Math 115 — Second Midterm

March 21, 2013

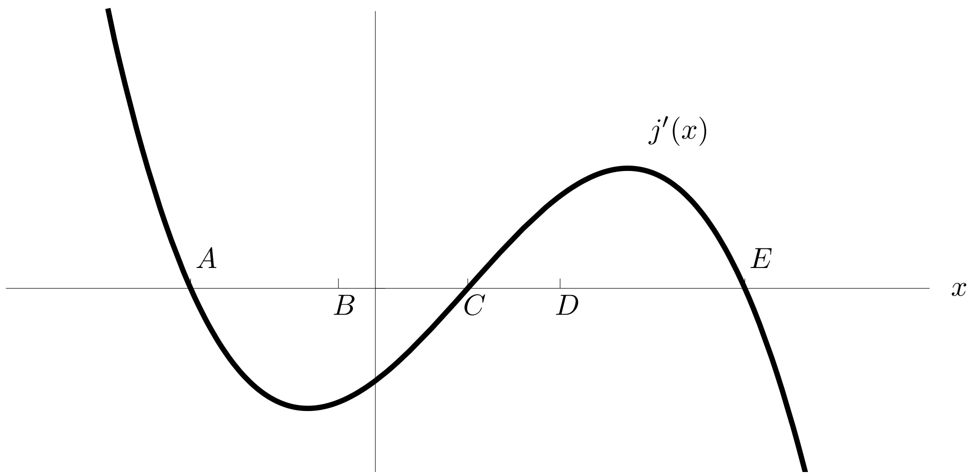
Name: EXAM SOLUTIONS

Instructor: ______________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 10 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	12	
2	10	
3	12	
4	11	
5	14	
6	9	
7	10	
8	12	
9	10	
Total	100	

1. [12 points] Consider the graph of $j'(x)$ given here. Note that this is not the graph of $j(x)$.



For each of (a)-(f) below, list **all** x-values labeled on the graph which satisfy the given statement in the blank provided. If the statement is not true at any of the labeled x-values, write "NP". You do not need to show your work. No partial credit will be given on each part of this problem.

(a) The function $j(x)$ has a local minimum at $x =$ ___**C**___.

(b) The function $j(x)$ has a local maximum at $x =$ ___**A, E**___.

(c) The function $j(x)$ is concave up at $x =$ ___**B, C, D**___.

(d) The function $j(x)$ is concave down at $x =$ ___**A, E**___.

(e) The function $j'(x)$ has a critical point at $x =$ ___**NP**___.

(f) The function $j''(x)$ is greatest at $x =$ ___**C**___.

2. [10 points] The velocity, in cm per second, of an inchworm moving in a straight line can be modeled by a **sinusoidal** function, $v(t)$, where t is the number of seconds since the inchworm started moving. Suppose the inchworm reaches its maximum velocity of 0.1 cm/sec one second after it starts moving and its minimum velocity of 0 cm/sec two seconds after it starts moving.

a. [5 points] Find a formula for $v(t)$ which is consistent with the information above.

Solution: The period is 2 (because the distance from the x-coordinate of a max to the x-coordinate of a min is 1), the amplitude is .05, and the midline is .05. The graph starts at a min, so it is a negative cosine function with no horizontal shift.

$$v(t) = -.05\cos(\pi t) + .05$$

b. [2 points] Based on your answer from (a), find a formula for $a(t)$, the acceleration of the inchworm t seconds after it started moving.

Solution: Acceleration is the derivative of velocity, so

$$a(t) = .05\pi \sin(\pi t)$$

c. [3 points] Based on your answers above, find a time when the inchworm's acceleration attains its largest positive value.

Solution: The maximum values of $\sin(x)$ are attained when $x = \ldots -3\pi/2, \pi/2, 5\pi/2, \ldots$, so (since we are looking at $\sin(\pi t)$) we can take $t = \frac{1}{2}$ (or many other answers). Alternatively, we could compute the derivative of $a(t)$ and set that equal to zero; if you did it this way, be sure that you wrote down a max and not a min.

3. [12 points] Consider the family of linear functions

$$L(x) = ax - 3$$

and the family of functions

$$M(x) = a\sqrt{x}$$

where a is a nonzero constant number. Note that the number a is the same for both equations. Find a value of a for which $L(x)$ is tangent to the graph of $M(x)$. Also find the x and y coordinates of the point of tangency. Write your answers in the blanks provided.

