Math 115 — First Midterm

February 9, 2016

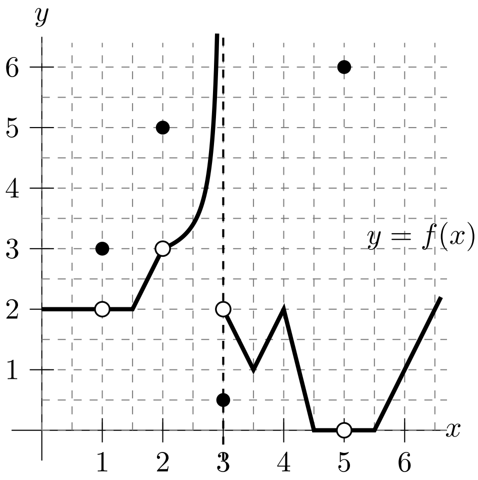
EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 11 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your initials (not name) on every page and point this out to your instructor when you hand in the exam.
4. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
5. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
6. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
7. The use of any networked device while working on this exam is not permitted.
8. You may use any calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
9. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
10. Include units in your answer where that is appropriate.
11. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches.
12. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	10	
2	11	
3	4	
4	10	
5	10	
6	12	

Problem	Points	Score
7	10	
8	4	
9	7	
10	9	
11	11	
Pre-Test	2	
Total	100	

1. [10 points] A portion of the graph of a function f is shown below.



Note: You may assume that pieces of the function that appear linear are indeed linear.

Use the graph above to evaluate each of the expressions below, and write your answer on the answer blank provided. If any of the quantities do not exist (including the case of limits that diverge to ∞ or $-\infty$), write DNE.

a. [1 point] $f(1)$

Answer: 3

b. [1 point] $\lim_{x\to 5} f(x)$

Answer: 0

c. [1 point] $\lim_{q\to 3} f(q)$

Answer: **DNE**

d. [1 point] $\lim_{z\to 2} f(2)$

Answer: 5

e. [1 point] $\lim_{r\to 6^-} f(r)$

Answer: 1

f. [1 point] $\lim_{h\to 0} \dfrac{f(4.25+h)-f(4.25)}{h}$

Answer: -4

g. [1 point] $\lim_{p\to 0.5} \dfrac{f(p)}{p}$

Answer: 4

h. [1 point] $\lim_{t\to 3} f(t)f(t+2)$

Answer: 0

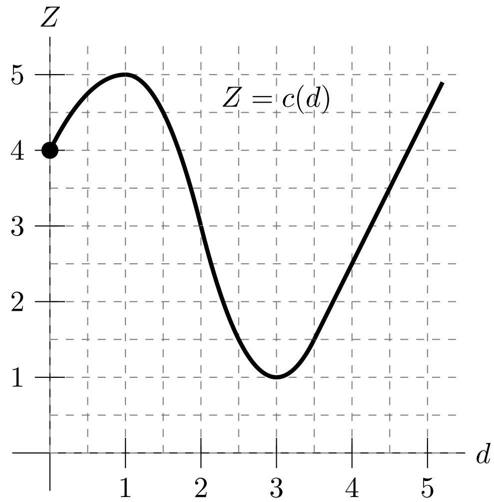
i. [1 point] $\lim_{x\to 3^+} f(f(x))$

Answer: 3

j. [1 point] $\lim_{s\to 1} f(f(s))$

Answer: 5

2. [11 points] In Townsville, USA, a vat of Chemical Z is spilled into Lake Townsville. Let $c(d)$ be the concentration of Chemical Z (in mg/L) at a depth of d meters below the surface in Lake Townsville. Assume that $c(d)$ is differentiable for $0 < d < 5$. A portion of the graph of $Z = c(d)$ is shown below.



a. [1 point] What is the concentration (in mg/L) of Chemical Z at the surface of Lake Townsville?

Answer: 4

b. [2 points] Circle all of the intervals below for which $c'(d)$ is positive over the entire interval. Circle NONE if there are no such intervals.

$\boxed{0.2 < d < 0.8}$ $1.2 < d < 1.8$ $2.2 < d < 2.8$ $\boxed{3.2 < d < 3.8}$ $\boxed{4.2 < d < 4.8}$ NONE

c. [3 points] What is the average rate of change of the concentration of Chemical Z over the interval from $d = 1$ to $d = 3$? *Remember to include units.*

Solution: $\frac{(1-5)\text{ mg/L}}{(3-1)\text{ m}} = -2\ (\text{mg/L})/\text{m}$

Answer: -2 **(mg/L)/m**

d. [2 points] Suppose that $c(d)$ is linear for $3.5 < d < 5$. Find $c'(3.5)$.

