Math 115 — Final Exam

April 23, 2015

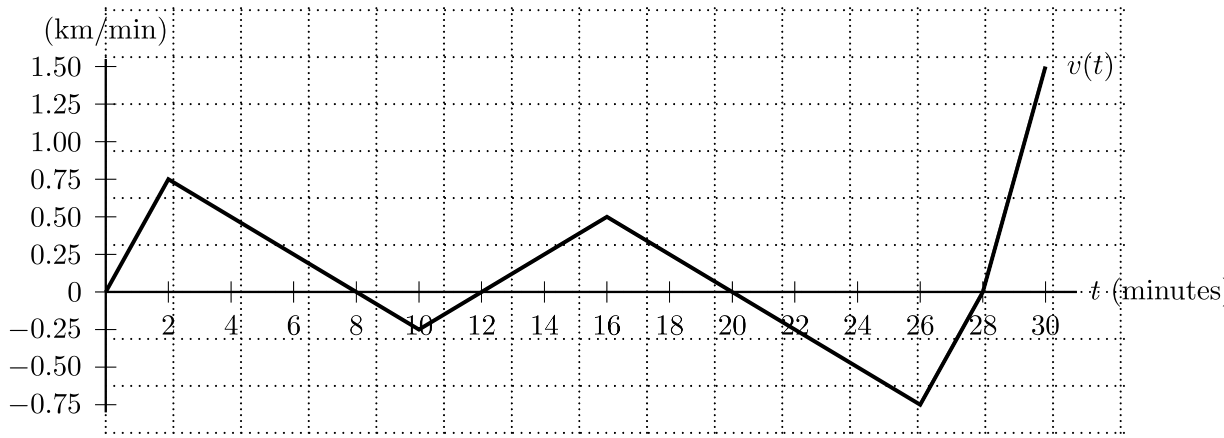
Last Name Only: EXAM SOLUTIONS

Instructor Name: ______________________ Section #: ________

1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 11 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
8. Include units in your answer where that is appropriate.
9. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones.
10. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	10	
2	13	
3	14	
4	12	
5	5	
6	6	
7	10	
8	11	
9	10	
10	5	
11	4	
Total	100	

1. [10 points] Unfortunately, Sebastian left the King's castle but never made it to Adam's manor because the brakes on his car were sabotaged. Sebastian was driving on a straight road between the King's castle and Adam's manor when he found himself unable to brake and racing down a hill. Let $v(t)$ be Sebastian's velocity (in kilometers per minute) t minutes after he left the King's castle. Note that $v(t)$ is positive when Sebastian is traveling towards Adam's manor. Sebastian suspected he was being followed so he occasionally backtracked.
Sebastian crashed 30 minutes into his journey. A graph of $v(t)$ is given below.



a. [3 points] How far from the King's castle was Sebastian 12 minutes into his journey? *Include units.*

Solution: Since Sebastian initially started at the King's castle, his distance from it after 12 minutes is given by $\int_0^{12} v(t)\,dt$. To calculate this we need to calculate the signed area between the graph of $v(t)$ and the t-axis. Therefore,

$$\int_0^{12} v(t)\,dt = (0.5)(8)(0.75) - (0.5)(4)(0.25) = 2.5 \text{ km}.$$

(Note that 0.5 is the area of each box in the graph.)

Answer: **2.5 km**

b. [2 points] What was Sebastian's average velocity during the first 12 minutes of his journey?

Solution: Sebastian's average velocity during the first 12 minutes is given by the equation

$$frac112 \int_0^{12} v(t)\,dt = \frac{2.5}{12}.$$

Answer: $\frac{2.5}{12}$ **km/min**

c. [2 points] Of the four times below, circle the one at which Sebastian's acceleration was the greatest (i.e. most positive).

$t = 6$ $\qquad$ $t = 13$ $\qquad$ $t = 20$ $\qquad$ $\boxed{t = 27}$

d. [3 points] In the interval $0 \le t \le 30$ when was Sebastian the closest to the King's castle? When was he the furthest from the King's castle?

