent line to the graph of $y = h(x)$ at $x = 3$. This is a line of slope 2 passing through the point $(3, 9)$. So $L(x) = 9 + 2(x - 3)$.

Answer: $L(x) =$ $9 + 2(x - 3)$

b. [3 points] Which of the following equalities could be true?
Circle all the statements that could be true or circle NONE OF THESE.
You do not need to explain your reasoning.

Solution: Since $h''(x) > 0$ for all x, the graph of $h(x)$ is concave up so lies above the graph of $L(x)$. Therefore, $h(-1) > L(-1) = 9 + 2(-4) = 1$.

$h(-1) = -1$

$h(-1) = 0$

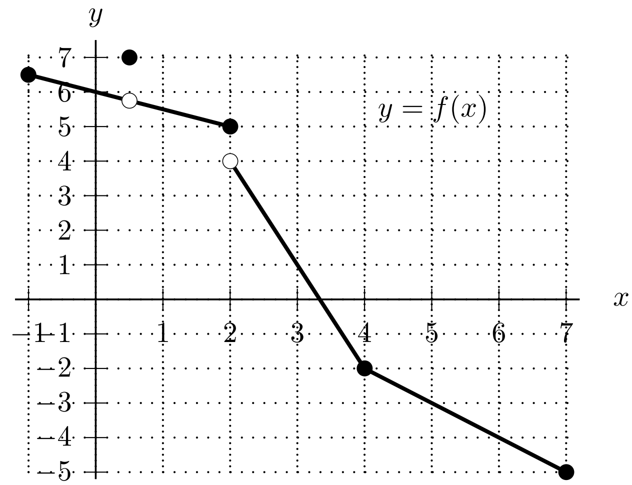
$h(-1) = 1$

$\boxed{h(-1) = 2}$

NONE OF THESE

2. [11 points]

Shown to the right is the graph of a function $f(x)$.



Note that you are not required to show your work on this problem. However, limited partial credit may be awarded based on work shown.

Find each of the following values. If the value does not exist, write DOES NOT EXIST.

a. [3 points] Let $h(x) = f(3x+1)$. Find $h'(1)$.

Solution: Since the graph $y = h(x)$ corresponds to the graph of $y = f(x)$ shifted left 1 unit and then horizontally compressed by a factor of $1/3$, $h(x)$ has a "sharp corner" at $x = 1$ so is not differentiable there.

Answer: $h'(1) =$ DOES NOT EXIST

b. [3 points] Let $k(x) = e^{f'(x)}$. Find $k'(6)$.

Solution: By the chain rule, $k'(x) = e^{f'(x)} f''(x)$. So $k'(6) = e^{f'(6)} f''(6) = e^{-1}(0) = 0$.

Answer: $k'(6) =$ 0

c. [2 points] Find $(f^{-1})'(0)$.

Solution: By the formula for the derivative of an inverse,

$$(f^{-1})'(0) = \frac{1}{f'(f^{-1}(0))} = \frac{1}{f'(10/3)} = \frac{1}{-3}.$$

Answer: $(f^{-1})'(0) =$ $-1/3$

d. [3 points] Let $j(x) = \dfrac{f(2x+1)}{x+1}$. Find $j'(1)$.

Solution: Applying the quotient and chain rules, we find that

$$j'(x) = \frac{2f'(2x+1)(x+1) - f(2x+1)(1)}{(x+1)^2}.$$

Thus,

$$j'(1) = \frac{2f'(3)(2) - f(3)}{2^2} = \frac{(2)(-3)(2) - (1)}{4} = \frac{-13}{4}.$$

Answer: $j'(1) =$ $-13/4$

3. [13 points] Let f be a function such that $f''(x)$ is defined for all real numbers. A table of some values of f' is given below.

x	2	3	4	6	9	11
$f'(x)$	4	1	0	2	0	-4

Assume that f' is continuous and either always decreasing or always increasing between consecutive values of x shown in the table.

a. [2 points] Using the table above, estimate $f''(11)$. *Show your work.*

Solution: Since f'' is the derivative of f', $f''(11) \approx \dfrac{f'(11) - f'(9)}{11 - 9} = \dfrac{-4 - 0}{11 - 9} = -2$.

Answer: $f''(11) \approx$ -2

For parts (b) through (e) below, find the indicated values.
Write NONE if there are no such values of x.
Write NOT ENOUGH INFO if there is not sufficient information to determine a value.
You do not need to explain your reasoning.

b. [3 points] Find the x-coordinates of all critical points of $f(x)$ on the interval $2 < x < 11$.

