Math 115 — Second Midterm

March 27, 2014

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 11 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
8. Include units in your answer where that is appropriate.
9. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones.
10. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	12	
2	9	
3	12	
4	8	
5	13	
6	10	
7	5	
8	6	
9	10	
10	10	
11	5	
Total	100	

1. [12 points] The table below gives several values of a differentiable function $f(x)$. Assume that both $f(x)$ and $f'(x)$ are invertible. Do not give approximations. If it is not possible to find the value exactly, write NOT POSSIBLE.

x	-3	-2	-1	0	1	2	3
$f(x)$	-8	-4	-1.2	0.5	1.4	1.8	2
$f'(x)$	5	3	2	1.2	0.5	0.3	0.1

a. [2 points] Let $g(x) = 3f(x) + 4$. Find $g'(1)$.

Solution: $g'(x) = 3f'(x)$, so $g'(1) = 3 \cdot 0.5 = 1.5$

Answer: $g'(1) =$ 1.5

b. [2 points] Find $(f^{-1})'(2)$.

Solution: $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$, so $(f^{-1})'(2) = \dfrac{1}{f'(f^{-1}(2))} = \dfrac{1}{f'(3)} = \dfrac{1}{0.1} = 10.$

Answer: $(f^{-1})'(2) =$ 10

c. [2 points] Let $h(x) = f(e^x)$. Find $h'(\ln 2)$.

Solution: $h'(x) = f'(e^x) \cdot e^x$, so $h'(\ln 2) = f'(e^{\ln 2}) \cdot e^{\ln 2} = f'(2) \cdot 2 = 0.3 \cdot 2 = 0.6.$

Answer: $h'(\ln 2) =$ 0.6

d. [2 points] Let $j(x) = e^{f(x)}$. Find $j'(-2)$.

Solution: $j'(x) = e^{f(x)} \cdot f'(x)$, so $j'(-2) = e^{f(-2)} \cdot f'(-2) = e^{-4} \cdot 3.$

Answer: $j'(-2) =$ $3e^{-4}$

e. [2 points] Let $k(x) = f(x)f(x-2)$. Find $k'(1)$.

Solution: $k'(x) = f'(x)f(x-2) + f(x)f'(x-2)$, so
$k'(1) = f'(1)f(1-2) + f(1)f'(1-2) = f'(1)f(-1) + f(1)f'(-1) = 0.5 \cdot (-1.2) + 1.4 \cdot 2 = -0.6 + 2.8 = 2.2.$

Answer: $k'(1) =$ 2.2

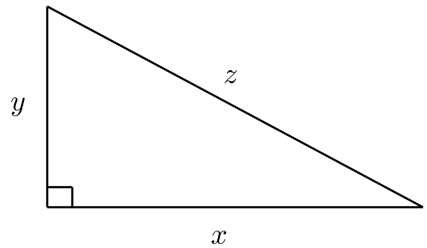
f. [2 points] Let $\ell(x) = \frac{f(x)}{f(x+3)}$. Find $\ell'(0)$.

Solution: $\ell'(x) = \dfrac{f'(x)f(x+3) - f'(x+3)f(x)}{(f(x+3))^2}$, so

$$\ell'(0) = \frac{f'(0)f(3) - f'(3)f(0)}{(f(3))^2} = \frac{1.2 \cdot 2 - 0.5 \cdot 0.1}{2^2} = \frac{2.4 - 0.05}{4} = \frac{2.35}{4} = 0.5875.$$

Answer: $\ell'(0) =$ 0.5875

2. [9 points] Consider a right triangle with legs of length x ft and y ft and hypotenuse of length z ft, as in the following picture:



a. [2 points] Suppose that the perimeter of the triangle is 8 ft. Let $A(x)$ give the area of the triangle, in ft^2, as a function of the side length x. In the context of this problem, what is the domain of $A(x)$? *Note that you do* <u>*not*</u> *need to find a formula for* $A(x)$.