Answer: Sebastian was the closest to the King's castle at $t =$ **0**.

Sebastian was the furthest from the King's castle at $t =$ **20**.

2. [13 points] For nonzero constants a and b with $b > 0$, consider the family of functions given by

$$f(x) = e^{ax} - bx\,.$$

Note that the derivative and second derivative of $f(x)$ are given by

$$f'(x) = ae^{ax} - b \qquad \text{and} \qquad f''(x) = a^2e^{ax}\,.$$

a. [6 points] Suppose the values of a and b are such that $f(x)$ has at least one critical point. For the domain $(-\infty, \infty)$, find all critical points of $f(x)$, all values of x at which $f(x)$ has a local extremum, and all values of x at which $f(x)$ has an inflection point.
Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all local extrema and inflection points.
(Note that your answer(s) may involve the constants a and/or b.)

Solution: To find critical points we (i) solve for x in the equation $f'(x) = 0$ and (ii) look for points in the domain of the function where the derivative is undefined. Solving $f'(x) = 0$ we have $ae^{ax} - b = 0$ so

$$e^{ax} = \frac{b}{a} \tag{1}$$

and we see that there is a critical point at $x = \frac{\ln(\frac{b}{a})}{a}$ (as long as $b/a > 0$).
The derivative is defined everywhere so this is the only critical point. (Since we are to assume there is at least one critical point, note that this implies that b/a must be positive.) To classify this critical point we can use the second derivative test. Since $a \neq 0$, we have that $f''(x) > 0$ for all x, so the function $f(x)$ is always concave up. Therefore, the critical point at $x = \frac{\ln(\frac{b}{a})}{a}$ is a local min.
To find points of inflection we (i) solve for x in the equation $f''(x) = 0$, (ii) look for any points in the domain of the function where the second derivative is undefined and (iii) check that $f''(x)$ changes sign at any points found. For the function $f(x)$, $f''(x)$ is never 0 and is defined for all x so we have no points of inflection.

(For each answer blank below, write NONE in the answer blank if appropriate.)

critical point(s) at $x =$ $x = \dfrac{\ln(\frac{b}{a})}{a}$ local min(s) at $x =$ $x = \dfrac{\ln(\frac{b}{a})}{a}$

inflection point(s) at $x =$ **NONE** local max(es) at $x =$ **NONE**

b. [2 points] Which of the following conditions on the constant a guarantee(s) that $f(x)$ has at least one critical point in its domain $(-\infty, \infty)$? Circle all the cases in which $f(x)$ definitely has at least one critical point. *Hint: There is at least one such condition listed.*

i. $a < 0$ ii. $\boxed{0 < a < b}$ iii. $\boxed{b < a}$

Solution: When finding the critical point in part (a), we had to solve equation (1), which only has a solution if $\frac{b}{a} > 0$. Since $b > 0$, this means that $f(x)$ has a critical point if and only if $a > 0$, which is true both if $0 < a < b$ and if $b < a$ (since $b > 0$).

c. [5 points] Find exact values of a and b so that $f(x)$ has a critical point at $(1, 0)$. Remember to show your work carefully.

Solution: First, $(1, 0)$ must lie on the curve so we get the equation $e^a - b = 0$ which implies $b = e^a$. Next, $x = 1$ must be a critical point so $ae^a - b = 0$. Substituting $b = e^a$ into the second equation we get $ae^a - e^a = 0$ which implies $a = 1$. So $a = 1$ and $b = e$.

Answer: $a =$ **1** and $b =$ **e**

3. [14 points] Let g be a differentiable function defined for all real numbers. A table of some values of g is given below.

w	-1	1	3	5
$g(w)$	-2	3	5	6

Assume that g is always strictly increasing on the interval $[-1, 5]$ and that g' is always strictly decreasing on the interval $[-1, 5]$.

a. [2 points] Estimate $g'(5)$.

Solution: $g'(5) \approx \frac{g(5)-g(3)}{5-3} = \frac{1}{2}$.

Answer: $g'(5) \approx$ $\frac{1}{2}$

b. [4 points] Rank the following quantities in order from least to greatest by filling in the blanks below with the options I-V.

