Math 115 — Second Midterm

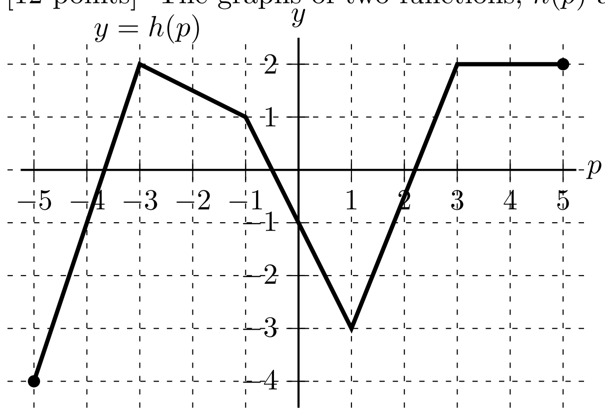
November 17, 2015

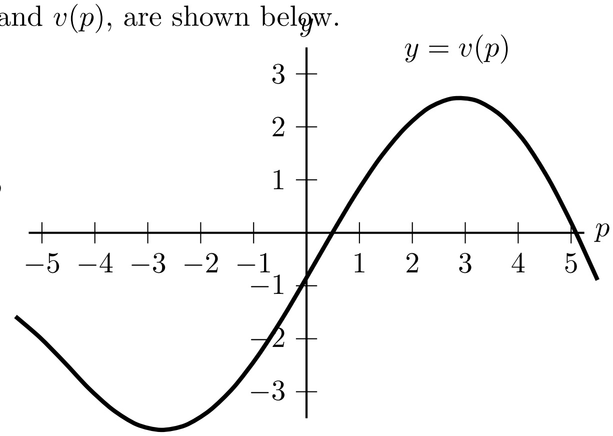
EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. This exam has 10 pages including this cover. There are 9 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your initials (not name) on every page and point this out to your instructor when you hand in the exam.
4. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
5. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
6. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
7. The use of any networked device while working on this exam is not permitted.
8. You may use any calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
9. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
10. Include units in your answer where that is appropriate.
11. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones and smartwatches.
12. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	12	
2	14	
3	11	
4	10	
5	12	
6	11	
7	6	
8	10	
9	14	
Total	100	

1. [12 points] The graphs of two functions, $h(p)$ and $v(p)$, are shown below.





The following questions concern the functions B, W, and Q defined as follows:

$$B(p) = \frac{h(2p)}{h(4p)}, \qquad W(p) = h(h(p)), \quad \text{and} \quad Q(p) = e^{-v(p)}.$$

Assume that the first and second derivatives of $v(p)$ are defined everywhere, i.e. that both v and v' are differentiable on $(-\infty, \infty)$. Note that the graph of $h(p)$ consists of line segments whose endpoints have integer (whole number) coordinates. Find the exact value of each of the quantities in **a.** and **b.** below. If the value does not exist, write DOES NOT EXIST. *Remember to show your work carefully.*

a. [4 points] $B'(-1)$

Solution: Applying the quotient and chain rules, we have

$$B'(p) = \frac{2h'(2p)h(4p) - 4h'(4p)h(2p)}{(h(4p))^2}.$$

So $\quad B'(-1) = \dfrac{2h'(-2)h(-4) - 4h'(-4)h(-2)}{h(-4)^2} = \dfrac{2(-\frac{1}{2})(-1) - 4(3)(\frac{3}{2})}{(-1)^2} = -17.$

Answer: $B'(-1) =$ -17

b. [4 points] $W'(2)$

Solution: By the chain rule, $W'(p) = h'(h(p))h'(p)$, so

$$W'(2) = h'(h(2))h'(2) = h'(-\frac{1}{2})h'(2) = (-2)(\frac{5}{2}) = -5.$$

Answer: $W'(2) =$ -5

c. [4 points] On the interval $-2 < p < 2$, is $Q(p)$ always increasing, always decreasing, or neither? Show your work and explain your reasoning.

