Math 115 — Second Midterm

March 20, 2012

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 10 pages including this cover. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	10	
2	10	
3	12	
4	12	
5	12	
6	16	
7	12	
8	16	
Total	100	

1. [10 points] For the following questions, circle "True" if the statement is **always** true, and otherwise circle "False". No justification is necessary.

a. [2 points] If $f(x)$ is a function with a local maximum at $x = c$, then $f'(c) = 0$.

True False

Solution: False

b. [2 points] If $g'(55) = g'(65) = 0$, then $g(x)$ is constant on the interval $55 \le x \le 65$.

True False

Solution: False

c. [2 points] The point $(\pi, 1)$ is on the curve defined by the implicit function $5\sin(xy) = \ln(y)$.

True False

Solution: True

d. [2 points] The function $A(x) = \dfrac{1}{R^2}\cos(Rx) + \dfrac{1}{2}x^2$ has an inflection point at $x = 0$, where R is a nonzero constant.

True False

Solution: False

e. [2 points] If $h'(x) < 0$ for all x in the interval $[2, 8]$, then the global maximum of $h(x)$ on that interval occurs at $x = 2$.

True False

Solution: True

2. [10 points] Lucy Lemon, the owner of a local lemonade stand, has observed that her lemonade sales are highly dependent on the temperature. Let $L(T)$ denote the number of cups of lemonade Lucy sells on a day whose average temperature is $T°$ Fahrenheit. Below is a table of values for the **derivative**, $L'(T)$. Assume that between each pair of consecutive T-values in the table, L' is either strictly increasing or strictly decreasing.

T	50	55	60	65	70	75	80	85	90
$L'(T)$	-5	-4	-2	2	5	6	3	-2	-3

a. [4 points] Approximate the values of T at which $L(T)$ has a critical point, and classify each as a local maximum, local minimum, or neither. (No explanation is necessary.)

Solution: The sign of the derivative goes from negative to positive at approximately $T = 62.5$, so by the first-derivative test, there is a local minimum at approximately that point. (Any T-value between 60 and 65 is an acceptable answer.) The sign of the derivative goes from positive to negative at approximately $T = 82.5$, so there is a local maximum at approximately that point. (Any T-value between 80 and 85 is an acceptable answer.)

b. [6 points] Suppose on a day when the temperature is $T°$ Fahrenheit, Lucy sells $R(T)$ dollars worth of lemonade. Assuming cups of lemonade sell for \$1.50 each, compute $R'(80)$, and write a sentence expressing the meaning of $R'(80)$ which would be understood by someone who knows no calculus.

Solution: The function $R(T)$ is given by the formula $R(T) = 1.5L(T)$, so

$$R'(80) = 1.5L'(80) = 4.5.$$

In practical terms, this means that Lucy earns about \$4.50 more on a day whose average temperature is 81°F than she does on a day whose average temperature is 80°F.

3. [12 points] The following questions relate to the implicit function

$$y^2 + 4x = 4xy^2.$$

a. [4 points] Compute $\dfrac{dy}{dx}$.

Solution: Differentiating the equation with respect to x, we have

$$2y\frac{dy}{dx} + 4 = 4y^2 + 8xy\frac{dy}{dx}.$$

Gathering terms involving $\frac{dy}{dx}$ to one side, the equation becomes

$$2y\frac{dy}{dx} - 8xy\frac{dy}{dx} = 4y^2 - 4$$

which gives the solution

$$\frac{dy}{dx} = \frac{4y^2 - 4}{2y - 8xy}.$$

b. [4 points] Find the equation for the tangent line to this curve at the point $(\frac{1}{3}, 2)$.

Solution: The slope is

$$\left.\frac{dy}{dx}\right|_{(\frac{1}{3},2)} = \frac{4 \cdot 2^2 - 4}{2 \cdot 2 - 8 \cdot \frac{1}{3} \cdot 2} = -9,$$

so by the point-slope formula, the equation is

$$y = -9x + 5.$$

c. [4 points] Find the x- and y-coordinates of all points at which the tangent line to this curve is vertical.