I. 0 II. $g'(1)$ III. $g(1) - g(-1)$ IV. $g'(3)$ V. $\dfrac{g(3) - g(1)}{2}$

$\mathbf{0}$ < $g'(3)$ < $\dfrac{g(3) - g(1)}{2}$ < $g'(1)$ < $g(1) - g(-1)$

c. [4 points] Find the best possible estimate of $\int_{-1}^{5} (g(w) + 1)\, dw$ using a right hand sum and the data provided. Be sure to write all of the terms in the sum.

Solution:

$$\begin{aligned}\int_{-1}^{5} (g(w) + 1)\, dw &\approx \Delta w((g(1) + 1) + (g(3) + 1) + (g(5) + 1)) \\ &= 2(4 + 6 + 7) \\ &= 34.\end{aligned}$$

d. [1 point] Is your estimate from part (c) an overestimate or underestimate of $\int_{-1}^{5} (g(w) + 1)\, dw$?
You do not need to explain your answer.

Underestimate $\boxed{\text{Overestimate}}$ Impossible to determine

Solution: The function $g(w) + 1$ is always increasing (since it is a vertical shift of $g(w)$, which is always increasing) so the right hand sum gives an overestimate.

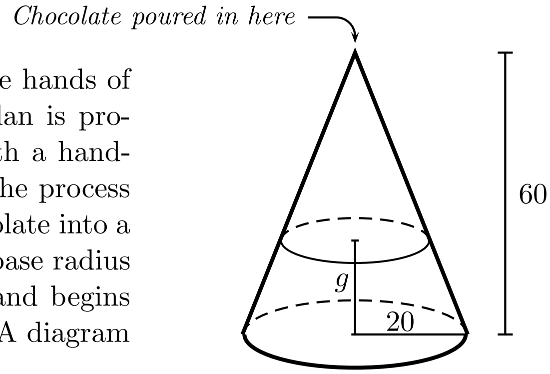
e. [3 points] Find the average value of $g'(w)$ on the interval $[-1, 5]$.

Solution: By definition, the average value of $g'(w)$ on $[-1, 5]$ is

$$\begin{aligned}g'(w) &= \frac{1}{6}\int_{-1}^{5} (g'(w))\, dw \\ &= \frac{1}{6}\left[g(5) - g(-1)\right] \\ &= \frac{8}{6} = \frac{4}{3}.\end{aligned}$$

(The average value of g' on the interval is the average rate of change of g over the interval.)

Answer: $\frac{4}{3}$



4. [12 points]
Having taken care of Sebastian and sent Erin into the hands of the *Illumisqati*, King Roderick is pleased that his plan is proceeding well. Our wicked villain decides to relax with a handmade chocolate before he heads to his farmhouse. The process of making the chocolate involves pouring molten chocolate into a mould. The mould is a cone with height 60 mm and base radius 20 mm. Roderick places the mould on the ground and begins pouring the chocolate through the apex of the cone. A diagram of the situation is shown on the right.

In case they are helpful, recall the following formulas for a cone of radius r and height h:

$$\text{Volume} = \frac{1}{3}\pi r^2 h \quad \text{and} \quad \text{Surface Area} = \pi r(r + \sqrt{h^2 + r^2}).$$

a. [6 points] Let g be the depth of the chocolate, in mm, as shown in the diagram above. What is the value of g when Roderick has poured a total of 20,000 mm^3 of chocolate into the mould? *Show your work carefully, and make sure your answer is accurate to at least two decimal places.*

Solution: The volume of the solid is given by $V = \frac{1}{3}\pi(20)^2 60 - \frac{1}{3}\pi r^2(60 - g)$ where r is the radius of the cross-section at height g. We want to rewrite r in terms of g. Using similar triangles we find the equation

$$\frac{r}{20} = \frac{60 - g}{60},$$

which implies $r = \frac{60-g}{3}$. Therefore, $V = 8000\pi - \frac{\pi}{27}(60 - g)^3$. So, to find the appropriate g we need to solve $8000\pi - \frac{\pi}{27}(60 - g)^3 =$20,000. Solving, we get

$$(60 - g)^3 = \frac{27}{\pi}(8000\pi - 20{,}000), \tag{2}$$

which implies $g = 60 - \sqrt[3]{\frac{27}{\pi}(8000\pi - 20{,}000)} \approx 24.67$. The chocolate is approximately 24.67 mm deep when he has poured a total of 20,000 mm^3 of chocolate into the mould.