Solution: By the chain rule, $Q'(p) = -v'(p)e^{-v(p)}$. Since $e^x > 0$ for all x, we know that $e^{-v(p)}$ is always positive. On the interval $-2 < p < 2$, we can see that $v(p)$ is increasing and never has a horizontal tangent line, which means that $v'(p) > 0$ on this interval. Thus $Q'(p) = -v'(p)e^{-v(p)}$ is always negative on that interval, which means that $Q(p)$ is always decreasing on this interval.

2. [14 points] Suppose $f(x)$ is a function defined for all real numbers whose derivative and second derivative are given by

$$f'(x) = (x-4)^3(x+2)^{2/3} \quad \text{and} \quad f''(x) = \frac{(11x+10)(x-4)^2}{3(x+2)^{1/3}}.$$

a. [7 points] Find all critical points of $f(x)$ and all values of x at which $f(x)$ has a local extremum. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all local extrema. For each answer blank below, write NONE if appropriate.

Solution: First we find the critical points, which occur when $f'(x) = 0$ or $f'(x)$ does not exist. From the formula, we can see that $f'(x) = 0$ when $x = 4$ or $x = -2$, and that $f'(x)$ always exists.
Next we must determine whether there is a local min, local max, or neither at each critical point. The second derivative test is inconclusive, because $f''(-2)$ does not exist and $f''(4) = 0$. So we must use the First Derivative Test. The factor $(x+2)^{2/3}$ is always positive, while $(x-4)$ is negative for $x < 4$ and positive for $x > 4$. This gives us the resulting signs:

Interval	$x < -2$	$-2 < x < 4$	$4 < x$
Sign of $f'(x)$	$- \cdot + = -$	$- \cdot + = -$	$+ \cdot + = +$

So $f(x)$ has a local minimum at $x = 4$ and no local maxima.

Answer: Critical point(s) at $x =$ $-2, 4$

Local max(es) at $x =$ NONE Local min(s) at $x =$ 4

b. [7 points] Find the x-coordinates of all inflection points of $f(x)$. If there are none, write NONE. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all inflection points.

Solution: First we find the candidate inflection points, which occur when $f''(x) = 0$ or $f''(x)$ does not exist. We can see that $f''(x) = 0$ at $x = 4$ and $-\frac{10}{11}$, and that $f''(x)$ is undefined when $x = -2$. To determine whether these are actually inflection points (where concavity changes), we must test the sign of the second derivative on either side of each of point. We find the following:

Interval	$x < -2$	$-2 < x < -\frac{10}{11}$	$-\frac{10}{11} < x < 4$	$4 < x$
Sign of $f''(x)$	$\frac{- \cdot +}{-} = +$	$\frac{- \cdot +}{+} = -$	$\frac{+ \cdot +}{+} = +$	$\frac{+ \cdot +}{+} = +$

The inflection points of $f(x)$ occur at the points where the sign of the second derivative changes, that is, at $x = -2$ and $x = -\frac{10}{11}$.

Answer: Inflection point(s) at $x =$ $-2, -\frac{10}{11}$

3. [11 points] For each of the problems below, circle <u>all</u> of the correct answers. If none of the answer choices provided are correct, circle NONE OF THESE.

a. [4 points] Let $s(t) = \begin{cases} t^3 + 8t^2 + 6t & \text{if } t \le c \\ 4t^2 + 2t & \text{if } t > c \end{cases}$

For which of the following values of c is $s(t)$ differentiable on $(-\infty, \infty)$?

i. $\boxed{-2}$

ii. $-\frac{2}{3}$

iii. 0

iv. $\frac{3}{2}$

v. 3

vi. NONE OF THESE

Solution: Note that the tangent lines to the graphs of $y = t^3 + 8t^2 + 6t$ and $y = 4t^2 + 2t$ are parallel at $t = c$ if and only if $3c^2 + 16c + 6 = 8c + 2$, i.e. if and only if $c = -2$ or $c = -\frac{2}{3}$. However, to be differentable at $t = c$, $s(t)$ must be continuous at $t = c$. It will be continuous when $c^3 + 8c^2 + 6c = 4c^2 + 2c$, which happens when $c = 0$ or $c = -2$. Therefore the only time the function is differentiable at c (and therefore on $(-\infty, \infty)$) is when $c = -2$.