Solution: The slope is undefined as these points, so we must have $2y - 8xy = 0$. Factoring out a $2y$ we get

$$2y(1 - 4x) = 0$$

which gives the solutions $y = 0$ or $x = \frac{1}{4}$. Plugging into the equation for the implicit function, $y = 0$ gives the point $(0, 0)$. However, when we plug in $x = \frac{1}{4}$, we get the equation $y^2 + 1 = y^2$, which has no solutions. Therefore, $(0, 0)$ is the only point at which the tangent line is vertical.

4. [12 points] Consider the family of functions

$$f(x) = ax - e^{bx},$$

where a and b are positive constants.

a. [4 points] Any function $f(x)$ in this family has only one critical point. In terms of a and b, what are the x- and y-coordinates of that critical point?

Solution: We find

$$f'(x) = a - be^{bx},$$

so setting this equal to zero and solving for x shows that there is a critical point at

$$x = \frac{1}{b}\ln\left(\frac{a}{b}\right).$$

The y-coordinate of the critical point is

$$y = f\left(\frac{1}{b}\ln\left(\frac{a}{b}\right)\right) = \frac{a}{b}\ln\left(\frac{a}{b}\right) - \frac{a}{b}.$$

b. [4 points] Is the critical point a local maximum or a local minimum? Justify your answer with either the first-derivative test or the second-derivative test.

Solution: The second derivative of $f(x)$ is $f''(x) = -b^2e^{bx}$, so

$$f''\left(\frac{1}{b}\ln\frac{a}{b}\right) = -b^2 \cdot \frac{a}{b} = -ab,$$

which is negative since a and b are both positive. Therefore, the second derivative test tells us that the critical point is a local maximum.

c. [4 points] For which values of a and b will $f(x)$ have a critical point at $(1, 0)$?

Solution: We need

$$\frac{1}{b}\ln\frac{a}{b} = 1 \quad \textbf{and} \quad \frac{a}{b}\ln\frac{a}{b} - \frac{a}{b} = 0.$$

The first equation rearranges to $\ln(\frac{a}{b}) = b$, and if we plug this into the second equation, we obtain

$$a - \frac{a}{b} = 0 \Rightarrow 1 - \frac{1}{b} = 0 \Rightarrow b = 1.$$

Plugging this into either equation and solving for a gives

$$a = e.$$

5. [12 points]

a. [3 points] Find the local linearization $L(x)$ of the function

$$f(x) = (1+x)^k$$

near $x = 0$, where k is a positive constant.

Solution: The derivative is $f'(x) = k(1+x)^{k-1}$, so the slope of the tangent line at $x = 0$ is

$$f'(0) = k.$$

Since $f(0) = 1^k = 1$, the tangent line passes through the point $(0, 1)$. Therefore, the point-slope formula shows that the equation of the tangent line is

$$y = kx + 1.$$

b. [3 points] For which values of k does this local linearization give underestimates of the actual value of $f(x)$? (Show your work.)

Solution: The local linearization gives underestimates of the actual value when $f''(0) > 0$. The second derivative is $f''(x) = k(k-1)(1+x)^{k-2}$, so

$$f''(0) = k(k-1).$$

Since $k > 0$, this is positive when the second factor is positive, which is when $k > 1$.

c. [2 points] Suppose you want to use $L(x)$ to find an approximation of the number $\sqrt{1.1}$. What number should k be, and what number should x be?

Solution: If $k = \frac{1}{2}$ and $x = 0.1$, then $f(0.1) = \sqrt{1.1}$, so $L(1.1)$ gives an approximation of $\sqrt{1.1}$.

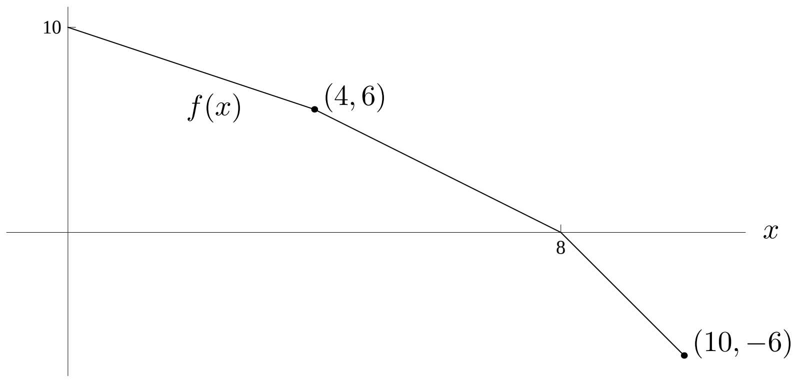
d. [2 points] Approximate $\sqrt{1.1}$ using $L(x)$.