Answer: critical point(s) at $x =$ 4, 9

c. [3 points] Find the x-coordinates of all local minima of $f(x)$ on the interval $2 < x < 11$.

Answer: local min(s) at $x =$ NONE

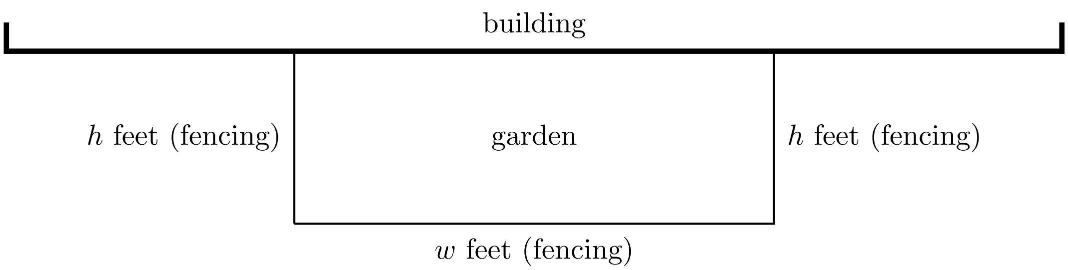
d. [3 points] Find the x-coordinates of all inflection points of $f(x)$ on the interval $2 < x < 11$.

Answer: inflection point(s) at $x =$ 4, 6

e. [2 points] Find all values of x at which $f(x)$ attains its global maximum on the interval $2 \leq x \leq 11$.

Answer: global max(es) at $x =$ 9

4. [12 points] Researchers are constructing a rectangular garden adjacent to their building. The garden will be bounded by the building on one side and by a fence on the other three sides. (See diagram below.) The fencing will cost them \$5 per linear foot. In addition, they will also need topsoil to cover the entire area of the garden. The topsoil will cost \$4 per square foot of the garden's area.
Assume the building is wider than any garden the researchers could afford to build.



a. [5 points] Suppose the garden is w feet wide and extends h feet from the building, as shown in the diagram above. Assume it costs the researchers a total of \$250 for the fencing and topsoil to construct this garden. Find a formula for w in terms of h.

Solution: A garden of these dimensions will require $2h+w$ feet of fencing and hw square feet of ground covered by topsoil. Thus,

$$250 = 5(2h + w) + 4hw\,.$$

Solving for w we find

$$w = \frac{250 - 10h}{4h + 5}\,.$$

Answer: $w = \frac{250 - 10h}{4h + 5}$

b. [3 points] Let $A(h)$ be the total area (in square feet) of the garden if it costs \$250 and extends h feet from the building, as shown above. Find a formula for the function $A(h)$. The variable w should not appear in your answer.

(Note that $A(h)$ is the function one would use to find the value of h maximizing the area. You should not do the optimization in this case.)

Solution: The area of the garden in square feet is given by hw. In part (a), a formula for w in terms of h was found when h and w are the dimensions of a garden that will cost \$250 in supplies to construct. Thus, $A(h) = h\left(\frac{250 - 10h}{4h + 5}\right)$.

Answer: $A(h) = h\left(\frac{250 - 10h}{4h + 5}\right)$

c. [4 points] In the context of this problem, what is the domain of $A(h)$?

Answer: $0 < h < 25$

5. [12 points] Let $f(x)$ be a differentiable function defined for all real x with derivative

$$f'(x) = \left(e^{x-1}\right) x^4 (x+4)(x-3)^2.$$

a. [3 points] Find the x-coordinates of all critical points of $f(x)$.

Solution: Critical points of $f(x)$ occur where $f'(x)$ is zero or undefined. Since we are given a formula for $f'(x)$ that is defined everywhere, $f'(x)$ is defined for all real numbers x. So the only critical points of $f(x)$ occur where $f'(x) = 0$, i.e. at $x = 0, -4, 3$.

Answer: critical point(s) at $x =$ $0,\ -4,\ 3$

b. [6 points] Find the x-coordinates of all local extrema of $f(x)$. If there are none of a particular type, write NONE.
Justify your answers, and be sure to show enough evidence to demonstrate that you have found all local extrema.