Answer: 2

e. [3 points] Using your answer to part (d), circle the appropriate choice and fill in the blank in the sentence below. *Remember to include units.*

Answer: If we go from a depth of 3.500 meters to a depth of 3.498 meters below the surface of Lake Townsville, the concentration of Chemical Z will

(circle one) INCREASE $\boxed{\text{DECREASE}}$

by approximately 0.004 **mg/L**.

3. [4 points] Let $h(x) = (x+3)e^{2x-2}$. Then the derivative of h is given by the formula $h'(x) = (2x+7)e^{2x-2}$. Find an equation for the tangent line to the graph of $y = h(x)$ at $x = 1$.

Solution: Because

$$h(1) = (1+3)e^{2(1)-2} = 4$$
$$h'(1) = (2(1)+7)e^{2(1)-2} = 9$$

the tangent line has slope 9 and goes through the point $(1,4)$, so to get the formula for the tangent line:

$$y - 4 = 9(x-1)$$
$$y = 9x - 5$$

Answer: $y =$ $9x - 5$

4. [10 points] Consider the function g defined by $g(x) = \begin{cases} \dfrac{1}{e^x - 1} & \text{if } x < \frac{1}{2} \\ \cos(x^x) & \text{if } \frac{1}{2} \le x < 5 \\ \dfrac{x^2}{(x-1)(6-x)} & \text{if } x \ge 5. \end{cases}$

a. [5 points] Use the limit definition of the derivative to write an explicit expression for $g'(3)$. *Your answer should not involve the letter g. Do not attempt to evaluate or simplify the limit.* Please write your final answer in the answer box provided below.

Solution: $g'(3) = \displaystyle\lim_{h\to 0} \frac{\cos\left((3+h)^{3+h}\right) - \cos\left(3^3\right)}{h}$

Answer: $g'(3) = \boxed{\displaystyle\lim_{h\to 0} \frac{\cos\left((3+h)^{3+h}\right) - \cos\left(3^3\right)}{h}}$

b. [3 points] Find all vertical asymptotes of the graph of $g(x)$. If there are none, write NONE.

Solution: Note that $x = 1$ is *not* a vertical asymptote because the third piece of the formula for $g(x)$ is only valid for $x \ge 5$. The vertical asymptotes are$x = 0$ and $x = 6$.

Answer: $x = 0$, $x = 6$

c. [2 points] Determine $\displaystyle\lim_{x\to\infty} g(x)$. If the limit does not exist, write DNE.

Solution: $\displaystyle\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{x^2}{(x-1)(6-x)} = \lim_{x\to\infty} \frac{x^2}{-x^2+7x-6} = -1.$

Answer: -1

5. [10 points] Vikram takes a non-stop train ride from Chennai straight to New Delhi. Let $g(t)$ be the distance (in km) of Vikram's train from Chennai t hours after his ride begins. Assume that the function g is increasing and invertible, and that g and g^{-1} are differentiable. Several values for $g(t)$ are shown in the table below.

t	0	2	5	6.5	10	11	16	28
$g(t)$	0	132	346	448	692	742	1152	2180

a. [3 points] Estimate the instantaneous velocity of Vikram's train 6 hours after his ride begins. *Show your work and include units.*

Solution: We estimate using average velocity based on nearby measurements:

$$\frac{448-346}{6.5-5} = 68$$

So we estimate the instantaneous velocity of Vikram's train 6 hours after his ride begins to be about 68 km/h.

Answer: 68 **km/h**

b. [5 points] Suppose $(g^{-1})'(700) = C$, where C is some constant.

(i) Using the data in the table above, find the best possible estimate of C.
Show your work.

Solution: We estimate the derivative based on nearby measurements:

$$\frac{11-10}{742-692} = 0.02 \text{ h/km}$$

So we estimate $C \approx 0.02$. 700 has units km, and C has units h/km.

Answer: 0.02

(ii) In interpreting the equation $(g^{-1})'(700) = C$, what are the units on 700 and C?

