Math 115 — Final Exam

April 25, 2011

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 10 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. Note that problems 6–9 will be graded giving very little partial credit.

Problem	Points	Score
1	10	
2	13	
3	10	
4	12	
5	11	
6	10	
7	10	
8	14	
9	10	
Total	100	

1. [10 points] Find a formula for a function of the form

$$f(x) = \frac{1}{a + x + bx^2}$$

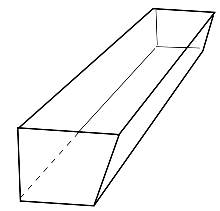
which has a local minimum at $(2, 1/2)$. Be sure to show that your function has a minimum at $(2, 1/2)$.

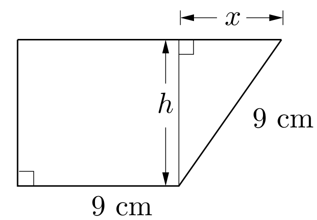
Solution: A local minimum will occur when $f'(x) = 0$ or is undefined. The former is when

$$-\frac{2bx + 1}{(bx^2 + x + a)^2} = 0.$$

The numerator is zero when $x = -\frac{1}{2b}$, and the denominator is zero on either side of this value, when $x = -\frac{1}{2b} \pm \frac{1}{2b}\sqrt{1 - 4ab}$. The first of these gives the local minimum we want: if we let $-\frac{1}{2b} = 2$, we have $b = -\frac{1}{4}$. This gives $f'(x) = -\frac{-\frac{1}{2}x+1}{(-\frac{1}{4}x^2+x+a)^2}$, so that if $x < 2$ we have $f'(x) < 0$, and if $x > 2$, $f'(x) > 0$. Thus if $b = -\frac{1}{4}$, $x = 2$ is a local minimum. To require that the minimum occur at $(2, 1/2)$, we want $f(2) = \frac{1}{a+2-1} = 1/2$, so that $a = 1$.

2. [13 points] A rain gutter attaches to the edge of a roof and collects the rain that falls on the roof. A common gutter design is shown in the figure to the right, and has a trapezoidal cross-section (also shown). In this problem we consider a gutter with base and side length 9 cm, as shown.





a. [1 point] Write an equation which relates the length x to the height h.

Solution: Using the pythagorean theorem, we have $h^2 = 81 - x^2$.

b. [4 points] Using your equation from (a), write an equation for the cross-sectional area of the gutter as a function of the length x (note that the area is the sum of a rectangular and right-triangular region).

Solution: We have $A(x) = 9h + \frac{1}{2}\,x\,h = 9\sqrt{81 - x^2} + \frac{1}{2}\,x\,\sqrt{81 - x^2}$.

c. [8 points] Find the length x that gives the maximum cross-sectional area. Be sure to show work that demonstrates that you have found the maximum.

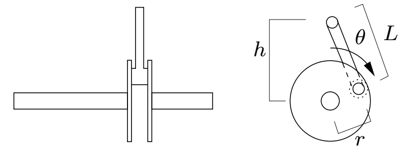
Solution: Note that the domain of interest for our area function is $0 \le x < 9$: taking $x < 0$ clearly gives a smaller cross-sectional area than $x > 0$, and $x = 9$ is clearly the largest value x can take (and the cross-sectional area goes to zero for $x = 9$). To find the maximum we first locate critical points. These are where $A'(x) = 0$ or is undefined. This gives

$$A'(x) = -\frac{9x}{\sqrt{81 - x^2}} + \frac{1}{2}\sqrt{81 - x^2} - \frac{1}{2}\frac{x^2}{\sqrt{81 - x^2}} = 0.$$

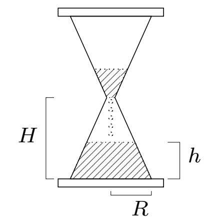
The value $x = 9$ lies outside our domain, so we need not worry about the derivative being undefined there. Multiplying through by $\sqrt{81 - x^2}$, we have $-x^2 - 9x + \frac{81}{2} = 0$. Thus, using the quadratic formula, critical points are where $x = -\frac{9}{2} \pm \frac{1}{2}\sqrt{3(81)}$.
We want a positive value of x, so take $x = x_c = \frac{9}{2}\,(\sqrt{3} - 1) \approx 3.294$. We can see from the derivative (which is a downward opening parabola divided by a positive square root) that this positive critical point must have A' going from positive to negative, and is thus a maximum. As the only critical point on the domain, $x = x_c$ must also be the global maximum.
Alternately, we can check the sign of A' on either side of x_c: $A'(0) = 9/2 > 0$ and $A'(4) \approx -1.43 < 0$. The second derivative test will also work, but the evaluation is more difficult.