Answer: $g \approx$ **24.67**

b. [6 points] How fast is the depth of the chocolate in the mould (g in the diagram above) changing when Roderick has already poured 20,000 mm^3 of chocolate into the mould if he is pouring at a rate of 5,000 mm^3 per second at this time? *Show your work carefully and make sure your answer is accurate to at least two decimal places. Be sure to include units.*

Solution: We want to find $\frac{dg}{dt}$ when $g = 60 - \sqrt[3]{\frac{27}{\pi}(8000\pi - 20{,}000)}$ (from part (a)) if $\frac{dV}{dt} = 5000$ at this time. Differentiating our formula (2) from part (a) with respect to t, we have

$$\frac{dV}{dt} = \frac{\pi}{9}(60 - g)^2 \frac{dg}{dt}.$$

Substituting $g = 60 - \sqrt[3]{\frac{27}{\pi}(8000\pi - 20{,}000)}$ and $\frac{dV}{dt} = 5000$ into this equation, we find

$$5000 = \frac{\pi}{9}\left(60 - \left[60 - \sqrt[3]{\frac{27}{\pi}(8000\pi - 20{,}000)}\right]\right)^2 \frac{dg}{dt} \quad \text{so}$$

$$\frac{dg}{dt} = \frac{5000 \cdot 9}{\pi}\left(\frac{27}{\pi}(8000\pi - 20{,}000)\right)^{-2/3} \approx 11.47.$$

(Using our approximation $g \approx 24.67$ instead gives us $\frac{dg}{dt} \approx 11.48$.)
So the depth of the chocolate is increasing at an instantaneous rate of about 11.47 mm/sec at that moment.

Answer: **11.47 mm/ sec.**

5. [5 points] After hearing of the *Illumisqati* activities from Erin and Elphaba, the Police storm the King's farmhouse and find ample evidence to convict him of kidnapping. However, since he is the King, charges can only be brought against him if the Police can show proficiency in mathematics. Help them by doing the following problem.

For c a constant, consider the function $B(u) = \arctan(u^c + 7)$.

Use the limit definition of the derivative to write an explicit expression for $B'(3)$.

Your answer should not involve the letter B. Do not attempt to evaluate or simplify the limit.

Answer: $B'(3) = \boxed{\lim_{h\to 0} \dfrac{\arctan((3+h)^c+7) - \arctan(3^c+7)}{h}}$

6. [6 points] Recall the following definitions:

- A function f is *even* if $f(-x) = f(x)$ for all x in the domain of f.
- A function f is *odd* if $f(-x) = -f(x)$ for all x in the domain of f.

Compute each of the integrals below. If not enough information is provided to answer the question, write NOT ENOUGH INFORMATION.

a. [2 points] Suppose g is a differentiable function on $(-\infty, \infty)$ and g' (the **derivative** of g) is a continuous odd function with $g(3) = 2$ and $g(7) = 9$. Find $\displaystyle\int_{-3}^{7} g'(x)\,dx$.

Solution: Since $g'(x)$ is odd, $\int_{-3}^{3} g'(x)\,dx = 0$ so we have

$$\int_{-3}^{7} g'(x)\,dx = \int_{3}^{7} g'(x)\,dx = g(7) - g(3) = 9 - 2 = 7.$$

Answer: $\displaystyle\int_{-3}^{7} g'(x)\,dx =$ 7

b. [2 points] Suppose that q is a continuous and even function on $(-\infty, \infty)$ and that $\displaystyle\int_{0}^{5} q(x)\,dx = -4$. Find $\displaystyle\int_{-5}^{5} (3q(x) + 7)\,dx$.

