Math 115 — Second Midterm

March 24, 2015

Last Name Only: EXAM SOLUTIONS

Instructor Name: ______________________ Section #: ________

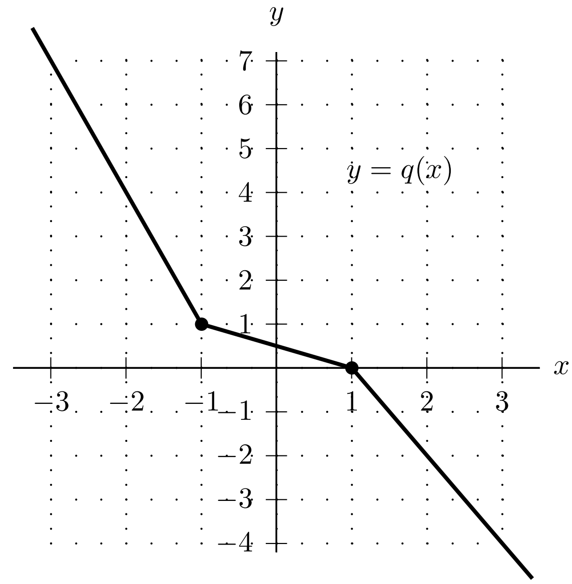
1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
8. Include units in your answer where that is appropriate.
9. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones.
10. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	11	
2	13	
3	8	
4	11	
5	9	
6	14	
7	10	
8	10	
9	9	
10	5	
Total	100	

1. [11 points]
Shown to the right is the graph of an invertible piecewise linear function $q(x)$. Note that the graph passes through the points $(-3,7)$, $(-1,1)$, $(1,0)$, and $(3,-4)$.

You are not required to show your work on this problem. However, limited partial credit may be awarded based on work shown.

Find the exact value of each of the quantities below. If there is not enough information provided to find the value, write "NOT ENOUGH INFO". If the value does not exist, write "DOES NOT EXIST".



a. [2 points] Let $r(x) = q^{-1}(x)$. Find $r'(2)$.

Solution: $r'(x) = \dfrac{1}{q'(q^{-1}(x))}$ so $r'(2) = \dfrac{1}{q'(q^{-1}(2))} = \dfrac{1}{-3} = -\dfrac{1}{3}$.

Answer: $r'(2) =$ $-\dfrac{1}{3}$

b. [3 points] Let $w(x) = \dfrac{x}{q(x+1)}$. Find $w'(-2)$.

Solution: By the quotient and chain rules, $w'(x) = \frac{q(x+1)-xq'(x+1)}{(q(x+1))^2}$ (where these quantities are defined). q' is not differentiable at $x = -1$, so $q'(x+1)$ is not defined at $x = -2$. (If $w'(-2)$ were to exist, then since $q(x+1) = \frac{w(x)}{x}$, we would have $q'(-1) = q'(-2+1) = \frac{(-2)w'(-2)-w(-2)}{(-2)^2}$.)

Answer: $w'(-2) =$ DOES NOT EXIST

c. [3 points] Let $v(x) = xq(\sin x)$. Find $v'(\pi)$.

Solution: By the product and chain rules we have $v'(x) = xq'(\sin x)\cos x + q(\sin x)$. So $v'(\pi) = \pi q'(\sin\pi)\cos\pi + q(\sin\pi) = \pi q'(0)(-1) + q(0) = \pi(-1/2)(-1) + (1/2) = \frac{\pi+1}{2}$.

Answer: $v'(\pi) =$ $\dfrac{\pi+1}{2}$

d. [3 points] Let $j(x) = \ln(q(2x))$. Find $j'(-1)$.