b. [4 points] Suppose f and f' are differentiable for all real numbers. Let $L(x)$ be the local linearization of f at $x = 3$. Suppose $f'(x) < 0$ for all $2.5 < x < 3.5$ and $f''(x) > 0$ for all $2.5 < x < 3.5$. Which of the following must be true?

i. $L(3) > f(3)$

ii. $\boxed{L(3) = f(3)}$

iii. $L(3) < f(3)$

iv. $L(3.1) > f(3.1)$

v. $L(3.1) = f(3.1)$

vi. $\boxed{L(3.1) < f(3.1)}$

vii. $L(3.9) > f(3.9)$

viii. $L(3.9) = f(3.9)$

ix. $L(3.9) < f(3.9)$

x. NONE OF THESE

Solution:

c. [3 points] Suppose that f is a differentiable function on $(-\infty, \infty)$ with no critical points, that both f and f' are invertible, and that $f(4) = 7$. Which of the following statements must be true?

i. f is an increasing function.

ii. f is a decreasing function.

iii. $f'(4) = \frac{1}{f^{-1}(7)}$.

iv. $\boxed{f'(4) = \frac{1}{(f^{-1})'(7)}}$.

v. $(f')^{-1}(4) = \frac{1}{(f^{-1})'(7)}$.

vi. $(f')^{-1}(7) = \frac{1}{(f^{-1})'(4)}$.

vii. $f'(4)(f^{-1})'(4) = 1$.

viii. $(f'(7))^{-1} = (f^{-1})'(7)$.

ix. NONE OF THESE

4. [10 points] A function $f(x)$ is defined and differentiable on the interval $0 < x < 10$. In addition, $f(x)$ and $f'(x)$ satisfy all of the following properties:

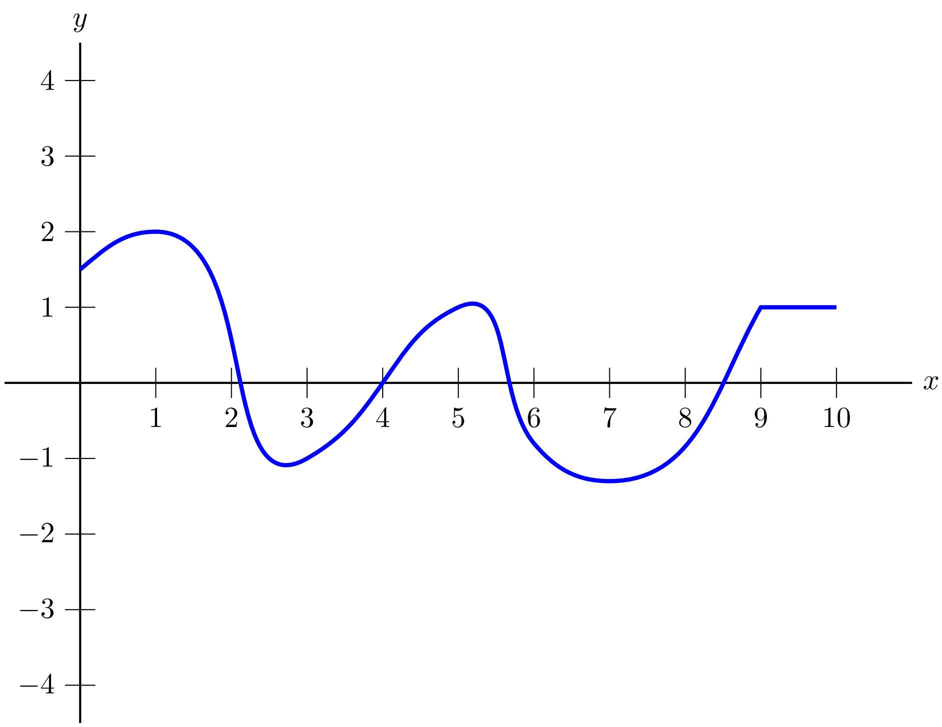
- $f'(x)$ is continuous on the interval $0 < x < 10$.
- $f'(1) = 2$.
- $f'(x)$ is differentiable on the interval $1 < x < 5$.
- $f(x)$ is concave up on the interval $3 < x < 5$.
- $f(x)$ has a local minimum at $x = 4$.
- $f(x)$ is decreasing on the interval $6 < x < 8$.
- $f(x)$ has an inflection point at $x = 7$.
- $f'(x)$ is not differentiable at $x = 9$.