Solution: Notice that we can let x be arbitrarily close to 0 and still have a perimeter of 8 ft by making y and z both very close to 4.
However, since z is always at least as big as x and since y is positive, x cannot be larger than 4 or or else $x + y + z$ would be greater than 8 ft.

Answer: $(0, 4)$

b. [7 points] Suppose instead that the perimeter of the triangle is allowed to vary, but the area of the triangle is fixed at 3 ft^2. Let $P(x)$ give the perimeter of the triangle, in ft, as a function of the side length x.

(i) In the context of this problem, what is the domain of $P(x)$?

Solution: x must be positive, but there is no upper bound on x. Even if x is very large, with a small enough y, it is still possible for the triangle to have area 3 ft^2.

Answer: $(0, \infty)$

(ii) Find a formula for $P(x)$. The variables y and z should <u>not</u> appear in your answer. (This is the equation one would use to find the value(s) of x minimizing the perimeter. You should <u>not</u> do the optimization in this case.)

Solution: The perimeter of the triangle is $x+y+z$ ft. Since we want it to be a function of x only, we need to use other information to eliminate the other variables.

The area is 3 ft^2, so we have $\frac{1}{2}xy = 3$. Solving for y yields $y = \frac{6}{x}$.

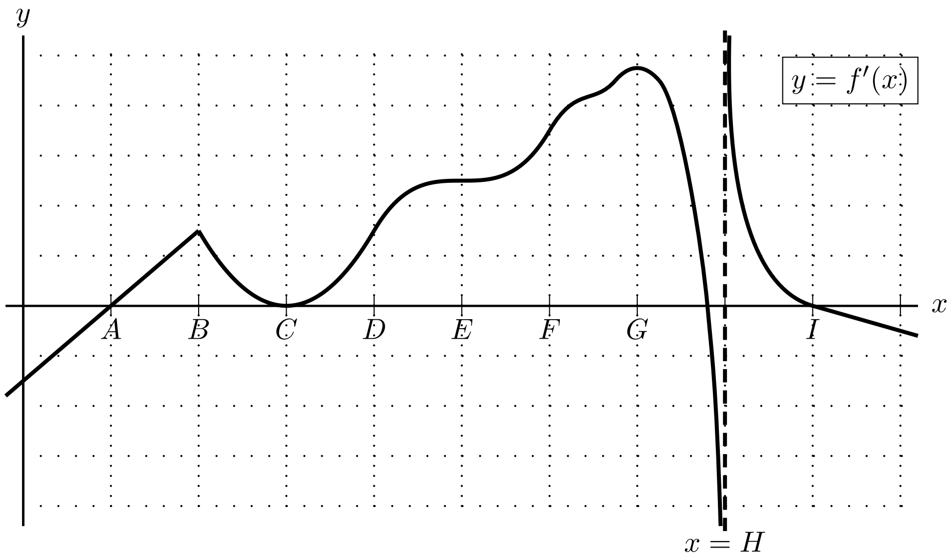
Since this is a right triangle, by the Pythagorean Theorem, we have $x^2 + y^2 = z^2$. Solving for z yields $z = \sqrt{x^2 + y^2}$. Using $y = \frac{6}{x}$ allows us to write z in terms of x as $z = \sqrt{x^2 + \left(\frac{6}{x}\right)^2}$.

Finally, then, we have

$$P(x) = x + y + z = x + \frac{6}{x} + \sqrt{x^2 + \left(\frac{6}{x}\right)^2}.$$

Answer: $P(x) =$ $x + \dfrac{6}{x} + \sqrt{x^2 + \left(\dfrac{6}{x}\right)^2}$

3. [12 points] The graph of a portion of $y = f'(x)$, the derivative of $f(x)$ is shown below. Note that there is a sharp corner at $x = B$ and that $x = H$ is a vertical asymptote.
The function $f(x)$ is continuous with domain $(-\infty, \infty)$.