Solution: By the chain rule, we have

$$j'(x) = \frac{1}{q(2x)} \cdot q'(2x) \cdot 2 = \frac{2q'(2x)}{q(2x)} \quad \text{so} \quad j'(-1) = \frac{2q'(-2)}{q(-2)} = \frac{2(-3)}{4} = -\frac{3}{2}.$$

Answer: $j'(-1) =$ $-\dfrac{3}{2}$

2. [13 points] A function $g(x)$ and its derivative are given by

$$g(x) = \frac{x^3 + 34x^2 + 732x + 5400}{(x+30)^4} \quad \text{and} \quad g'(x) = \frac{-(x-6)^2(x-10)}{(x+30)^5}.$$

a. [8 points] Find all critical points of $g(x)$ and all values of x at which $g(x)$ has a local extremum. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all local extrema.

Solution: To find critical points we (i) solve for x in the equation $g'(x) = 0$ and (ii) look for points in the domain of the function where the derivative is undefined. The critical points occur at $x = 6$ and $x = 10$. Note that $x = -30$ is not a critical point since it is not in the domain of the function.

To test whether each of these critical points is a local extremum, we consider the sign of $g'(x)$ on the intervals on each side of the critical points and then apply the First Derivative Test. Note that $(x-6)^2$ is positive for all $x \neq 6$, and $(x+30)^5$ is positive for all $x > -30$. Finally, note that $x - 10$ is positive for $x > 10$ and negative for $x < 10$. We summarize the resulting signs of the derivative in the table below.

Interval	$-30 < x < 6$	$6 < x < 10$	$x > 10$
Sign of $g'(x)$	$\frac{-\cdot+\cdot-}{+} = +$	$\frac{-\cdot+\cdot-}{+} = +$	$\frac{-\cdot+\cdot+}{+} = -$

We see that $g(x)$ does not have a local extremum at $x = 6$ and that, by the First Derivative Test, $g(x)$ has a local maximum at $x = 10$.

(For each answer blank below, write NONE in the answer blank if appropriate.)

Answer: critical point(s) at $x =$ 6, 10

Answer: local min(s) at $x =$ **NONE**

Answer: local max(es) at $x =$ 10

b. [5 points] Find the values of x that minimize and maximize $g(x)$ on the interval $[0, \infty)$. Use calculus to find your answers, and be sure to show enough evidence that the points you find are indeed global extrema.

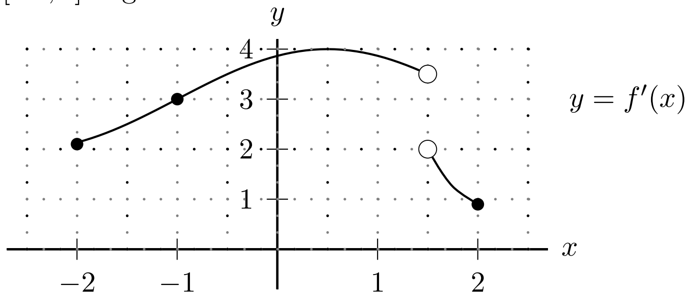
Solution: The only local extremum is the local maximum at $x = 10$ so since g is continuous on the interval $[0, \infty)$,this must also be the global max. (Alternatively, note that from part (a), we know that $g(x)$ is increasing for $0 \leq x \leq 10$ and decreasing for $x \geq 10$, so $g(10)$ must be the global maximum value of $g(x)$ on the interval $[0, \infty)$.) To check for a global min we need to consider the ends of the interval. Now, $g(0) = \frac{1}{150}$ and $\lim_{x\to\infty} g(x) = 0$, which is less than $\frac{1}{150}$. This implies that there is no global minimum of $g(x)$ on the interval $[0, \infty)$.

(For each answer blank below, write NONE in the answer blank if appropriate.)

Answer: global min(s) at $x =$ **NONE**

Answer: global max(es) at $x =$ 10

3. [8 points] Suppose $f(x)$ is a function that is continuous on the interval $[-2, 2]$. The graph of $f'(x)$ on the interval $[-2, 2]$ is given below.



a. [3 points] Let $L(x)$ be the local linearization of $f(x)$ at $x = -1$. Using the fact that $f(-1) = 4$, write a formula for $L(x)$.

Solution: $f(-1) = 4$ and $f'(-1) = 3$, so $L(x) = 4 + 3(x - (-1)) = 4 + 3(x + 1)$.