On the axes provided below, sketch a possible graph of $f'(x)$ (the **derivative** of $f(x)$) on the interval $0 < x < 10$.

Make sure your sketch is large and unambiguous.

Solution: One possible solution is shown below.

Graph of $y = f'(x)$



5. [12 points] In Srebmun Foyoj, Maddy and Cal are eating lava cake. Let $T(v)$ be the time (in seconds) it takes Maddy to eat a v cm^3 serving of lava cake. Assume $T(v)$ is invertible and differentiable for $0 < v < 1000$. Several values of $T(v)$ and its first and second derivatives are given in the table below.

v	10	15	60	100	150	200	300
$T(v)$	11	22	84	194	393	513	912
$T'(v)$	2.4	1.9	1.8	3.6	3.7	0.9	17.5
$T''(v)$	-0.11	-0.08	0.05	0.04	-0.04	-0.05	0.59

Remember to show your work carefully.

a. [4 points] Use an appropriate linear approximation to estimate the amount of time it takes Maddy to eat a 64 cm^3 serving of lava cake. *Include units.*

Solution: The closest point in the table to $v = 64$ is $v = 60$, so this is the appropriate choice for the tangent line approximation. Based on the table, the line will go through $(60, 84)$ and have slope 1.8, so it must be $L(v) = 84 + 1.8(v - 60)$. Plugging in 64 for v, we get an estimate of 91.2 seconds.

Answer: 91.2 **seconds**

b. [4 points] Use the quadratic approximation of $T(v)$ at $v = 200$ to estimate $T(205)$. (Recall that a formula for the quadratic approximation $Q(x)$ of a function $f(x)$ at $x = a$ is $Q(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$.)

Solution: Let $Q(v)$ be the quadratic approximation of $T(v)$ at $v = 200$. Then

$$Q(v) = T(200)+T'(200)(v-200)+\frac{T''(200)}{2}(v-200)^2 = 513+0.9(v-200)+\frac{-0.05}{2}(v-200)^2.$$

So the resulting approximation of $T(205)$ is given by

$$T(205) \approx Q(205) = 513+0.9(205-200)-\frac{0.05}{2}(205-200)^2 = 513+4.5-0.625 = 516.875.$$

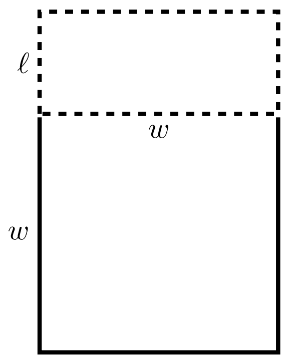
Answer: $T(205) \approx$ 516.875

c. [4 points] Let $C(v)$ be the time (in seconds) it takes Cal to eat a v cm^3 serving of lava cake, and suppose $C(v) = T(\sqrt{v})$. Let $L(v)$ be the local linearization of $C(v)$ at $v = 100$. Find a formula for $L(v)$. Your answer should not include the function names T or C.

Solution: We know $L(v) = C(100)+C'(100)(v-100)$. We also know $C(100) = T(10) = 11$. So we need to find $C'(100)$.
Since $C(v) = T(\sqrt{v})$, we apply the chain rule and see that $C'(v) = \frac{1}{2\sqrt{v}}T'(\sqrt{v})$. Using the table above, we then find that $C'(100) = \frac{1}{20}T'(10) = \frac{2.4}{20} = 0.12$.
So $L(v) = 11 + 0.12(v - 100)$.