Solution: By the linearity properties of definite integrals,

$$\int_{-5}^{5} (3q(x)+7)\,dx = 3\left(\int_{-5}^{5} q(x)\,dx\right) + \int_{-5}^{5} 7\,dx = 3\left(\int_{-5}^{5} q(x)\,dx\right) + 70.$$

Since $q(x)$ is even, $\displaystyle\int_{-5}^{5} q(x)\,dx = 2\int_{0}^{5} q(x)\,dx = -8$.

Therefore, $\displaystyle\int_{-5}^{5} (3q(x)+7)\,dx = 3(-8) + 70 = 46$.

Answer: $\displaystyle\int_{-5}^{5} (3q(x)+7)\,dx =$ 46

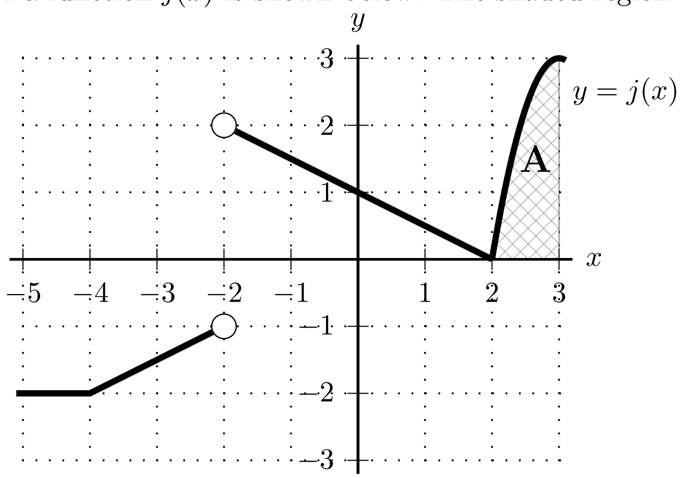
c. [2 points] Let $h(x) = \ln x$ and suppose p is a differentiable function on $(-\infty, \infty)$ with $p(4) = 7$. Find $\displaystyle\int_{4}^{1} (h(x)p'(x) + h'(x)p(x))\,dx$.

Solution: Note that $h(x)p(x)$ is an antiderivative of $h(x)p'(x) + h'(x)p(x)$ so by the Fundamental Theorem of Calculus,

$$\int_{4}^{1} (h(x)p'(x) + h'(x)p(x))\,dx = h(1)p(1) - h(4)p(4) = \ln(1)p(1) - \ln(4)p(4) = -7\ln(4).$$

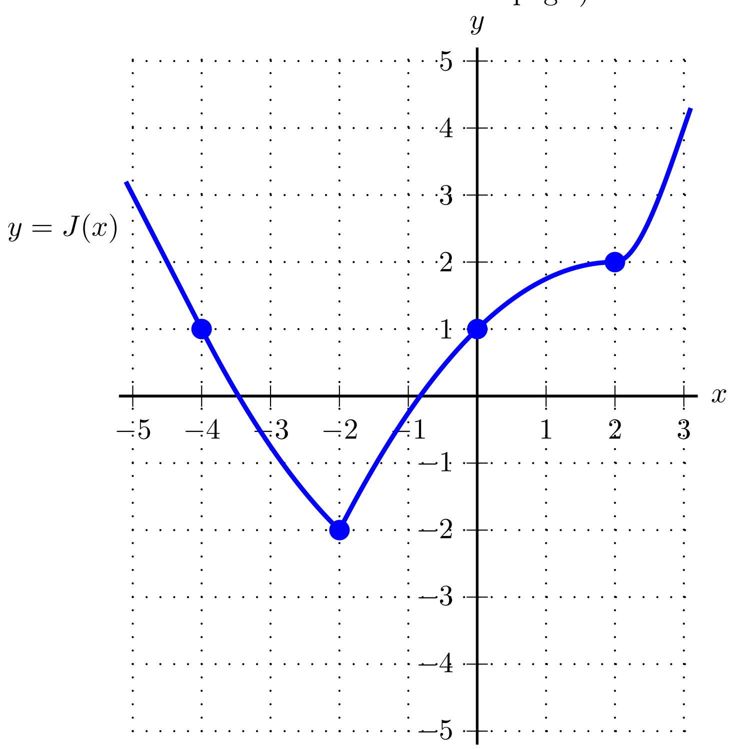
Answer: $\displaystyle\int_{4}^{1} (h(x)p'(x) + h'(x)p(x))\,dx =$ $-7\ln(4)$

7. [10 points] The graph of a function $j(x)$ is shown below. The shaded region A has area 2.