Answer: $L(x) =$ $4 + 3(x + 1)$ **or** $3x + 7$

b. [2 points] Use your formula for $L(x)$ to approximate $f(-0.5)$.

Solution: Since -0.5 is close to -1 we have

$$f(-0.5) \approx L(-0.5) = 4 + 3(-0.5 + 1) = 43(0.5) = 5.5.$$

Answer: $f(-0.5) \approx$ 5.5

c. [3 points] Is your answer from part (b) an overestimate or an underestimate of the actual value of $f(-0.5)$? Justify your answer.

Circle one: overestimate [underestimate] CANNOT BE DETERMINED

Justification:

Solution: The function $f'(x)$ is increasing between -2 and 0 so $f(x)$ is concave up over this interval. Therefore the tangent line to the graph of $f(x)$ at $x = -1$ lies below the graph of $f(x)$ between $x = -2$ and $x = 0$. In particular, the local linearization $L(x)$ of $f(x)$ at $x = -1$ gives an underestimate of f on that interval.

4. [11 points] Elphaba has found a corrupt prison guard, Mert, to sell her metal piping to use to dig a tunnel out of the prison. Mert can sell Elphaba steel piping and copper piping, and he provides the following information.

- The number of kilograms (kg) of soil that Elphaba can dig with steel piping is proportional to the number of centimeters (cm) of steel piping that she buys. She can dig 50 kg of soil per cm of steel piping, and her cost (in dollars) of buying x cm of steel piping is given by $A(x) = x^2 + x$.
- The number of kilograms (kg) of soil that Elphaba can dig with copper piping is proportional to the number of centimeters (cm) of copper piping that she buys. She can dig 30 kg of soil per cm of copper piping, and her cost (in dollars) of buying y cm of copper piping is given by $B(y) = 2y$.

a. [1 point] How many kilograms of soil can Elphaba dig with x cm of steel piping?

Answer: $50x$

For parts (b)-(d) below, suppose Elphaba buys w cm of steel piping and k cm of copper piping and that this is exactly the right amount of piping so that she can dig through 2700 kg of soil to dig her escape tunnel.

b. [3 points] Write a formula for k in terms of w.

Solution: With w cm of steel piping, she can dig through $50w$ kg of soil and with k cm of copper piping, she can dig through $30k$ kg of soil. So $50w + 30k = 2700$, and solving for k, we find $k = \frac{2700-50w}{30}$.

Answer: $k = \dfrac{2700 - 50w}{30} = 90 - \dfrac{5}{3}w$

c. [4 points] Let $T(w)$ be the total cost (in dollars) of all the piping Elphaba buys to dig her escape tunnel. Find a formula for the function $T(w)$. The variable k and the function names A and B should not appear in your answer.
(Note that $T(w)$ is the function one would use to minimize Elphaba's costs. You should not do the optimization in this case.)

Solution: $T(w) = A(w) + B\left(\dfrac{2700 - 50w}{30}\right) = w^2 + w + 2\left(\dfrac{2700 - 50w}{30}\right)$.

Answer: $T(w) = w^2 + w + 2\left(\dfrac{2700 - 50w}{30}\right)$ **or** $w^2 - \dfrac{7}{3}w + 18$

d. [3 points] What is the domain of $T(w)$ in the context of this problem?

Solution: In the context of this problem, the smallest possible value of w is 0, which would occur if Elphaba were to buy only copper piping. The largest possible value of w would occur if Elphaba were to buy only steel piping. In that case, $50w = 2700$ so $w = \frac{2700}{50} = 54$. In the context of this problem, the domain of $T(w)$ consists of all values of w between 0 and 54, i.e. the interval $[0, 54]$.

Answer: $[0, 54]$

5. [9 points] Consider the curve $\mathcal{C}$ defined by

$$e^{\pi xy} = ay^2 + x^2$$

where a is a positive constant.

a. [6 points] For this curve $\mathcal{C}$, find a formula for $\frac{dy}{dx}$ in terms of x and y. The constant a may appear in your answer. Remember to show every step of your work clearly.