3. [10 points] For each of the following determine the indicated quantity.

Crank shaft diagram (part a)



Hourglass diagram (part b)



a. [4 points] In an internal combustion engine, pistons are pushed up and down by a crank shaft similar to the diagram shown to the right. As the shaft rotates the height of the piston, h, is related to the rotational angle θ of the shaft by $h = r\cos\theta + \sqrt{L^2 - r^2\sin^2\theta}$, where r and L are constant lengths. If $r = 10$ cm, $L = 15$ cm, and h is decreasing at a rate of 5000 cm/s when $\theta = 3\pi/4$, how fast is θ changing then?

Solution: Using the chain rule, we know that $h'(t) = \left(-r\sin\theta - \frac{r^2\sin\theta\cos\theta}{\sqrt{L^2-r^2\sin^2\theta}}\right)\cdot\frac{d\theta}{dt}$. Thus if $\theta = 3\pi/4$, $h' = -5000$, $r = 10$ and $L = 15$, we have $-5000 = \left(-\frac{10}{\sqrt{2}} - \frac{50}{\sqrt{225-50}}\right)\cdot\frac{d\theta}{dt} \approx -3.29\cdot\frac{d\theta}{dt}$. Thus $\frac{d\theta}{dt} \approx 1500$ radians/sec

b. [6 points] The lower chamber of an hourglass is shaped like a cone with height H in and base radius R in, as shown in the figure to the right, above. Sand falls into this cone. Write an expression for the volume of the sand in the lower chamber when the height of the sand there is h in *(Hint: A cone with base radius r and height y has volume $V = \frac{1}{3}\pi r^2 y$, and it may be helpful to think of a difference between two conical volumes.)*. Then, if $R = 0.9$ in, $H = 2.7$ in, and sand is falling into the lower chamber at 2 in^3/min, how fast is the height of the sand in the lower chamber changing when $h = 1$ in?

Solution: The whole volume of the lower chamber is $V_{tot} = \frac{1}{3}\pi R^2 H$. The volume of the empty space above the sand is similarly $V_{emp} = \frac{1}{3}\pi r^2 (H-h)$, where r is the radius at the height h. By comparing the similar triangles delimiting the full lower chamber and the empty top section, we see that $r = \frac{R}{H}(H-h)$. Thus $V_{emp} = \frac{1}{3}\pi\frac{R^2}{H^2}(H-h)^3$. The volume of the lower, sand-filled region is therefore

$$V = \frac{1}{3}\pi\frac{R^2}{H^2}\left(H^3 - (H-h)^3\right).$$

Then, differentiating, we have

$$\frac{dV}{dt} = \pi\frac{R^2}{H^2}(H-h)^2\frac{dh}{dt}.$$

Thus, when $\frac{dV}{dt} = 2$, $R = 0.9$, $H = 2.7$, and $h = 1$,

$$2 = \pi\frac{1}{3^2}(1.7)^2\frac{dh}{dt},$$

so that $\frac{dh}{dt} = \frac{18}{\pi(1.7)^2} \approx 1.98$ in/min.

4. [12 points] Suppose $P(\theta)$ is the power, in kilojoules per hour (kJ/h), produced by a solar panel when the angle between the sun and the panel is θ, measured in degrees. Suppose $C(t)$ is the power, in kJ/h, produced by the solar panel t hours after sunrise on a typical summer day. Give practical interpretations of the following.

a. [4 points] $P'(30) = 9$.

Solution: If the angle between the sun and the panel changes from 30 to 31 degrees, the power output of the panel increases by about 9 kJ/h.