Solution: Since the graphs of M and L intersect are tangent at the relevant point, we have

$$M'(x) = L'(x). \qquad (\dagger)$$

Computing the derivatives, we get

$$\frac{1}{2}ax^{-\frac{1}{2}} = a$$

Canceling the a from this equation and doing some algebra, we see $x = \frac{1}{4}$. Since the two graphs must also intersect at this point, we must have

$$M\left(\frac{1}{4}\right) = L\left(\frac{1}{4}\right),$$

and using this, we can solve for a; we get $a = -12$. Finally, we can recover $y = -6$ by plugging these values into either the equation for $M(x)$ or the equation for $L(x)$.

$a =$ **-12**

$x =$ **.25**

$y =$ **-6**

4. [11 points]

a. [4 points] Find the tangent line approximation of the function

$$p(x) = 1 + x^k$$

near $x = 1$, where k is a positive constant.

Solution:

$$L(x) = k(x-1) + 2$$

b. [2 points] Suppose you want to use your tangent line from (a) to approximate the number $1 + \sqrt{0.95}$. What values of k and x would you plug in to your answer from (a)?

Solution: We'd take $k = \frac{1}{2}$ and $x = .95$.

c. [2 points] Approximate $1 + \sqrt{0.95}$ using your tangent line from (a).

Solution: We have

$$1 + \sqrt{.95} \approx .5(-.05) + 2 = 1.975.$$

d. [3 points] Determine whether your approximation in (c) is an over- or underestimate. Be sure your reasoning is clear.

Solution: The graph of $1 + x^{.5}$ is just the graph of the square root function shifted up by one, so it's concave down everywhere. It follows that the linear approximation is an overestimate.

5. [14 points] Consider the family of functions

$$g(x) = \frac{ax^b}{\ln(x)}$$

where a and b are nonzero constants.

a. [4 points] Calculate $g'(x)$.

Solution: Using the quotient rule, we get

$$g'(x) = \frac{abx^{b-1}\ln(x) - ax^{b-1}}{(\ln(x))^2}.$$

b. [6 points] Find values for a and b so that $g(e) = 1$ and $g'(e) = 0$.

Solution: Since $g(e) = 1$ we have

$$ae^b = 1. \qquad (\star)$$

Since $g'(e) = 0$, the numerator of the answer to (a) must be zero, which says

$$abe^{b-1} - ae^{b-1}. \qquad (\dagger)$$

The equation ($\dagger$) simplifies to $b = 1$, and then from ($\star$) we deduce $a = \frac{1}{e}$.

c. [4 points] With the values of a and b you found in (b), is $x = e$ a local minimum of g, a local maximum of g or neither? Justify your answer.

Solution: It is a local minimum. To see this, we use the first derivative test. The denominator of our expression for g' is always positive, and (with our values of a and b) the numerator is

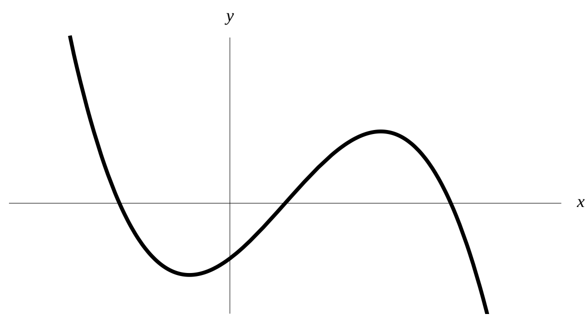
$$\frac{1}{e}(\ln(x) - 1).$$

This expression changes signs from negative to positive around $x = e$.

6. [9 points] In each of the following problems, draw a graph of a function with all of the indicated properties. If there is no such function, then write "NO SUCH FUNCTION EXISTS". You do not need to write any explanations. No partial credit will be given on each part of this problem.

a. [3 points] A continuous function $f(x)$, whose domain is all real numbers, with the following four properties:
(i.) $f(x)$ attains a local minimum somewhere.
(ii.) $f(x)$ attains a local maximum somewhere.
(iii.) $f(x)$ does not attain a global minimum.
(iv.) $f(x)$ does not attain a global maximum.

Solution:

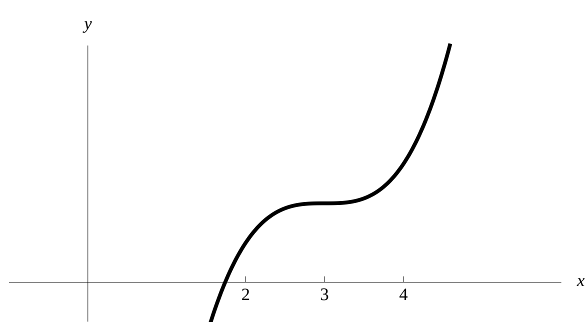


b. [3 points] A continuous function $g(x)$, whose domain is the closed interval $[0, 1]$, with the following two properties:
(i.) $g(x)$ does not attain a global maximum on the interval $[0, 1]$
(ii.) $g(x)$ attains a global minimum on the interval $[0, 1]$.