Solution: We differentiate both sides of the equation defining $\mathcal{C}$ with respect to x and then solve for $\frac{dy}{dx}$.

$$\begin{aligned}
\frac{d}{dx}\left(e^{\pi xy}\right) &= \frac{d}{dx}\left(ay^2 + x^2\right) \\
e^{\pi xy}\frac{d}{dx}(\pi xy) &= 2ay\frac{dy}{dx} + 2x \\
\pi e^{\pi xy}\left(x\frac{dy}{dx} + y\right) &= 2ay\frac{dy}{dx} + 2x \\
\pi x e^{\pi xy}\frac{dy}{dx} + \pi y e^{\pi xy} &= 2ay\frac{dy}{dx} + 2x \\
\pi x e^{\pi xy}\frac{dy}{dx} - 2ay\frac{dy}{dx} &= 2x - \pi y e^{\pi xy} \\
\frac{dy}{dx}\left(\pi x e^{\pi xy} - 2ay\right) &= 2x - \pi y e^{\pi xy} \\
\frac{dy}{dx} &= \frac{2x - \pi y e^{\pi xy}}{\pi x e^{\pi xy} - 2ay}
\end{aligned}$$

Answer: $\frac{dy}{dx} =$ $\frac{2x - \pi y e^{\pi xy}}{\pi x e^{\pi xy} - 2ay}$

b. [1 point] Let $a = 1$. Exactly one of the following points (x, y) lies on the curve $\mathcal{C}$. Circle that one point.

$(0, 3)$ $\quad$ $(1, 2)$ $\quad$ $(2, -1)$ $\quad$ **[$(0, -1)$]** $\quad$ $(e^{\pi}, 0)$

Solution: When $a = 1$, the point $(0, -1)$ satisfies the equation defining the curve $\mathcal{C}$.

c. [2 points] With $a = 1$ as above, is the tangent line to the curve $\mathcal{C}$ at the point you chose in (b) increasing, decreasing, or is there not enough information to determine this? Circle your one choice and then justify your answer.

The tangent line to the curve $\mathcal{C}$ at the point circled in (b) is

i. **[increasing.]** $\quad$ ii. decreasing. $\quad$ iii. NOT ENOUGH INFORMATION

Justification:

Solution: The slope of this tangent line is $\frac{dy}{dx}$ evaluated at $(0, -1)$, i.e. the slope of the tangent line is $\frac{2(0)-\pi(-1)e^{\pi(0)(-1)}}{\pi(0)e^{\pi(0)(-1)}-2(1)(-1)} = \frac{\pi}{2}$. Since $\frac{\pi}{2} > 0$, the tangent line has positive slope so is increasing.

6. [14 points] Let p be a function such that $p''(x)$ is defined for all real numbers x. A table of some values of $p'(x)$ is given below.

x	-9	-5	-1	3	7	11
$p'(x)$	-3	0	-4	0	2	1

Assume that p' is either always strictly decreasing or always strictly increasing between consecutive values of x shown in the table.

For each of the questions below, circle ALL of the appropriate choices. If none of the choices are correct, circle NONE OF THESE.

a. [2 points] At which, if any, of the following values of x does $p(x)$ definitely have a local maximum in the interval $-9 < x < 11$?

-5 -1 3 7 NONE OF THESE

Solution: The only critical points of $p(x)$ in this interval are at $x = -5$ and $x = 3$. We now classify these critical points. $p(x)$ does not have a local extremum at $x = -5$ since $p'(x)$ is negative both immediately before and after $x = -5$. $p(x)$ has a local minimum at $x = 3$ since the sign of $p'(x)$ changes from negative to positive there.

b. [2 points] At which, if any, of the following values of x does $p(x)$ definitely attain its global minimum on the interval $-9 \le x \le 11$?

-9 -5 -1 3 7 11 NONE OF THESE

Solution: $p(x)$ is decreasing for $-9 \le x \le 3$ and increasing for $3 \le x \le 11$.

c. [2 points] At which, if any, of the following values of x does $p'(x)$ (the derivative of $p(x)$) definitely attain its global maximum on the interval $-9 \le x \le 11$?