Answer: Units on 700 are **km**

Answer: Units on C are **h/km**

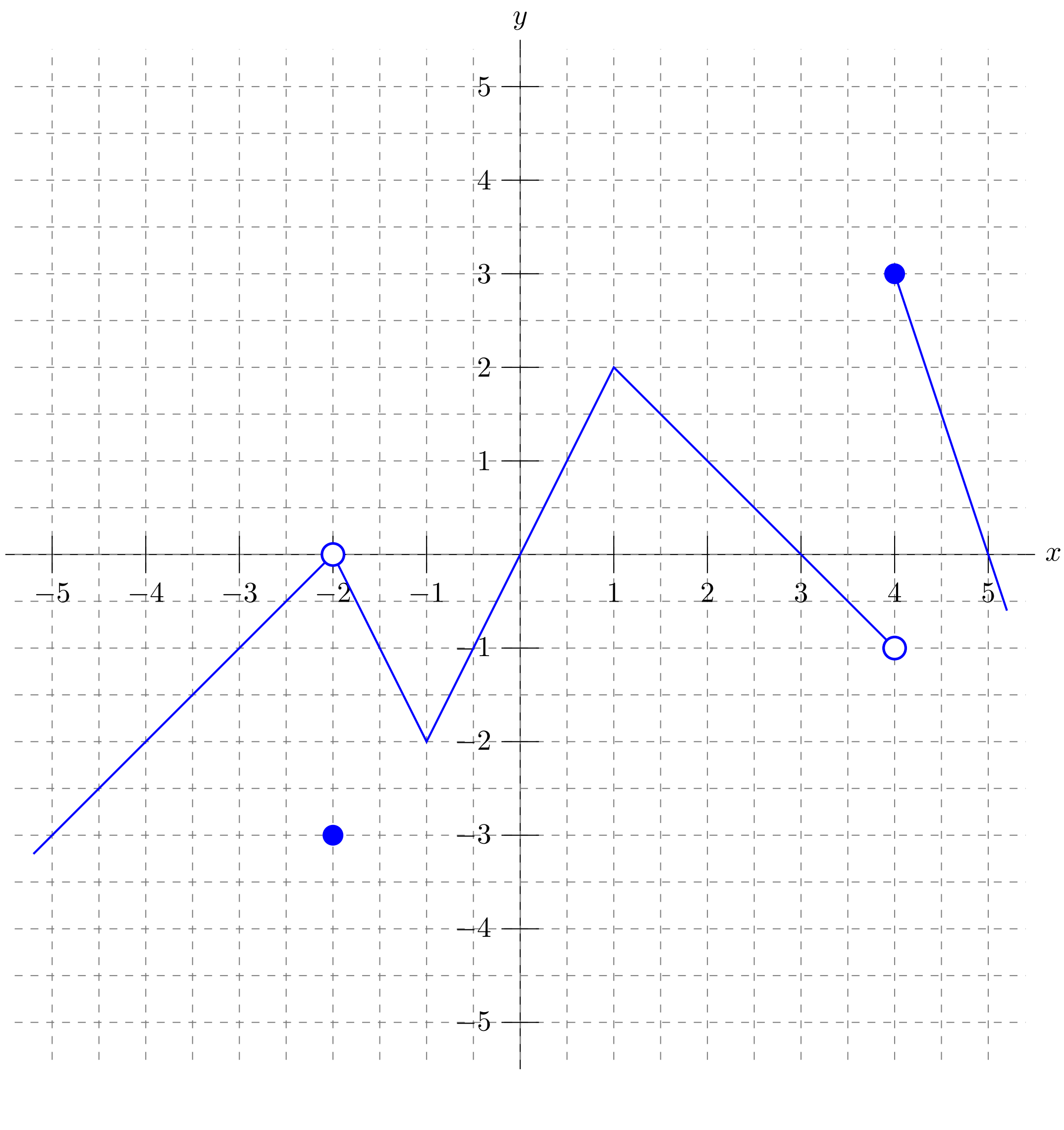
c. [2 points] Let $R(t)$ be the total rainfall (in cm) in New Delhi during the first t hours of Vikram's train ride. Express the following statement with a single mathematical equation: "Over the first 900 km of Vikram's train ride, it rained 3.6 cm in New Delhi."

Answer: $R(g^{-1}(900)) = 3.6$

6. [12 points] On the axes provided below, sketch the graph of a single function $y = h(x)$ satisfying all of the following:

- $h(x)$ is defined for all x in the interval $-5 < x < 5$.
- $h'(x) > 0$ for all $x < -3$.
- $\lim_{x\to -2} h(x) = 0$.
- $h(-2) = -3$.
- The average rate of change of $h(x)$ between $x = -1$ and $x = 1$ is 2.
- $h(1) = 2$.
- $h(x)$ is linear between $x = 1$ and $x = 3$.
- $h'(2) = -1$.
- $\lim_{x\to 4^-} h(x) = -1$.
- $\lim_{x\to 4} h(x)$ does not exist.
- $h'(x) < 0$ for all $x > 4$.

Make sure that your sketch is large and unambiguous.