Solution: If k and x are as above, then $\sqrt{1.1} \approx L(0.1) = 1.05$.

e. [2 points] What is the error in the approximation from part (d)?

Solution: The error is the actual value minus the approximate value which is $\sqrt{1.1} - 1.05 \approx -0.00119$.

6. [16 points] Consider the piecewise linear function $f(x)$ graphed below:



For each function $g(x)$, find the value of $g'(3)$:

a. [4 points] $g(x) = \sin\left([f(x)]^3\right)$

Solution:

$$g'(x) = \cos(f(x)^3) \cdot 3f(x)^2 \cdot f'(x)$$
$$g'(3) = \cos(7^3) \cdot 3 \cdot 7^2 \cdot (-1) = 124.0442.$$

b. [4 points] $g(x) = \dfrac{f(x^2)}{x}$

Solution:

$$g'(x) = \frac{x \cdot f'(x^2) \cdot 2x - f(x^2)}{x^2}$$
$$g'(3) = \frac{3(-3)6 - (-3)}{9} = -5.667.$$

c. [4 points] $g(x) = \ln(f(x)) + f(2)$

Solution:

$$g'(x) = \frac{1}{f(x)} f'(x) + 0$$
$$g'(3) = \frac{1}{7} \cdot (-1) = \frac{-1}{7}.$$

d. [4 points] $g(x) = f^{-1}(x)$

Solution:

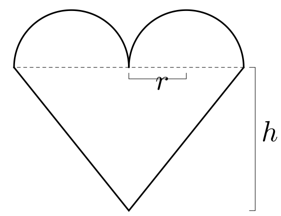
$$g'(x) = \frac{1}{f'(f^{-1}(x))}$$
$$g'(3) = \frac{1}{f'(6)} = -\frac{2}{3}.$$

7. [12 points] For Valentine's Day, Jason decides to make a heart-shaped cookie for Sophie to try to win her over. Being mathematically-minded, the only kind of heart that Jason knows how to construct is composed of two half-circles of radius r and an isosceles triangle of height h, as shown below.

Jason happens to know that Sophie's love for him will be determined by the dimensions of the cookie she receives; if given a cookie as described above, her love L will be

$$L = hr^2,$$

where r and h are measured in centimeters and L is measured in pitter-patters, a standard unit of affection. If Jason wants to make a cookie whose area is exactly 300cm^2, what should the dimensions be to maximize Sophie's love?



Solution: The area of the heart shape is

$$A = \pi r^2 + 2rh.$$

Setting this equal to 300 and solving for h gives the formula

$$h = \frac{300 - \pi r^2}{2r} = 150r^{-1} - \frac{\pi}{2}r.$$

Therefore, the formula for L can be written in terms of r alone:

$$L(r) = (150r^{-1} - \tfrac{\pi}{2}r)r^2 = 150r - \tfrac{\pi}{2}r^3.$$

We need to find the global maximum of $L(r)$. We have

$$L'(r) = 150 - \frac{3\pi}{2}r^2 = 0 \Rightarrow r = \frac{10}{\sqrt{\pi}}.$$

This critical point is a local maximum of L by the second-derivative test, since

$$L''(r) = -3\pi r \Rightarrow L''(10/\sqrt{\pi}) = -30\sqrt{\pi} < 0.$$

Since it is the only critical point, it must therefore be the global maximum.
Plugging this value of r into our formula, we can find the value of h, as well. We find that the dimensions that maximize Sophie's love are:

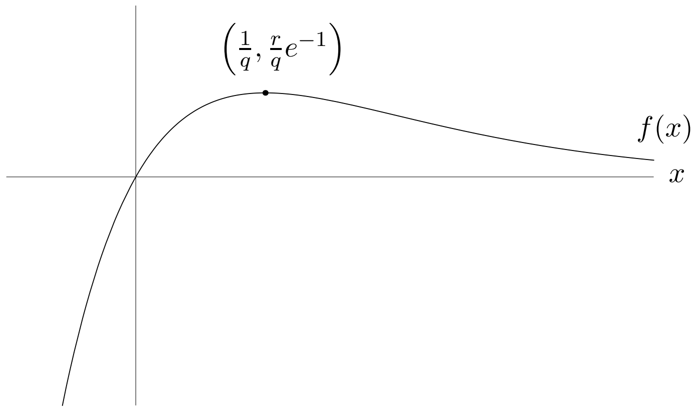
$$r = 10/\sqrt{\pi} \approx 5.6419\text{cm},$$