-9 -5 -1 3 7 11 NONE OF THESE

Solution: Since $p'(x)$ is always increasing of decreasing between points in the table, the global max must occur at one of the values of x in the table. Realizing this, we need to choose the value of x in the table that gives the largest value of $p'(x)$, which is $x = 7$.

d. [3 points] On which of the following intervals is $p(x)$ definitely always concave up?

$-9 < x < -5$ $-5 < x < -1$ $-1 < x < 3$ $3 < x < 7$ $7 < x < 11$ NONE OF THESE

Solution: The function $p(x)$ is concave up whenever the derivative $p'(x)$ is increasing.

e. [3 points] At which, if any, of the following values of x does $p(x)$ definitely have an inflection point in the interval$-9 < x < 11$?

-5 -1 3 7 NONE OF THESE

Solution: We are looking for places where $p'(x)$ changes from increasing to decreasing or vice versa.

f. [2 points] Which, if any, of the following must be true?

$p''(7) \ge p''(-3)$ $p''(7) = p''(-3)$ $p''(7) \le p''(-3)$ NONE OF THESE

Solution: The function $p'(x)$ is increasing between 3 and 7 and decreasing between 7 and 11 so $p''(7) = 0$. The function $p'(x)$ is decreasing between -5 and -1, so $p''(-3) \le 0$.

7. [10 points] To aid in Elphaba's escape, Walt has concocted a supplement that will make her stronger and more agile. The concentration of the supplement in Elphaba's system, in mg/ml, t minutes after it is administered is given by the following formula:

$$T(t) = \begin{cases} at^3 & 0 \le t \le 5 \\ b(t-6)^2 + 10 & 5 < t \le 7 \end{cases}$$

where a and b are constants.

a. [7 points] Given that $T(t)$ is differentiable, find a and b. Give your answers in exact form.

Solution: Because the two pieces of the function are polynomials, the only point at which the function could fail to be differentiable is at $t = 5$. In order to be differentiable at $t = 5$, $T(t)$ must be continuous there. So $\lim_{t\to 5^-} T(t) = \lim_{t\to 5^+} T(t)$. Now,

$$\lim_{t\to 5^-} T(t) = \lim_{t\to 5^-} at^3 = a(5^3) = 125a$$

and

$$\lim_{t\to 5^+} T(t) = \lim_{t\to 5^+} b(t-6)^2 + 10 = b(5-6)^2 + 10 = b + 10.$$

So we must have $125a = b + 10$.

In order for $T(t)$ to be differentiable at $t = 5$, the slope of the tangent line to the graph of $y = at^3$ and $t = 5$ must be the same as the slope of the tangent line to the graph of $y = b(t-6)^2 + 10$ at $t = 5$. The slope of the tangent line to the graph of $y = at^3$ at $t = 5$ is $3at^2$ evaluated at $t = 5$, which is $75a$. The slope of the tangent line to the graph of $y = b(t-6)^2 + 10$ at $t = 5$ is $2b(t-6)$ evaluated at $t = 5$, which is $-2b$. Hence $75a = -2b$.

So in order for $T(t)$ to be differentiable, we must have $125a = b + 10$ and $75a = -2b$. Solving these two equations simultaneously we find $a = \frac{4}{65}$ and $b = -\frac{30}{13}$.

Answer: $a =$ $\frac{4}{65}$ and $b =$ $-\frac{30}{13}$

b. [3 points] Using the values of a and b you found in part (a), give a formula for the tangent line to the graph of $y = T(t)$ at $t = 5$.

Solution: $T(5) = a(5^3) = \frac{4}{65}(125) = \frac{100}{13}$ and $T'(5) = a(3(5^2)) = \frac{4}{65}(75) = \frac{60}{13}$. So a formula for the tangent line to the graph of $y = T(t)$ at $t = 5$ is $y = \frac{100}{13} + \frac{60}{13}(t-5)$.