For each of the questions below, circle all of the available correct answers.
(Circle NONE if none of the available choices are correct.)

a. [2 points] At which of the following six values of x is the function $f(x)$ not differentiable?

B C E F $\boxed{H}$ I NONE

b. [2 points] At which of the following six values of x does the function $f'(x)$ appear to be not differentiable?

A $\boxed{B}$ C D E F NONE

c. [2 points] At which of the following nine values of x does $f(x)$ have a critical point?

$\boxed{A}$ B $\boxed{C}$ D E F G $\boxed{H}$ $\boxed{I}$ NONE

d. [2 points] At which of the following nine values of x does $f(x)$ have a local minimum?

$\boxed{A}$ B C D E F G $\boxed{H}$ I NONE

e. [2 points] At which of the following nine values of x is $f''(x) = 0$?

A B $\boxed{C}$ D $\boxed{E}$ F $\boxed{G}$ H I NONE

f. [2 points] At which of the following nine values of x does $f(x)$ have an inflection point?

A $\boxed{B}$ $\boxed{C}$ D E F $\boxed{G}$ H I NONE

4. [8 points] A ship's captain is standing on the deck while sailing through stormy seas. The rough waters toss the ship about, causing it to rise and fall in a sinusoidal pattern. Suppose that t seconds into the storm, the height of the captain, in feet above sea level, is given by the function

$$h(t) = 15\cos(kt) + c$$

where k and c are nonzero constants.

a. [3 points] Find a formula for $v(t)$, the vertical velocity of the captain, in feet per second, as a function of t. The constants k and c may appear in your answer.

Solution: The velocity is the derivative of the height function, so we compute

$$v(t) = h'(t) = -15k\sin(kt).$$

Notice that the Chain Rule gives us a factor of k out front, and since c is an additive constant, it disappears when we take the derivative.
Notice also that $v(t) = \frac{dh}{dt}$ does indeed have units of feet per second, as required.

Answer: $v(t) =$ $-15k\sin(kt)$

b. [2 points] Find a formula for $v'(t)$. The constants k and c may appear in your answer.

Answer: $v'(t) =$ $-15k^2\cos(kt)$

c. [3 points] What is the maximum vertical acceleration experienced by the captain? The constants k and c may appear in your answer. You do not need to justify your answer or show work. *Remember to include units.*

Solution: The acceleration is just the derivative of the velocity function, which was just computed in the previous part.
Since $v'(t) = -15k^2\cos(kt)$ is sinusoidal with midline 0 and amplitude $15k^2$, the maximum value it achieves is $15k^2$.
Since $v'(t) = \frac{dv}{dt}$, the units on the acceleration are feet per second per second, or feet per second squared.

Answer: Max vertical acceleration: $15k^2$ **ft/s^2**

5. [13 points] Suppose $f(x)$ is a function defined for all x whose derivative and second derivative are given by

$$f'(x) = \frac{(x+2)^2(x-3)}{(x+1)^{1/3}} \qquad \text{and} \qquad f''(x) = \frac{2(x+2)(x-1)(4x+3)}{3(x+1)^{4/3}}.$$

a. [2 points] Find the x-coordinates of all critical points of $f(x)$. If there are none, write NONE.

Solution: Critical points of $f(x)$ occur where $f'(x)$ is zero or undefined. $f'(x)$ is zero when the numerator is zero, at $x = -2$ and $x = 3$. $f'(x)$ is undefined when the denominator is zero, at $x = -1$. Therefore, $x = -2, -1, 3$ are the critical points of $f(x)$.

Answer: Critical point(s) at $x =$ $-2, -1, 3$

b. [6 points] Find the x-coordinates of all local extrema of $f(x)$.
If there are none of a particular type, write NONE.
Justify your answers, and be sure to show enough evidence to demonstrate that you have found all local extrema.