Solution: All local extrema occur at critical points. Since f is continuous, we use the First Derivative test to classify the critical points from part (a) as local minima, local maxima, or neither.
We determine the sign of $f'(x)$ for $x < -4$, $-4 < x < 0$, $0 < x < 3$, and $x > 3$. On each of these intervals, e^{x-1}, x^4 and $(x-3)^2$ are positive. Thus, the sign of $f'(x)$ is determined by $(x+4)$, which is positive for $x > -4$ and negative for $x < -4$. Thus,

Interval	$x < -4$	$-4 < x < 0$	$0 < x < 3$	$x > 3$
Sign of $f'(x)$	-	+	+	+

By the First Derivative Test, $f(x)$ has a local minimum at $x = -4$. No other critical points are local extrema.

Answer: local min(s) at $x =$ -4

Answer: local max(es) at $x =$ NONE

c. [3 points] Suppose $f(1) = -7$. Use the tangent line approximation to $f(x)$ at $x = 1$ to estimate $f(1.1)$.

Solution: To find the tangent line approximation of $f(x)$ at $x = 1$, we first calculate

$$f'(1) = e^{(1-1)}(1^4)(1+4)(1-3)^2 = 20\,.$$

Thus, $f(1.1) \approx f(1) + f'(1)(1.1-1) = -7 + (20)(0.1) = -5\,.$

Answer: $f(1.1) \approx$ -5

6. [11 points] Consider the curve $\mathcal{C}$ defined by

$$e^{xy} = 4x - y^2 + 2\,.$$

a. [6 points] For this curve $\mathcal{C}$, find a formula for $\frac{dy}{dx}$ in terms of x and y.

Solution: Applying $\frac{d}{dx}$ to both sides of the equation for the curve, we have

$$e^{xy}\left(x\frac{dy}{dx} + y(1)\right) = 4 - 2y\frac{dy}{dx}\,.$$

Collecting all terms involving $\frac{dy}{dx}$ to the left hand side and factoring out $\frac{dy}{dx}$ gives

$$\frac{dy}{dx}\left(xe^{xy} + 2y\right) = 4 - ye^{xy}\,.$$

Thus, $\dfrac{dy}{dx} = \dfrac{4 - ye^{xy}}{xe^{xy} + 2y}$.

Answer: $\dfrac{dy}{dx} = \dfrac{4 - ye^{xy}}{xe^{xy} + 2y}$

b. [2 points] Exactly one of the points below lies on the curve $\mathcal{C}$. Circle that one point.

$(2, 0)$ $\quad$ $(1, -2)$ $\quad$ $(1, 1)$ $\quad$ $\boxed{(0, -1)}$

c. [3 points] Find an equation for the tangent line to the curve $\mathcal{C}$ at the point you chose in part (b).

Solution: The slope of the tangent line to $\mathcal{C}$ at (0,-1) is given by plugging $x = 0$ and $y = -1$ into the formula we found for $\frac{dy}{dx}$, which gives

$$\frac{4 - (-1)e^{(0)(-1)}}{0e^{(0)(-1)} + 2(-1)} = -\frac{5}{2}\,.$$

Thus, the tangent line is given by the equation

$$y = -1 - \frac{5}{2}(x - 0)\,.$$

Answer: $y = -1 - \dfrac{5}{2}x$

7. [11 points] Let g be a differentiable function defined for all real numbers satisfying all of the following properties:

- $g(5) = 4$.
- $g(x)$ has a local maximum at $x = -2$ and $g(-2) = 3$.
- $g(x)$ has a local minimum at $x = 1$ and $g(1) = -1$.
- g has exactly two critical points.
- $\lim_{x\to\infty} g(x) = +\infty$.
- $\lim_{x\to-\infty} g(x) = 0$.

a. [3 points] Circle all of the following intervals on which $g'(x)$ must be always positive.

[$x < -2$] $-2 < x < -1$ $-1 < x < 1$ [$1 < x < 3$] [$3 < x < 5$] [$5 < x$]

b. [4 points] Find all the values of x at which $g(x)$ attains global extrema on $-2 \leq x \leq 5$. If not enough information is provided, write NOT ENOUGH INFO. If there are no such values of x, write NONE. Briefly indicate your reasoning.

Solution: Since $g(x)$ has local extrema at $x = -2$ and $x = 1$, $g(x)$ has critical points at these values. As stated in the problem, g has exactly 2 critical points. Thus, these are the only critical points.
Since $g(x)$ is continuous and $-2 \leq x \leq 5$ is a closed interval, $g(x)$ attains global extrema on this interval, and they occur at critical points or end points. By comparing the values of $g(x)$ at the critical points and endpoints ($g(-2) = 3$, $g(1) = -1$, and $g(5) = 4$), we conclude the following:

Answer: global min(s) at $x =$ 1

Answer: global max(es) at $x =$ 5

c. [4 points] Find all the values of x at which $g(x)$ attains global extrema on its domain. If not enough information is provided, write NOT ENOUGH INFO. If there are no such values of x, write NONE. Briefly indicate your reasoning.