Answer: $y =$ $\frac{100}{13} + \frac{60}{13}(t-5)$ **or** $\frac{60}{13}t - \frac{200}{13}$

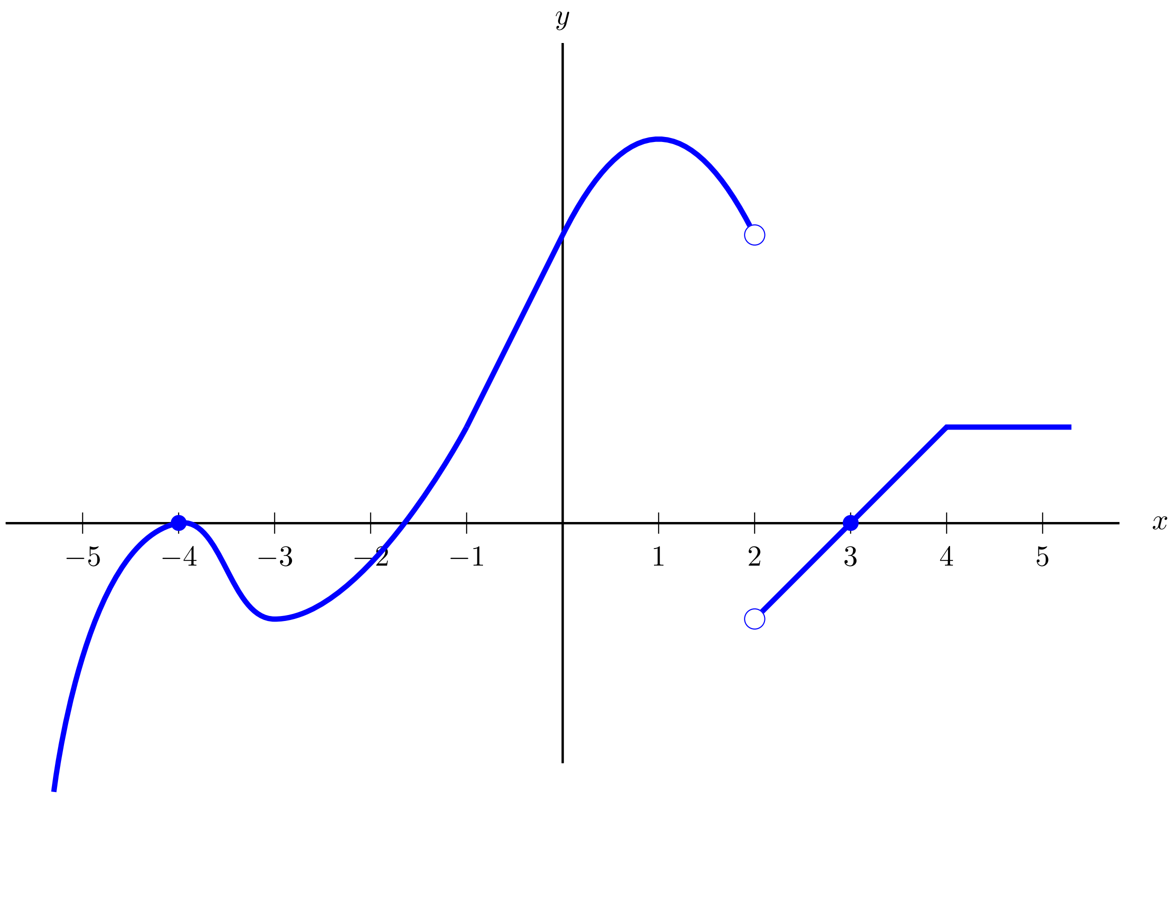
8. [10 points] A function $h(x)$ satisfies all of the following:

- $h(x)$ is continuous on the interval $-5 < x < 5$.
- $h(x)$ is differentiable for all x in the interval $-5 < x < 5$ except at $x = 2$.
- $h(x)$ is decreasing for $-5 < x < -2$.
- $h(x)$ has a critical point at $x = -4$.
- $h(x)$ is concave up for $-3 < x < -1$.
- $h(x)$ has an inflection point at $x = 1$.
- $h(x)$ has a local minimum at $x = 3$.
- $h(x)$ is increasing at a constant rate for $4 < x < 5$.