Solution: To classify the critical points, we use the First Derivative Test, so we look at the sign of $f'(x)$ before and after each critical point.
For $x < -2$, $(x+2)$, $(x-3)$, and $(x+1)$ are all negative, so $f'(x) = \frac{-^2 \cdot -}{-} = +$ is positive.
Similarly, for $-2 < x < -1$, we have $f'(x) = \frac{+^2 \cdot -}{-} = +$.
For $-1 < x < 3$, we have $f'(x) = \frac{+^2 \cdot -}{+} = -$.
For $x > 3$, we have $f'(x) = \frac{+^2 \cdot +}{+} = +$.
Summarizing on a number line, we see:

$f'(x)$: $+$ | -2 | $+$ | -1 | $-$ | 3 | $+$

We see that $f'(x)$ does not change sign at $x = -2$, so this point is not a local extremum. At $x = -1$, $f'(x)$ changes from positive to negative, so $x = -1$ is a local maximum. Finally, at $x = 3$, $f'(x)$ changes from negative to positive, so $x = 3$ is a local minimum.

Note that we could use the Second Derivative Test as well, but it would be inconclusive at $x = -2$, so we would have to resort to the First Derivative Test to classify that critical point.

Answer: Local min(s) at $x =$ 3

Answer: Local max(es) at $x =$ -1

c. [5 points] Find the x-coordinates of all inflection points of $f(x)$. If there are none, write NONE.
Justify your answers, and be sure to show enough evidence to demonstrate that you have found all inflection points.

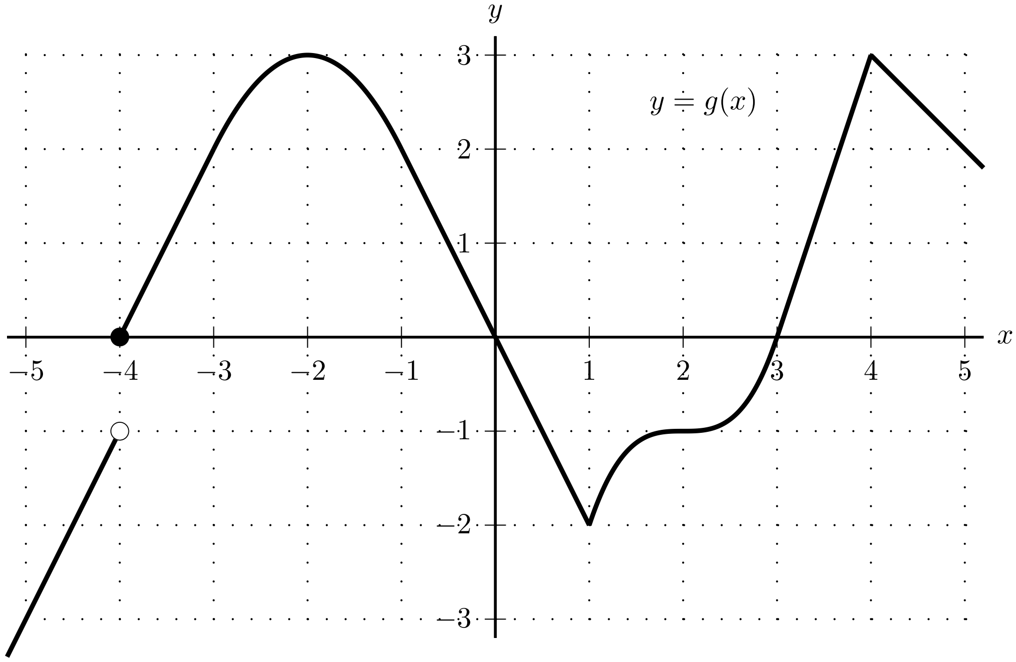
Solution: Inflection points of $f(x)$ can occur whenever $f''(x)$ is zero or undefined. In this case, $f''(x)$ is zero at $x = -2$, $-3/4$, and 1 and undefined at $x = -1$. We must check whether the sign of $f''(x)$ actually changes at each of these points.
For $x < -2$, we have $f''(x) = \frac{- \cdot - \cdot -}{+} = -$.
For $-2 < x < -1$, we have $f''(x) = \frac{+ \cdot - \cdot -}{+} = +$.
For $-1 < x < -3/4$, we have $f''(x) = \frac{+ \cdot - \cdot -}{+} = +$.
For $-3/4 < x < 1$, we have $f''(x) = \frac{+ \cdot - \cdot +}{+} = -$.
For $x > 1$, we have $f''(x) = \frac{+ \cdot + \cdot +}{+} = -$.
Summarizing on a number line, we see:

$f''(x)$: $-$ | -2 | $+$ | -1 | $+$ | $-\frac{3}{4}$ | $-$ | 1 | $+$

Because $f''(x)$ does not change sign at $x = -1$, this is not an inflection point. Since $f''(x)$ changes sign at $x = -2$, $-3/4$, and 1, we conclude that these are the inflection points of $f(x)$.

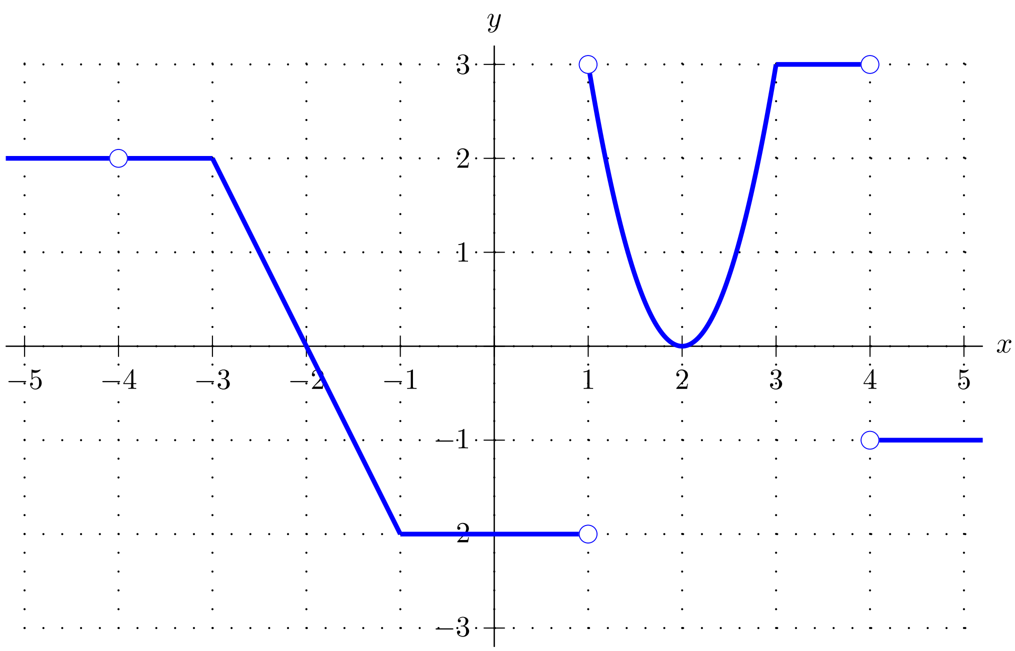
Answer: Inflection point(s) at $x =$ $-2, -3/4, 1$

6. [10 points] A portion of the graph of $y = g(x)$ is shown below.



On the axes below, sketch the graph of $y = g'(x)$.
Be sure that you pay close attention to each of the following:

- where g' is defined
- the value of $g'(x)$ near each of $x = -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5$
- the sign of g'
- where g' is increasing/decreasing/constant



7. [5 points] Let

$$s(t) = \begin{cases} 5t^2 & \text{if } t \le 3 \\ p + c(t-3) & \text{if } t > 3 \end{cases}$$

be a differentiable function, where p and c are constants.

a. [3 points] Find the values of p and c.