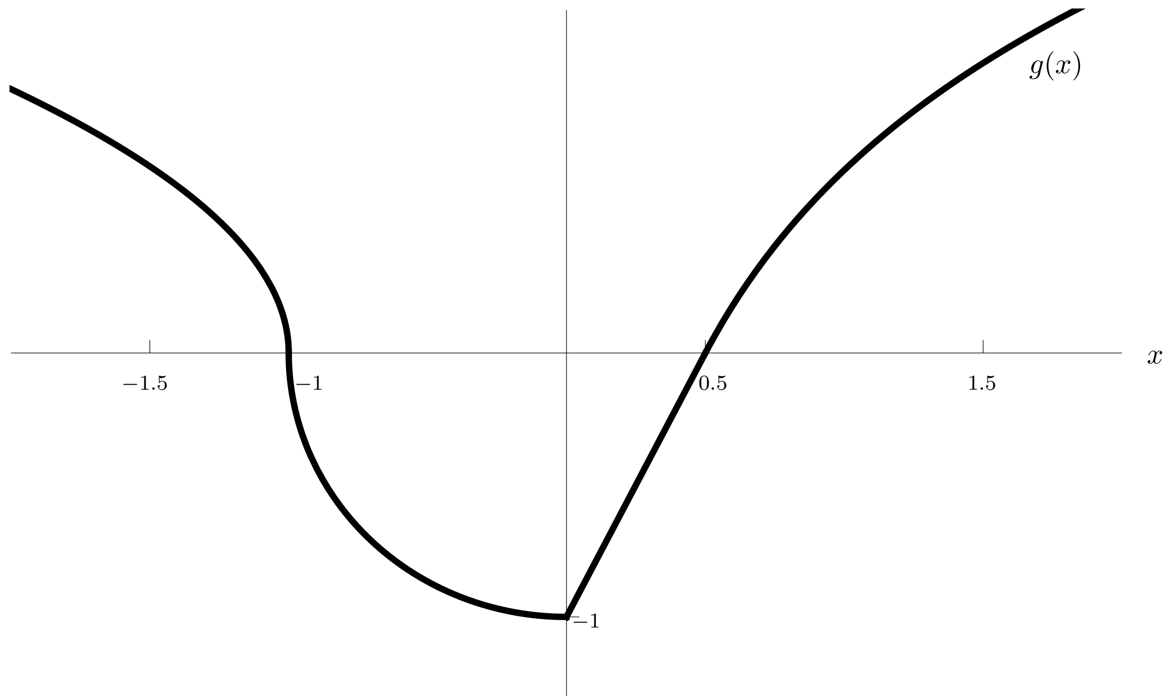
b. [4 points] $\int_0^2 C(t)\,dt = 270$.

Solution: In the two hours after sunrise on a typical summer day, the solar panel produces 270 kJ of energy.

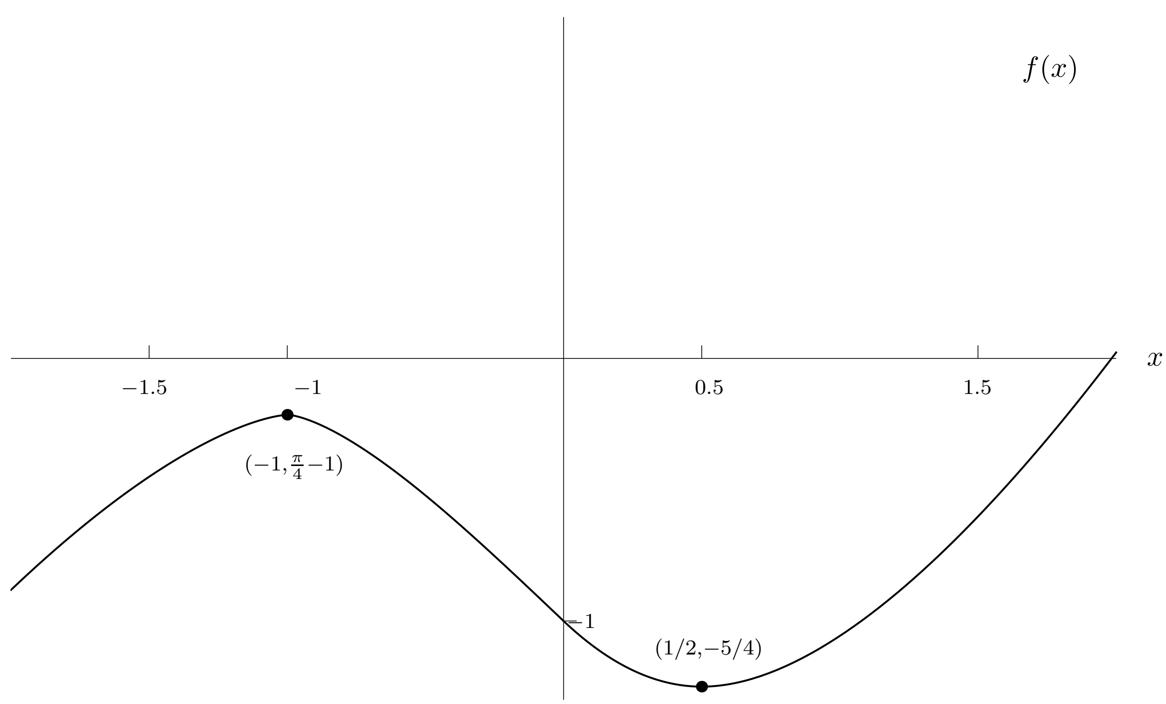
c. [4 points] $\frac{1}{12}\int_0^{12} C(k)\,dk = 288$.