On the axes provided below, sketch a well-labeled graph of an antiderivative of $J(x)$ of $j(x)$ that is defined and continuous on the interval $-5 \leq x \leq 3$ and that satisfies $J(0) = 1$.
Be sure that you pay close attention to each of the following:

- the value of $J(x)$ at each of its critical points and inflection points (Be sure to also write this data in the answer blanks at the bottom of the page.)
- where J is/is not differentiable
- where J is increasing/decreasing/constant
- the concavity of the graph of $y = J(x)$



On the answer blanks below, write both the x– **and** y–coordinates of all critical points and all inflection points of $J(x)$. Write NONE if $J(x)$ has no such points.

Both coordinates of all critical points: $(-2,-2)$, $(2,2)$

Both coordinates of all inflection points: $(-2,-2)$, $(2,2)$

8. [11 points] Public opinion has swung against the King since his arrest. Elphaba has been travelling the Sovereign lands collecting donations of acorns to help launch an attack against the King. Let $P(x)$ be the total mass (in kg) of acorns that Elphaba has collected after she has travelled a total of x km. Let $Q(t)$ be Elphaba's velocity (in km/day) when she has been travelling for t days. You may assume that $Q(t)$ is continuous and always positive and that $P(x)$ is an increasing, differentiable function.

For each of questions (a) through (d) below, circle the one best answer. No points will be given for ambiguous or multiple answers.

a. [2 points] Circle the one equation below that best supports the following statement:
When Elphaba has travelled 100 km, she has collected approximately 3 kg less acorns than she will have collected when she has travelled 100.5 km.

i. $\boxed{P'(100) = 6}$

ii. $P'(100) = -3$

iii. $P'(100) = 1.5$

iv. $P'(100.5) = -6$

v. $P'(100.5) = 3$

vi. $P'(100.5) = -1.5$

b. [2 points] Which one of the following expressions is equal to the amount (in kg) by which Elphaba's collection of acorns increases over the course of the 50th km of her travels?

i. $P(50)$

ii. $P'(49)$

iii. $\displaystyle\int_{49}^{50} P(t)\,dt$

iv. $\boxed{\displaystyle\int_{49}^{50} P'(x)\,dx}$

c. [2 points] Which one of the following expressions is equal to the mass (in kg) of acorns that Elphaba collected during the 4th day of her travels?

i. $P'(4)$

ii. $\boxed{P\left(\displaystyle\int_0^4 Q(t)\,dt\right) - P\left(\displaystyle\int_0^3 Q(t)\,dt\right)}$

iii. $P(4) - P(3)$

iv. $P\left(\displaystyle\int_3^4 Q(t)\,dt\right)$

d. [2 points] Let m be a positive constant and let $R(t)$ be the antiderivative of $Q(t)$ such that $R(0) = 0$. Assuming that both $P(t)$ and $R(t)$ are invertible, which one of the following expressions is equal to the time (in days) it takes Elphaba to collect m kg of acorns?