Solution: Since $s(t)$ is differentiable, it is also continuous. By continuity, the two parts must agree at $t = 3$, so we have

$$5 \cdot 3^2 = p + c(3-3) = p,$$

or $p = 45$.

By differentiability, $s'(t)$ must exist at $t = 3$. For $t < 3$, we have $s'(t) = 10t$, and for $t > 3$, we have $s'(t) = c$. To be differentiable at $t = 3$, these two must agree at $t = 3$, so we have

$$10 \cdot 3 = c,$$

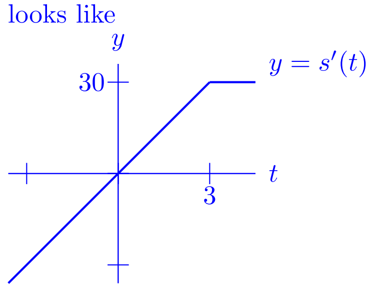
or $c = 30$.

Answer: $p =$ 45 and $c =$ 30

b. [2 points] Is $s'(t)$ differentiable at $t = 3$?
To receive any credit on this question, you must justify your answer.

Solution: No. We saw above that $s'(t) = \begin{cases} 10t & \text{if } t \le 3 \\ 30 & \text{if } t > 3 \end{cases}$.

The graph of $y = s'(t)$ therefore looks like



which has a sharp corner at $t = 3$.

8. [6 points] Find a formula for $\frac{dy}{dx}$ for the implicit function $ax^2 + xy^2 + b \ln y = c$.
The constants a, b, and c may appear in your answer.

Solution: Applying $\frac{d}{dx}$ to both sides of the given equation, we have

$$2ax + y^2 + 2xy\frac{dy}{dx} + \frac{b}{y}\frac{dy}{dx} = 0.$$

Collecting all the terms involving $\frac{dy}{dx}$ on one side and then factoring it out, we find

$$\frac{dy}{dx}\left(2xy + \frac{b}{y}\right) = -2ax - y^2$$

and hence

$$\frac{dy}{dx} = \frac{-2ax - y^2}{2xy + \frac{b}{y}}.$$

Answer: $\frac{dy}{dx} = \frac{-2ax - y^2}{2xy + \frac{b}{y}}$

9. [10 points] After a long, cold winter, a ship's captain sails across Lake Michigan to Chicago. Upon arrival, the captain hosts a party on board to celebrate the arrival of spring. The party begins at exactly 6 pm and ends at exactly midnight. Let $N(t)$ be the noise level, in decibels, of the ship captain's party t hours after it begins. During the party, a formula for $N(t)$ is given by

$$N(t) = 0.5t^4 - 4t^3 + 7t^2 + 60.$$

a. [8 points] Find the exact values of t that minimize and maximize $N(t)$ on the interval $[0, 6]$. Use calculus to find your answers, and be sure to show enough evidence that the points you find are indeed global extrema.

Solution: By the Extreme Value Theorem, there will be both a global minimum and a global maximum, and they will occur at either the end points or the critical points. So we begin by finding the critical points.
We have $N'(t) = 2t^3 - 12t^2 + 14t = 2t(t^2 - 6t + 7)$. There are no points where this is undefined, so our only critical points are at the zeros. We immediately see that $t = 0$ is a zero; for the others, we use the Quadratic Formula to find two more zeros at

$$t = \frac{6 \pm \sqrt{6^2 - 28}}{2} = 3 \pm \sqrt{2}.$$

Our critical points are therefore $t = 0$, $3 - \sqrt{2}$, and $3 + \sqrt{2}$.
To determine the global extrema, then, we compare the values of $N(t)$ at all critical points and end points of our interval:

t	0	$3-\sqrt{2}$	$3+\sqrt{2}$	6
$N(t)$	60	64.8	42.2	96

Since the smallest value of $N(t)$ occurs at $t = 3 + \sqrt{2}$, this is our global minimum, and since the largest value of $N(t)$ occurs at $t = 6$, this is our global maximum.

(For each answer blank below, write NONE in the answer blank if appropriate.)