$$h = 10\sqrt{\pi} \approx 17.7245\text{cm}.$$

8. [16 points] Below is the graph of the function

$$f(x) = rxe^{-qx},$$

where r and q are constants. Assume that both r and q are greater than 1. The function $f(x)$ passes through the origin and has a local maximum at the point $P = \left(\frac{1}{q}, \frac{r}{q}e^{-1}\right)$, as shown in the graph.



a. [4 points] Justify, using either the first-derivative test or second-derivative test, that the point P is a local maximum.

Solution: To apply the first-derivative test, first compute:

$$f'(x) = re^{-qx}(-qx + 1).$$

Thus, there is indeed a critical point at $x = \frac{1}{q}$. Plugging in $x = 0$ (which is less than $\frac{1}{q}$), we find $f'(0) = r > 0$, while plugging in $x = \frac{2}{q}$ (which is greater than $\frac{1}{q}$), we find $f'(\frac{2}{q}) = -re^{-2} < 0$. Thus, f' changes from positive to negative at $x = \frac{1}{q}$, so it is a local maximum.
To apply the second-derivative test, compute $f''(x) = -rqe^{-qx}(-qx + 2)$, so

$$f''(\tfrac{1}{q}) = -rqe^{-1} < 0.$$

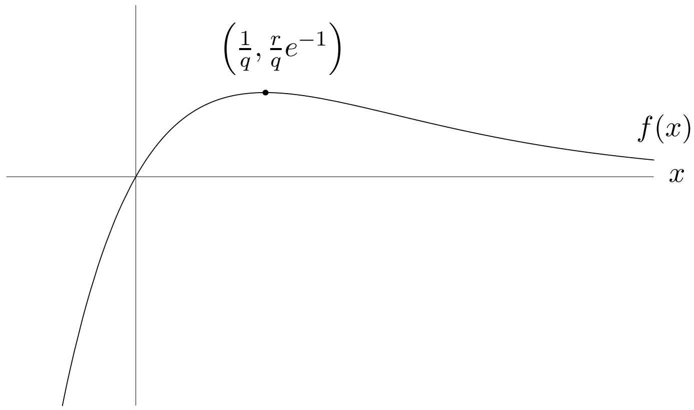
b. [2 points] What are the x-coordinates of the global maximum and minimum of $f(x)$ on the domain $[0, 1]$? (If $f(x)$ does not have a global maximum on this domain, say "no global maximum", and similarly if $f(x)$ does not have a global minimum.)

Solution: Since $q > 1$, the local maximum at $x = \frac{1}{q}$ is within this domain. Therefore, the global maximum occurs at $x = \frac{1}{q}$ and the global minimum occurs at $x = 0$.

c. [2 points] What are the x-coordinates of the global maximum and minimum of $f(x)$ on the domain $(-\infty, \infty)$? (If $f(x)$ does not have a global maximum on this domain, say "no global maximum", and similarly if $f(x)$ does not have a global minimum.)

Solution: The global maximum is at $x = \frac{1}{q}$ and there is no global minimum.

8. (continued) For your convenience, the graph of $f(x)$ is repeated below.



d. [4 points] Suppose that $g(x)$ is a function with $g'(x) = f(x)$. Find x-values of all local maxima and minima of $g(x)$. Justify that each maximum you find is a maximum and each minimum is a minimum.

Solution: The function $g(x)$ has critical points wherever $g'(x) = f(x) = 0$, which is only at $x = 0$. Since $f(x)$ changes from negative to positive at this point, the critical point at $x = 0$ is a local minimum by the first-derivative test.

e. [4 points] If $g(x)$ is as in part (d), for which x-values does $g(x)$ have inflection points? Show that these x-values are indeed inflection points.

Solution: The function $g(x)$ has inflection points when $g''(x) = f'(x)$ changes sign. This occurs precisely when $f(x)$ changes from increasing to decreasing (or vice versa), which is at $x = \frac{1}{q}$.