Solution: NO SUCH FUNCTION EXISTS

c. [3 points] A differentiable function $j(x)$ with the following two properties:
(i.) The linear approximation to $j(x)$ at $x = 3$ gives an overestimate when used to approximate $j(2)$.
(ii.) The linear approximation to $j(x)$ at $x = 3$ gives an underestimate when used to approximate $j(4)$.

Solution:



7. [10 points] For each real number k, there is a curve in the plane given by the equation

$$e^{y^2} = x^3 + k.$$

a. [4 points] Find $\frac{dy}{dx}$.

Solution: We have

$$2ye^{y^2}\frac{dy}{dx} = 3x^2,$$

so

$$\frac{dy}{dx} = \frac{3x^2}{2ye^{y^2}}$$

b. [3 points] Suppose that $k = 9$. There are two points on the curve where the tangent line is horizontal. Find the x and y coordinates of each one.

Solution: Horizontal tangent lines occur when the numerator of the derivative is zero, so in this case $x = 0$. To solve for the y-coordinate, we have

$$e^{y^2} = 9$$

so $y = \pm\sqrt{\ln(9)}$.

c. [3 points] Now suppose that $k = \frac{1}{2}$. How many points are there where the curve has a horizontal tangent line?

Solution: Again we get $x = 0$. Now if we try to solve for y we have

$$y^2 = \ln\left(\frac{1}{2}\right) < 0$$

and so there are no points where the curve has a horizontal tangent line.

8. [12 points] In the following table, both f and g are differentiable functions of x. In addition, $g(x)$ is an invertible function. Write your answers in the blanks provided. You do not need to show your work.

x	2	3	4	5
$f(x)$	7	6	2	9
$f'(x)$	-2	1	3	2
$g(x)$	1	4	7	11
$g'(x)$	1	2	3	2

a. [3 points] If $h(x) = \frac{g(x)}{f(x)}$, find $h'(4)$.

$h'(4) =$ **-15/4**

b. [3 points] If $k(x) = f(x)g(x)$, find $k'(2)$.

$k'(2) =$ **5**

c. [3 points] If $m(x) = g^{-1}(x)$, find $m'(4)$.

$m'(4) =$ **1/2**

d. [3 points] If $n(x) = f(g(x))$, find $n'(3)$.

$n'(3) =$ **6**

9. [10 points] The function $f(x)$ is twice-differentiable. Some values of f and f' are given in the following table. In addition, it is known that $f''(x)$ is positive.

x	0	1	2	3	4
$f(x)$	7	6	7	9	12
$f'(x)$	-2	$\frac{1}{2}$	1	2	4

No partial credit will be given on any part of this problem.

a. [4 points] **Circle** any statement which is true, and **draw a line through** any statement which is false.

(i.) For some value of x with $0 < x < 1$, f has a critical point.

(ii.) ~~For some value of x with $1 < x < 2$, f has a critical point.~~

(iii.) ~~For some value of x with $2 < x < 3$, f has a critical point.~~

(iv.) ~~For some value of x with $3 < x < 4$, f has a critical point.~~

b. [3 points] If possible, find the global minimum value of $f(x)$ on the closed interval $[0, 4]$. (Give the y-coordinate, not the x-coordinate.) Do not give an approximation. If it is not possible to find it exactly, write "IT IS NOT POSSIBLE TO FIND IT EXACTLY."

Solution: We know that $f(x)$ is decreasing until some critical point p between 0 and 1 and is increasing after that (because we know that f' goes from negative to positive between 0 and 1 and never becomes negative again, since $f'' > 0$). The minimum occurs at some point that's not included in the table, so IT IS NOT POSSIBLE TO FIND IT EXACTLY.

c. [3 points] If possible, find the global maximum value of $f(x)$ on the closed interval $[0, 4]$. (Give the y-coordinate, not the x-coordinate.) Do not give an approximation. If it is not possible to find it exactly, write "IT IS NOT POSSIBLE TO FIND IT EXACTLY."

Solution: The only critical point is a local minimum, so the maximum value must be at one of the endpoints. Looking at the table, we see the maximum value is 12 (when $x = 4$).