Answer: Global min(s) at exactly $t =$ $3 + \sqrt{2}$

Answer: Global max(es) at exactly $t =$ 6

b. [2 points] How loud does the captain's party get? *Remember to include units.*

Solution: We just saw that the maximum value of this function occurs at $t = 6$, when $N(6) = 96$. Therefore, the loudest the captain's party gets is 96 decibels.

Answer: 96 **decibels**

10. [10 points] Let $f(x)$ be a function with $f(1) = 5$, $f'(1) = -2$, and $f''(1) = 3$.

a. [2 points] Use the local linearization of $f(x)$ at $x = 1$ to estimate $f(0.9)$.

Solution: The local linearization of $f(x)$ at $x = 1$ is $5 - 2(x - 1)$. Plugging in $x = 0.9$ yields $5 - 2(0.9 - 1) = 5 + 0.2 = 5.2$.

Answer: $f(0.9) \approx$ 5.2

b. [2 points] Do you expect your estimate from Part (a) to be an overestimate or underestimate? To receive any credit on this question, you must justify your answer.

Solution: Since $f''(1) = 3$, the graph of $y = f(x)$ is concave up near $x = 1$. Therefore, the tangent line at $x = 1$ lies under the graph of $f(x)$ near $x = 1$, so we expect this to be an underestimate.

c. [2 points] Use the tangent line approximation of $f'(x)$ near $x = 1$ to estimate $f'(1.1)$.

Solution: The tangent line to $f'(x)$ at $x = 1$ passes through the point $(1, f'(1))$ and has slope $f''(1)$ (as the slope of the derivative function is given by the <u>second</u> derivative). Therefore, the tangent line to $f'(x)$ at $x = 1$ is given by the equation

$$L = -2 + 3(x - 1).$$

Plugging in $x = 1.1$ yields $-2 + 3(1.1 - 1) = -1.7$.

Answer: $f'(1.1) \approx$ -1.7

d. [4 points] Suppose that the tangent line approximation of $f(x)$ near $x = 8$ estimates $f(8.05)$ to be 3.75 and $f(8.1)$ to be 3.6. Find $f(8)$ and $f'(8)$.

Solution: Since the tangent line passes through the points $(8.05, 3.75)$ and $(8.1, 3.6)$, it has slope

$$\frac{3.6 - 3.75}{8.1 - 8.05} = \frac{-0.15}{0.05} = -3.$$

Hence $f'(8) = -3$. Moreover, an equation for this tangent line is therefore

$$L = 3.75 - 3(x - 8.05),$$

so plugging in $x = 8$, it passes through the point $(8, 3.9)$.
By definition of the tangent line, then, we have that $f(x)$ also passes through the point $(8, 3.9)$ and also has slope -3, so we conclude that $f(8) = 3.9$ and $f'(8) = -3$.

Answer: $f(8) =$ 3.9 and $f'(8) =$ -3

11. [5 points] A curve $\mathcal{C}$ gives y as an implicit function of x. The curve $\mathcal{C}$ passes through the point $(1,2)$ and satisfies