Answer: $L(v) =$ $11 + 0.12(v - 100)$

6. [11 points]
The engineer Elur Niahc has been commissioned to build a park for the citizens of Srebmun Foyoj. The park will consist of a square attached to a rectangular dog park (as shown in the diagram on the right).
The fencing for the dog park (bold, dashed line) costs \$4 per linear meter, and the fencing for the three remaining sides of the square portion of the park (bold, solid line) costs \$6 per linear meter.



a. [5 points] Assume that Elur spends \$2400 on fencing. The resulting park will have width w meters, and the length of the dog park will be ℓ meters, as shown in the diagram above. Find a formula for ℓ in terms of w.

Solution: The cost of the fencing for the dog park is $4 \cdot (2\ell + 2w) = 8\ell + 8w$, and the cost of the fencing for the remaining three sides is $6 \cdot (3w) = 18w$. So the total cost of fencing is $8\ell + 8w + 18w = 8\ell + 26w$. Since Elur spends \$2400 on fencing, we have $2400 = 8\ell + 26w$. Solving for ℓ, we find $\ell = \dfrac{2400 - 26w}{8} = 300 - \dfrac{13}{4}w = 300 - 3.25w$.

Answer: $\ell = \dfrac{2400 - 26w}{8} = 300 - 3.25w$

b. [3 points] Let $A(w)$ be the total area (in square meters) of the resulting park (including the dog park) if the width is w meters and Elur spends \$2400 on fencing. Find a formula for the function $A(w)$. The variable ℓ should not appear in your answer.
(Note: This is the function that Elur would use to find the value of w maximizing the area of the park, but you should not do the optimization in this case.)

Solution: The total area of the park in terms of w and ℓ is given by

$$A(w) = w \cdot (w + \ell) = w^2 + w\ell.$$

Using our expression for ℓ in terms of w, we find
$A(w) = w^2 + w\left(\dfrac{2400 - 26w}{8}\right) = w^2 + w(300 - 3.25w) = 300w - 2.25w^2.$

Answer: $A(w) = w^2 + w\left(\dfrac{2400 - 26w}{8}\right) = 300w - 2.25w^2$

c. [3 points] In the context of this problem, what is the domain of $A(w)$?

Solution: Since w is a length, $w \geq 0$. Since ℓ is also a length, $\ell = \dfrac{2400 - 26w}{8}$ must also be at least 0. This means the biggest w can be is when $\dfrac{2400 - 26w}{8} = 0$, or when

$$w = \frac{2400}{26} = \frac{1200}{13} = \frac{300}{3.25} \approx 92.3.$$

Note that if $w = 0$, then the park has no area and if $\ell = 0$ then there is no dog park. In the answer shown below, we have excluded these degenerate cases.

Answer: **The interval** $\left(0, \dfrac{2400}{26}\right)$

7. [6 points] A curve $\mathcal{C}$ gives y as an implicit function of x. This curve passes through the point $(-2, 1)$ and satisfies

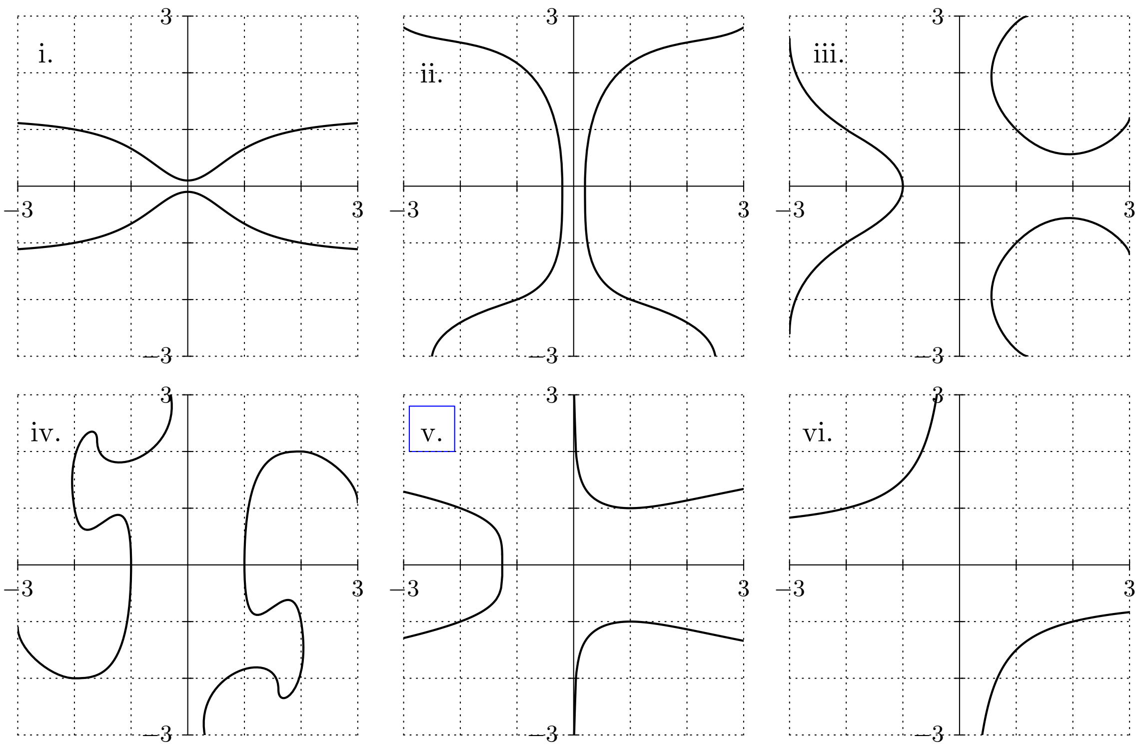
$$\frac{dy}{dx} = \frac{x^2 - y^4}{2xy^3}.$$