Solution: Since $\lim_{x\to\infty} g(x) = +\infty$, g has no global maximum on its domain.

We claim that the function $g(x)$ has a global minimum value of -1 at $x = 1$. To see this, first note that $g(x)$ is increasing for $x > 1$. So $g(x) > g(1) = -1$ for $x > 1$. Now, $g(x)$ is decreasing for $-2 < x < 1$, so $g(x) > g(1) = -1$ for $-2 < x < 1$. Finally, $g(x)$ is increasing for $x < -2$ and $\lim_{x\to\infty} = 0$, so $g(x) > -1$ for $x < -2$. (In fact, $g(x) \geq 0$ for $x < -2$.)
Combining this information, we conclude that $g(1) = -1$ is the minimum value achieved by $g(x)$ on its domain.

Answer: global min(s) at $x =$ 1

Answer: global max(es) at $x =$ NONE

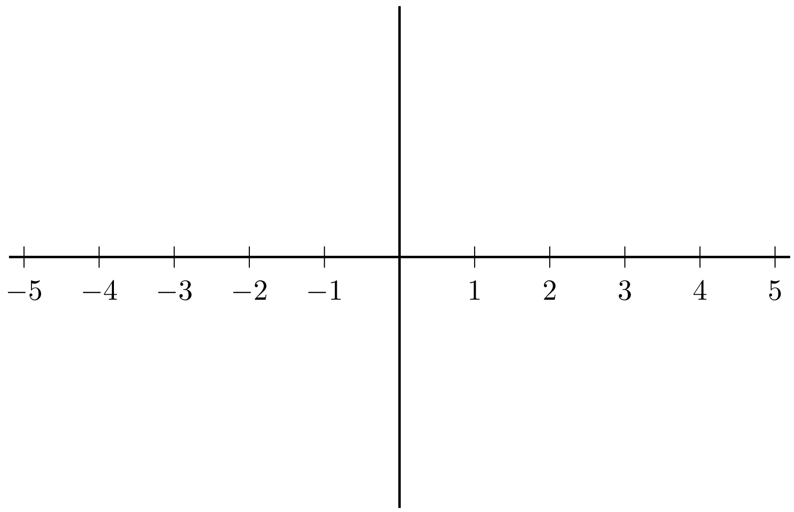
8. [7 points] For each of parts (a) and (b) below, draw a graph of a single function with all of the listed properties. If there is no function satisfying all the properties, circle NO SUCH FUNCTION EXISTS.

Note: If "NO SUCH FUNCTION EXISTS." is circled, the graph will not be graded.

a. [3 points] A function $j(x)$ defined on the interval $-5 < x < 5$ with the following two properties:

- $j''(x) > 0$ everywhere.
- $j(x)$ has a local max at $x = 0$.

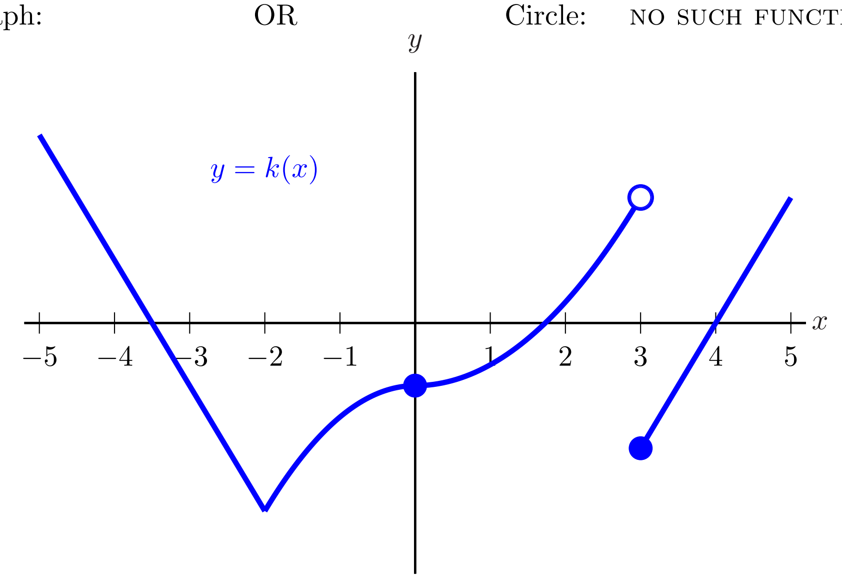
Draw a graph: OR Circle: NO SUCH FUNCTION EXISTS.