Solution: In the first twelve hours after sunrise on a typical summer day, the average power output of the panel is 288 kJ/h.

5. [11 points] A function $g(x)$ is graphed below. The curve forms a quarter circle between $x = -1$ and $x = 0$ and a line between $x = 0$ and $x = 0.5$



On the axes below, sketch a well-labeled graph of $f(x)$, an antiderivative of $g(x)$, satisfying $f(0) = -1$. Be sure to label the coordinates of $f(x)$ at $x = -1$ and $x = 0.5$.



6. [10 points] The table below gives the expected growth rate, $g(t)$, in ounces per week, of the weight of a baby in its first 54 weeks of life (which is slightly more than a year).[1] Assume for this problem that $g(t)$ is a decreasing function.

week t	0	9	18	27	36	45	54
growth rate $g(t)$	6	6	4.5	3	3	3	2

a. [6 points] Using six subdivisions, find an overestimate and underestimate for the total weight gained by a baby over its first 54 weeks of life.

Solution: The gain rate is a decreasing function, so a left-hand sum will be an overestimate and a right-hand sum an underestimate. The left-hand sum is

$$\int_0^{54} g(t)\,dt \approx (6+6+4.5+3+3+3)(9) = 229.5 \quad \text{oz},$$

and the right-hand sum

$$\int_0^{54} g(t)\,dt \approx (6+4.5+3+3+3+2)(9) = 193.5 \quad \text{oz}.$$

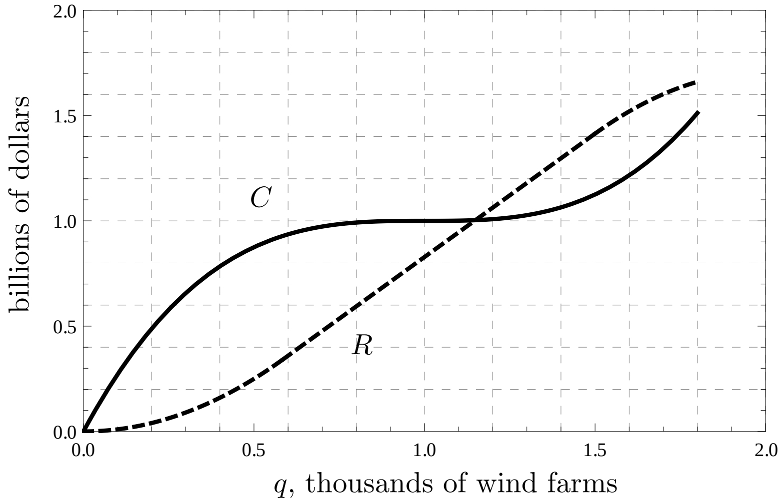
That is, we expect the weight gain to be between 12 and 14 lb!

b. [4 points] How frequently over the 54 week period would you need the data for $g(t)$ to be measured to find overestimates and underestimates for the total weight gain over this time period that differ by 0.5 lb (8 oz)?

Solution: We know that the difference between the over- and underestimates is over − under = $|g(54) - g(0)|\Delta t$. Thus we need $\Delta t \leq 8/(6-2) = 2$ weeks. So we would need data for $g(t)$ every two weeks.