a. [1 point] One of the values below is the slope of the curve $\mathcal{C}$ at the point $(-2, 1)$. Circle that one value.

Answer: The slope at $(-2, 1)$ is

$-\frac{3}{16}$ $-\frac{1}{4}$ $-\frac{3}{8}$ $-\frac{1}{2}$ $-\frac{5}{8}$ $\boxed{-\frac{3}{4}}$ $-\frac{15}{16}$

b. [5 points] One of the following graphs is the graph of the curve $\mathcal{C}$. Which of the graphs i-vi is it? To receive any credit on this question, you must circle your answer next to the word "Answer" below.



Remember: To receive any credit on this question, you must circle your answer next to the word "Answer" below.

Answer: i. ii. iii. iv. $\boxed{\text{v.}}$ vi.

Solution: The curve must pass through the point $(-2, 1)$, which rules out (ii). As seen in part (a), the slope at $(-2, 1)$ is negative, which rules out (vi). The tangent lines must be horizontal when the curve crosses the x- or y-axis, which rules out (i). Graph (iv) can be ruled out in a number of ways: the magnitude of the slope is too large at $(-2, 1)$, there should not be vertical tangent lines away from the axes, and there should not be a horizontal tangent line at $(2, 2)$. Finally, there should be a horizontal tangent through $(1, 1)$, ruling out (iii). This leaves graph (v).
Note: The slope at the point $(-2, 1)$ in graph (v) as it appears here is not sufficiently steep. For this reason, full credit was also awarded for choosing graph (iii).

8. [10 points] The citizens of Srebmun Foyoj have decided to put a bed of mumertxe flowers in their new park. The floral density D (in flowers per square meter) of a flowerbed of area A square meters is given by $D = f(A)$. Formulas for $f(A)$ and its derivative $f'(A)$ are given below.

$$f(A) = 30\left(\frac{A^3 - 4.5A^2 + 4.5A - 0.5}{e^A}\right) + 15, \quad \text{and} \quad f'(A) = -30\left(\frac{(A-0.5)(A-2)(A-5)}{e^A}\right).$$

a. [5 points] The citizens intend to make the area of the flowerbed between 1.5 and 3.5 square meters. What area A (with $1.5 \le A \le 3.5$) should they make the flowerbed in order to maximize the density of the flowers in the flowerbed? Use calculus to find and justify your answer, and be sure to show enough evidence to demonstrate that the area you find does indeed maximize the density of the flowers.