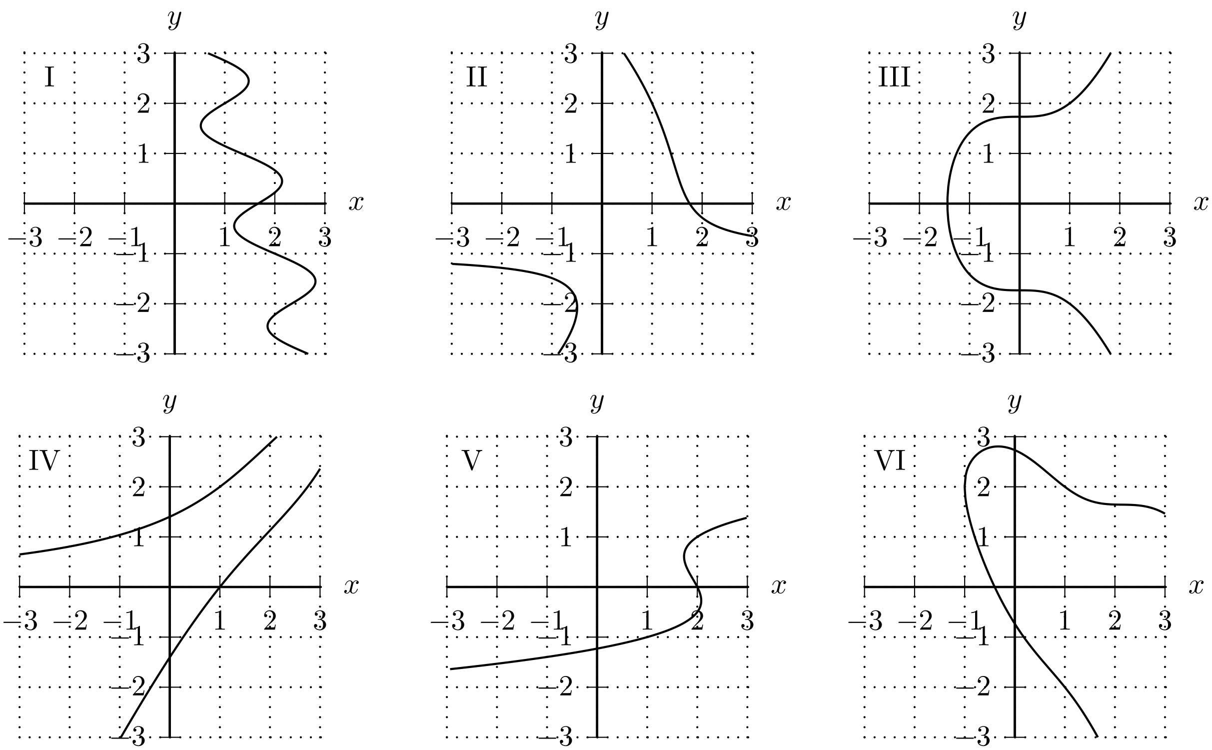
$$\frac{dy}{dx} = \frac{y^2 - 2xy + 4y - 5}{4(y-x)}.$$

a. [1 point] One of the values below is the slope of the curve $\mathcal{C}$ at the point $(1,2)$. Circle that one value.

Solution: Plugging $x = 1$ and $y = 2$ into the given formula for $\frac{dy}{dx}$ yields $3/4$.

Answer: The slope at $(1,2)$ is $\frac{1}{4}$ $\frac{1}{3}$ $\frac{1}{2}$ $\frac{5}{8}$ $\frac{2}{3}$ $\boxed{\frac{3}{4}}$ $\frac{4}{5}$

b. [4 points] One of the following graphs is the graph of the curve $\mathcal{C}$.
Which of the graphs I-VI is it? To receive any credit on this question, you must circle your answer next to the word "Answer" below.



Solution: We know that the desired curve passes through the point $(1,2)$ with slope $3/4$. This allows us to eliminate Graph V (which doesn't pass through $(1,2)$) and Graphs II and VI (which have negative slope at $(1,2)$).
To decide between Graphs I, III, and IV, we look at other points on the graphs.
Graph I passes through the point $(2,-1)$ with negative slope, but the above formula for $\frac{dy}{dx}$ says that it should have positive slope there, so Graph I is incorrect.
Graph III passes through the point $(1,-2)$ with negative slope, but the above formula for $\frac{dy}{dx}$ says that it too should have positive slope there, so Graph III is incorrect.
The only remaining possibility is Graph IV.
(Note that we could have also eliminated all but Graph IV by checking for vertical tangent lines at points (x,y) with $y = x$.)

Remember: To receive any credit on this question, you must circle your answer next to the word "Answer" below.

Answer: I II III $\boxed{\text{IV}}$ V VI