On the axes provided below, sketch a possible graph of $h'(x)$ (the derivative of $h(x)$).
Make sure that your sketch is large and unambiguous.

Graph of $y = h'(x)$

Solution: Below is one possible graph.



9. [9 points] Elphaba and Walt are planning to break out of prison. They would like to escape no later than 20 hours after devising their plan, and they would like to attempt their escape during the noisiest part of the day. Let $N(t)$ be the noise level (in decibels) in the prison t hours after Elphaba and Walt have devised their escape plan.
On the interval $[0, 20]$, a formula for $N(t)$ is given by

$$N(t) = 60 + 1.01^{p(t)} \quad \text{where} \quad p(t) = \frac{1}{3}t^3 - 9t^2 + 56t + 200.$$

a. [8 points] Find the values of t that minimize and maximize $N(t)$ on the interval $[0, 20]$. Use calculus to find your answers, and be sure to show enough evidence that the points you find are indeed global extrema.

Solution: Since $N(t)$ is continuous on the interval $[0, 20]$, we can apply the Extreme Value Theorem and compare the values of $N(t)$ at the critical points and endpoints of the interval.
We first need to find the critical points in this interval. Taking the derivative of $N(t)$ and setting it equal to zero we have $N'(t) = \ln(1.01)p'(t)(1.01)^{p(t)} = 0$ so critical points occur when $0 = p'(t) = t^2 - 18t + 56$. Solving we determine that the only critical points of $N(t)$ occur at $t = 4$ and $t = 14$, which are both in the interval $[0, 20]$.
To find the global extrema we need to evaluate the function at $t = 0, 4, 14, 20$. We find $N(0) \approx 67.316$, $N(4) \approx 80.0528$, $N(14) \approx 63.819$ and $N(20) \approx 106.874$. Choosing the largest and smallest values, by the Extreme Value Theorem, we see that the global minimum occurs at $t = 14$ and the global maximum occurs at $t = 20$.

(For each answer blank below, write NONE in the answer blank if appropriate.)

Answer: global min(s) at $t =$ 14

Answer: global max(es) at $t =$ 20

b. [1 point] As mentioned above, Elphaba and Walt would like to escape no later than 20 hours after devising their plan, and they would like to escape during the noisiest part of the day. When should Elphaba and Walt attempt their escape?

Answer: They should try to escape 20 hours after devising their plan.

10. [5 points] Suppose $g(x)$ is a differentiable function defined for all real numbers that satisfies the following properties:

- $g(x)$ has exactly two critical points.
- $g(x)$ has a local maximum at $x = 0$ and $g(0) = 4$.
- $g(x)$ has a local minimum at $x = 2$ and $g(2) = 1$.
- $\lim_{x \to -\infty} g(x) = 3$.
- $\lim_{x \to \infty} g'(x) = 1$.

Circle <u>all</u> of the statements below that must be true about the function g or circle NONE OF THESE if none of the statements must be true.

i. The function $g(x)$ is increasing on the entire interval $x < 0$.

ii. The function $g(x)$ is increasing on the entire interval $0 < x < 2$.

iii. The function $g(x)$ is increasing on the entire interval $x > 2$.

iv. On its domain $(-\infty, \infty)$, the function $g(x)$ attains its global maximum at $x = 0$.

v. On its domain $(-\infty, \infty)$, the function $g(x)$ attains its global minimum at $x = 2$.

vi. On its domain $(-\infty, \infty)$, the function $g(x)$ does not have a global maximum value.

Solution: Note that the fact that $\lim_{x \to \infty} g'(x) = 1$ tells us that $g(x) \to \infty$ as $x \to \infty$, since if $g(x)$ approached a finite limit, its derivative would have to approach zero.

vii. On its domain $(-\infty, \infty)$, the function $g(x)$ does not have a global minimum value.

viii. NONE OF THESE