b. [4 points] A function $k(x)$ defined on the interval $-5 < x < 5$ with the following three properties:

- $k(x)$ is continuous everywhere except at $x = 3$.
- $k(x)$ is differentiable everywhere except at $x = -2$ and $x = 3$.
- $k(x)$ has an inflection point at $x = 0$.

Draw a graph: OR Circle: NO SUCH FUNCTION EXISTS.



9. [10 points] Our friend Oren, the Math 115 student, wants to minimize how long it will take him to complete his upcoming web homework assignment. Before starting the assignment, he buys a cup of tea containing 55 milligrams of caffeine.
Let $H(x)$ be the number of minutes it will take Oren to complete tonight's assignment if he consumes x milligrams of caffeine. For $10 \le x \le 55$

$$H(x) = \frac{1}{120}x^2 - \frac{4}{3}x + 20\ln(x)\,.$$

Instead of immediately starting the assignment, he solves a calculus problem to determine how much caffeine he should consume.

a. [8 points] Find all the values of x at which $H(x)$ attains global extrema on the interval $10 \le x \le 55$. Use calculus to find your answers, and be sure to show enough evidence that the points you find are indeed global extrema.

Solution: Since $H(x)$ is continuous on the interval $10 \le x \le 55$, by the Extreme Value Theorem, $H(x)$ attains both a global minimum and a global maximum on this interval. These will occur at either endpoints or critical points.
Now,

$$H'(x) = \frac{x}{60} - \frac{4}{3} + \frac{20}{x} = \frac{x^2 - 80x + 1200}{60x} = \frac{(x-60)(x-20)}{60x}\,.$$

Thus, $H(x)$ has exactly one critical point on the interval $10 \le x \le 55$, and it is at $x = 20$. To determine the global extrema, we compare the values of $H(x)$ at all critical points and endpoints

x	10	20	55
$H(x)$	≈ 33.55	≈ 36.58	≈ 32.02

Thus, the global minimum is at $x = 55$, and the global maximum is at $x = 20$.

(For each answer blank below, write NONE in the answer blank if appropriate.)

Answer: global min(s) at $x =$ 55

Answer: global max(es) at $x =$ 20

b. [2 points] Assuming Oren consumes at least 10 milligrams and at most 55 milligrams of caffeine, what is the shortest amount of time it could take for him to finish his assignment? *Remember to include units.*

Solution: The minimum of $H(x)$ occurs at $x = 55$, where $H(55) \approx 32.02$.

Answer: ≈ 32 **minutes**

10. [8 points] You are not required to show your work on this page.

a. [2 points] Circle the <u>one</u> option that correctly fills in the blank.

The local linearization of $B(x) = e^{x^2}$ at $x = 5$ is given by $L(x) =$ _____________.

$e^{25} + (2xe^{x^2})(x-5)$	$e^{x^2} + (2xe^{x^2})(x-5)$	$2e^{25}x - 5$
$B'(a)(x-a) + B(x)$	$\boxed{e^{25}(10x-49)}$	$e^{x^2} + (10e^{25})(x-5)$

b. [3 points] Suppose $g(x)$ is a function such that $g''(x)$ exists for all real numbers x. Suppose further that $g'(x)$ (the derivative of $g(x)$) has a critical point at $x = 2$.
Circle all the statements below that <u>must</u> be true or circle NONE OF THESE.

$g(x)$ has a local extremum at $x = 2$.

$g(x)$ has an inflection point at $x = 2$.

$g'(2) = 0$.

$\boxed{g''(2) = 0.}$

NONE OF THESE

c. [3 points] Let $f(x)$ be a differentiable function such that for all real numbers x, $f(x) < 0$ and $f'(x) < 0$. Let $j(x) = f(f(x))$.
Circle all the statements below that <u>must</u> be true or circle NONE OF THESE.

$j(x) > 0$ for all x.	$j'(x) < 0$ for all x.
$\boxed{j(x) < 0 \text{ for all } x.}$	$\boxed{j(x) \text{ has no local extrema.}}$
$\boxed{j'(x) > 0 \text{ for all } x.}$	NONE OF THESE