[1] Riordan J. *Breastfeeding and Human Lactation*, 3rd ed. Boston: Jones and Bartlett, 2005, p.103, 512-513.

7. [10 points] Airwatt Construction company builds large-scale farms of wind turbines. A graph of cost C and revenue R for the company at different production levels q is shown below. Here cost and revenue are measured in billions of dollars and production level is measured in thousands of wind farms. *In each of the following parts, be sure it is clear how you obtain your answers.*



a. [3 points] Approximate the marginal revenue at $q = 0.8$. Show how you obtain your estimate.

Solution: The marginal revenue is the slope of the tangent line to the revenue curve. At $q = 0.8$ the slope is approximately $0.25/0.2 = 1.25$. This means the marginal revenue is 1.25 billion dollars per thousand wind farms. Or if we reduce the units, it's $1,250,000 per wind farm.

b. [3 points] Approximate the cost of producing one additional wind farm when $q = 1.6$.

Solution: The marginal cost is the slope of the tangent line to the cost curve. At $q = 1.6$ the slope is approximately 1. This means the marginal cost is 1 billion dollars per thousand wind farms. Reducing units, the cost to produce an additional wind farm when $q = 1.6$ is about $1 million.

c. [4 points] Approximate the maximum profit which can be achieved by Airwatt. At what production level does this occur?

Solution: The maximum profit occurs at a point where the revenue curve is above the cost curve and MR=MC. This means the slopes of the tangent lines to the curves C and R are equal at the production level q in question. Looking on the graph, it appears this occurs somewhere between $q = 1.5$ and $q = 1.7$, say $q = 1.6$. The vertical distance between C and R at this point is about 0.3. So the maximum profit is about $300 million and it occurs when 1600 wind farms have been built.

8. [14 points] A car speeds up at a constant rate from 10 to 70 mph over a period of half an hour, between $t = 0$ and $t = 1/2$. Its fuel efficiency, $E(v)$, measured in miles per gallon, depends on its speed, v, measured in miles per hour.

a. [4 points] Write an integral which gives the total distance traveled by the car during the half hour.

Solution: The total distance traveled by the car during the half hour is

$$\int_0^{1/2} v(t)\, dt.$$

b. [5 points] Write an integral which gives the average fuel efficiency of the car during the half hour.

Solution: The average fuel efficiency of the car during the half hour is

$$\frac{1}{60}\int_{10}^{70} E(v)\, dv.$$

Or, in terms of t, the integral

$$\frac{1}{1/2}\int_0^{1/2} E(v(t))\, dt$$

is equivalent.

c. [5 points] For speeds v greater than 70 mph suppose the relationship between E and v is given by

$$E(v) = 2 + v^{-av}$$

for some constant a. Using this formula, write an expression for the definition of the derivative $E'(82)$. Do not evaluate your expression.

Solution: The derivative of E at $v = 82$ is

$$E'(82) = \lim_{h\to 0} \frac{(2 + (82+h)^{-(82+h)a}) - (2 + 82^{-82a})}{h}.$$

9. [10 points] For each of the statements below, circle TRUE if the statement is always true and circle FALSE otherwise. The letters a, b and c below represent real number constants. Any ambiguous marks will be marked as incorrect. No partial credit will be given on this problem.

a. [2 points] Let $f(x)$ and $g(x)$ be continuous functions which are defined for all real numbers. If $f(x) \geq g(x)$ for all real numbers x, then $\int_a^b f(x)\ dx \geq \int_a^b g(x)\ dx$ whenever $a < b$.

[True] False

b. [2 points] If a is a positive, then the function $h(x) = \frac{\ln(ax^2)+x}{x}$ is an antiderivative of $j(x) = \frac{2-\ln(ax^2)}{x^2}$.

[True] False

c. [2 points] Suppose a differentiable function $\ell(x)$ is concave down and defined for all real numbers. If $a < b$, then

$$\frac{\ell(b) - \ell(a)}{b - a} < \ell'(b).$$

True [False]

d. [2 points] If $x = a$ is a critical point of a function $m(x)$, then $m'(a) = 0$.

True [False]

e. [2 points] If $n(x)$ and $p(x)$ are continuous functions which are defined for all real numbers, then

$$\int_a^b (c\,n(x) - p(x))\,dx = c\int_a^b n(x)\,dx + \int_b^a p(x)\,dx$$

[True] False