i. $R(P(m))$

ii. $\boxed{R^{-1}(P^{-1}(m))}$

iii. $R(P^{-1}(m))$

iv. $P(R(m))$

v. $P^{-1}(R^{-1}(m))$

vi. $P(R^{-1}(m))$

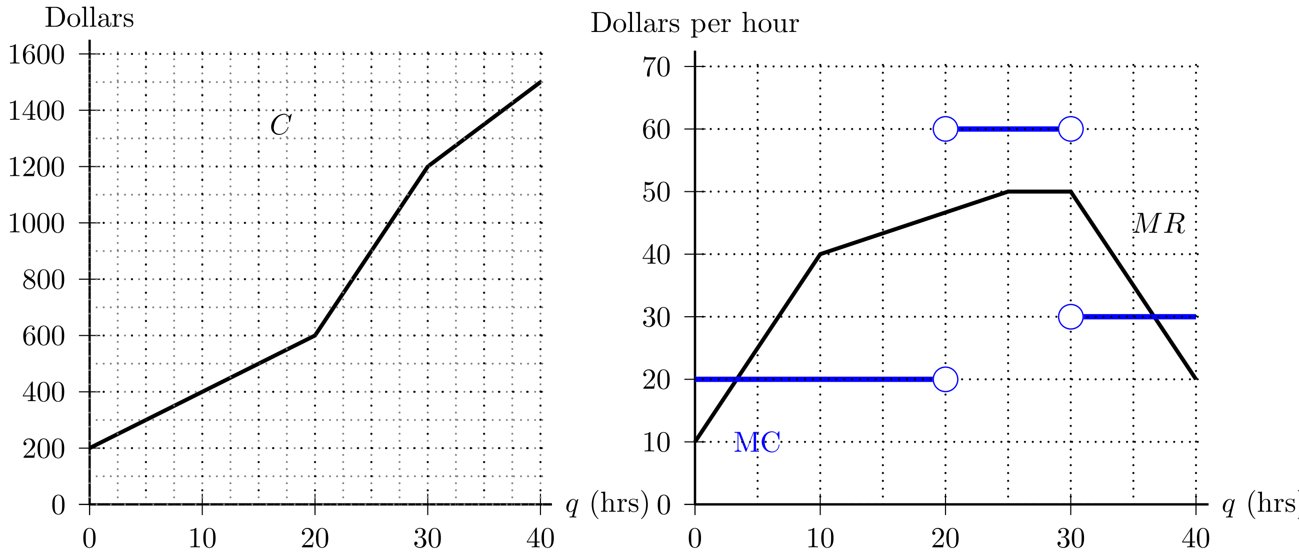
e. [3 points] Write an equation that expresses the following statement:
After Elphaba has been travelling for a total of 5 days, she has collected a total of 200 kg of acorns.

Answer: $P\left(\displaystyle\int_0^5 Q(t)\,dt\right) = 200$ **or** $\displaystyle\int_0^5 Q(t)\,dt = P^{-1}(200)$

9. [10 points] Months later, the now infamous Roderick has been dethroned. Before Erin returns to the University of Michigan, she visits Roderick to hear his side of the story. He encourages her to share his story. Erin is in fact quite a good storyteller, so she begins to consider a career as a travelling storyteller. She decides to charge clients for her time (in hours).

a. [3 points] Shown below are graphs of the cost, C, and marginal revenue, MR, of Erin's potential storytelling business. Note that both graphs are continuous and piecewise linear.

Carefully sketch the graph of Erin's marginal cost function on the same axes as the given graph of her marginal revenue. (That is, draw the graph of marginal cost on the set of axes on the right.)



b. [3 points] Let $\pi(q)$ be Erin's profit from q hours of work as a travelling storyteller. Estimate all the critical points of $\pi(q)$ for $0 < q < 40$.

Solution: To find critical points of $\pi(q)$ we (i) look for points where MC=MR and (ii) look for points where $\pi'(q) = \text{MC} - \text{MR}$ is undefined. The marginal cost equals marginal revenue when $q = \frac{10}{3}$ and $q = \frac{110}{3}$ and the derivative $\pi'(q)$ is undefined when $q = 20$ and $q = 30$.