Solution: Since $f'(A)$ is always defined (as e^A is never zero), the critical points of f are at $A = 0.5, 2,$ and 5. Of these, only $A = 2$ is in the interval in question. Since f is continuous and our domain is a closed interval, we can apply the Extreme Value Theorem, and need only evaluate $f(A)$ at $A = 1.5, 2,$ and 3.5 and choose the value of A for which $f(A)$ is largest. We find

$$f(1.5) \approx 11.65$$
$$f(2) \approx 8.91$$
$$f(3.5) \approx 17.72.$$

Therefore $f(A)$ attains its maximum value on the interval $[1.5, 3.5]$ when $A = 3.5$. So the density of the flowers in the flowerbed will be maximized when the flowerbed has area 3.5 square meters.

Answer: Maximum density when area $A =$ 3.5

b. [5 points] Suppose instead that the citizens can make the flowerbed any area greater than or equal to 1.5 square meters. What are the largest and smallest densities this flowerbed could have? Use calculus to find your answer and be sure to show enough evidence to demonstrate that you have found the minimum and maximum densities.

Solution: From part (a), we know that $f(1.5) \approx 11.65$ and $f(2) \approx 8.91$. Since $A = 5$ is now in the domain we are considering, we need to also consider $f(5) \approx 21.98$. Together with our data from part (a) above, this implies that on the closed interval $1.5 \le A \le 5$, the minimum value of f is $f(2)$ (≈ 8.91), and the maximum value of f is $f(5)$ (≈ 21.98).

Note that for $A > 5$, the sign of $f'(A)$ is given by $- \cdot \frac{+\cdot+\cdot+}{+}$ so $f'(A) < 0$ on this interval. Thus we know that $f(A)$ is decreasing on the interval $(5, \infty)$. Since $\lim_{A\to\infty} f(A) = 15$, we see that for $A > 5$, we have $15 < f(A) < f(5)$ (so the value of $f(A)$ is always between $f(5)$ (≈ 21.98) and 15).

Therefore, for $A \ge 1.5$, we have a maximum density of about 21.98 flowers per square meter and a minimum density of about 8.91 flowers per square meter.

(For each answer blank below, write NONE if appropriate.)

Answer: Maximum density: $D =$ 21.98

Answer: Minimum density: $D =$ 8.91

9. [14 points]

a. [8 points] Consider functions f satisfying <u>all</u> of the following conditions:

- $f(x)$ is differentiable on the interval $0 < x < 8$.
- The critical points of $f(x)$ in the interval $0 < x < 8$ are $x = 2, 4$, and 6. ($f(x)$ has no other critical points in this interval.)
- The table below shows some values of $f(x)$ and of its derivative $f'(x)$.

x	1	3	5	7
$f(x)$	3	6	11	0
$f'(x)$	-1	?	?	-1

For each of the statements below, decide whether the statement is true for ALL functions f satisfying all of the conditions described above, for SOME of these functions f, or for NONE of these functions f. Circle the <u>one</u> correct choice for each statement.

(i) $f(x)$ has a local minimum at $x = 2$.

[ALL] SOME NONE

(ii) $f'(3) > 0$.

[ALL] SOME NONE

(iii) $f(x)$ has a local maximum at $x = 4$.

ALL [SOME] NONE

(iv) There is exactly one value of a with $3 < a < 7$ such that $f(x)$ has a local maximum at $x = a$.

[ALL] SOME NONE

b. [6 points] Consider functions g satisfying <u>all</u> of the following conditions:

- $g(z)$ and $g'(z)$ are differentiable on the interval $12 < z < 18$.
- The critical points of $g(z)$ in the interval $12 < z < 18$ are $z = 14$ and $z = 16$. ($g(z)$ has no other critical points in this interval.)
- The table below shows some values of $g(z)$ and of its <u>second derivative</u> $g''(z)$.

z	13	14	15	16	17
$g(z)$	8	?	6	?	2
$g''(z)$	?	-1	?	0	?

For each of the statements below, decide whether the statement is true for ALL functions g satisfying all of the conditions described above, for SOME of these functions g, or for NONE of these functions g. Circle the <u>one</u> correct choice for each statement.

(i) $g(z)$ has a local extremum at $z = 14$.

[ALL] SOME NONE

(ii) $g'(15) > 0$.

ALL SOME [NONE]

(iii) $g(z)$ has an inflection point at $z = 16$.

ALL [SOME] NONE