Answer: critical point(s) at $q =$ $\frac{10}{3}$**, 20, 30,** $\frac{110}{3}$

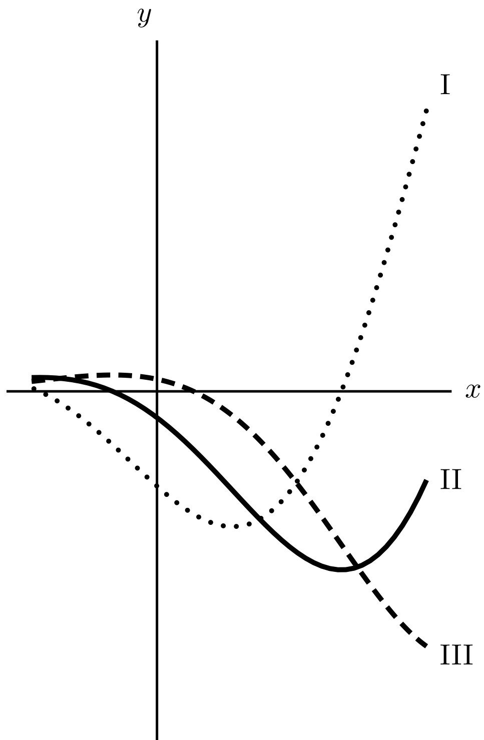
c. [4 points] If she can spend at most 40 hours on this venture, how many hours of work as a travelling storyteller should Erin do in order to maximize her profit? What is her maximum possible profit (in dollars)? (Assume that her revenue is 0 if she spends 0 hours storytelling.) *Briefly indicate your reasoning.*

Solution: By the extreme value theorem, since profit is continuous on the closed interval, maximum profit occurs and must occur at a critical point or endpoint of the interval. Since marginal profit is negative just before $q = 10/3$, $q = 30$, and $q = 40$, none of these can result in max profit. The area between the MR and MC curves between $q = 20$ and $q = 30$ (when $MR < MC$) is larger than that between $q = 30$ and $q = 110/3$ (when $MR > MC$), so profit decreases between $q = 20$ and $q = 110/3$. Hence max profit occurs at either $q = 20$ or $q = 0$. Similarly, comparing the areas between $q = 0$ and $q = 10$ to that between $q = 10/3$ and $q = 20$ we see that profit is greater at $q = 20$ than at $q = 0$, and max profit occurs at $q = 20$.

Answer: Maximum profit occurs at $q =$ 20

Maximum profit: $83 + \frac{1}{3}$ **dollars**

10. [5 points] Shown on the axes below are the graphs of $y = f(x)$, $y = f'(x)$, and $y = f''(x)$.



Determine which graph is which and circle the <u>ONE</u> correct response below.

i. • $f(x)$: I, $f'(x)$: II, and $f''(x)$: III

ii. • $f(x)$: I, $f'(x)$: III, and $f''(x)$: II

iii. • $f(x)$: II, $f'(x)$: I, and $f''(x)$: III

iv. • $f(x)$: II, $f'(x)$: III, and $f''(x)$: I

v. • $f(x)$: III, $f'(x)$: I, and $f''(x)$: II

vi. • $f(x)$: III, $f'(x)$: II, and $f''(x)$: I

11. [4 points] Suppose w and r are continuous functions on $(-\infty, \infty)$, $W(x)$ is an invertible antiderivative of $w(x)$, and $R(x)$ is an antiderivative of $r(x)$.
Circle <u>all</u> of the statements I-VI below that <u>must</u> be true.
If none of the statements must be true, circle NONE OF THESE.

I. $W(x) + R(x) + 2$ is an antiderivative of $w(x) + r(x)$.

II. $W(x) + R(x)$ is an antiderivative of $w(x) + r(x) + 2$.

III. $\cos(W(x))$ is an antiderivative of $\sin(w(x))$.

IV. $e^{W(x)}$ is an antiderivative of $w(x)e^{w(x)}$.

V. $e^{R(x)}$ is an antiderivative of $r(x)e^{R(x)}$.

VI. If w is never zero, then $W^{-1}(R(x))$ is an antiderivative of $\dfrac{r(x)}{w(W^{-1}(R(x)))}$.

Solution: To see that VI is true, we check that

$$\frac{d}{dx}\left(W^{-1}(R(x))\right) = R'(x)\frac{1}{W'(W^{-1}(R(x)))} = \frac{r(x)}{w(W^{-1}(R(x)))}.$$

VII. NONE OF THESE