7. [10 points] Note that the situations described in parts **a.** and **b.** on this page are not related to each other.

a. [6 points] A dose of a total of 1.2 milliliters of a drug is injected into a patient steadily for 0.3 seconds. At the end of this time, the quantity of the drug in the body starts to decay exponentially, decreasing by 0.18 percent per second. Let $Q(t)$ be the quantity of the drug in the body, in milliliters, t seconds after the injection begins.

The function $Q(t)$ can be described using a piecewise-defined formula, as shown below. Use the description above to fill in the four answer blanks provided below with appropriate formulas and bounds so that the function $Q(t)$ is continuous for all $t > 0$.

Answer: $$Q(t) = \begin{cases} \underline{\quad 4t \quad} & \text{if } 0 < t \leq \underline{\quad 0.3 \quad} \\ \underline{\quad 1.2(0.9982)^{t-0.3} \quad} & \text{if } \underline{\quad 0.3 \quad} < t. \end{cases}$$

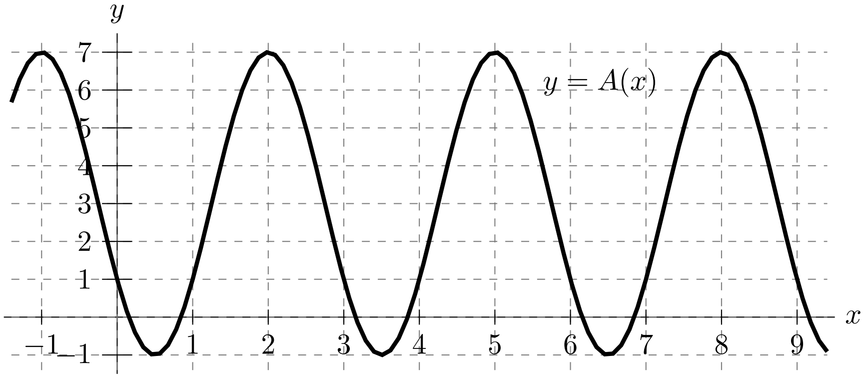
b. [4 points] Suppose that someone studying parking habits at U-M during the 2015-16 school year makes the following statement:

"During this school year, the number of cars that arrive on campus before 8 am has increased by 25% every thirty days."

Let $C(d)$ be the number of cars that arrive on campus before 8 am on the dth day of the school year. Which of the formulas below model the situation described in the quote above, where K is some positive constant? (Circle all correct answers. Or circle NONE OF THESE.)

$C(d) = K(0.25)^{d/30}$	$C(d) = K(1.25/30)^d$	$C(d) = K + (0.25/30)^d$
$\boxed{C(d) = K(1.25)^{d/30}}$	$\boxed{C(d) = K(0.8)^{-d/30}}$	$C(d) = K + (1.25/30)^d$
$C(d) = Ke^{1.25d}$	$C(d) = K(4)^{-d/30}$	$C(d) = K + 0.25d$
$C(d) = Ke^{0.25d}$	$C(d) = Kd^{1.25}$	$C(d) = K + 0.25d/30$
$\boxed{C(d) = Ke^{\ln(1.25)d/30}}$	$C(d) = Kd^{0.25}$	$C(d) = 1.25\sin(\frac{\pi d}{15}) + K$
$C(d) = Ke^{\ln(0.25)d/30}$	$C(d) = K + (0.25)^{d/30}$	$C(d) = 1.25\cos(\frac{\pi d}{15}) + K$
$C(d) = K(0.25/30)^d$	$C(d) = K + (1.25)^{d/30}$	NONE OF THESE

8. [4 points] Let $A(x)$ be a sinusoidal function, a portion of which is shown in the graph below.



Write a formula for $A(x)$.

Solution: There are many possible formulas. Among the possbilities are the following:

$A(x) = 4\cos\left(\frac{2\pi}{3}(x-2)\right)+3$ $\qquad$ $A(x) = -4\cos\left(\frac{2\pi}{3}\left(x-\frac{1}{2}\right)\right)+3$

$A(x) = 4\sin\left(\frac{2\pi}{3}\left(x-\frac{5}{4}\right)\right)+3$ $\qquad$ $A(x) = -4\sin\left(\frac{2\pi}{3}\left(x-\frac{11}{4}\right)\right)+3$

Answer: $A(x) =$ $4\cos\left(\frac{2\pi}{3}(x-2)\right)+3$

9. [7 points] Consider the function $f(x)$ defined by

$$f(x) = \begin{cases} xe^{Ax} + B & \text{if } x < 3 \\ C(x-3)^2 & \text{if } 3 \le x \le 5 \\ \dfrac{130}{x} & \text{if } x > 5. \end{cases}$$

Suppose $f(x)$ satisfies all of the following:

- $f(x)$ is continuous at $x = 3$.
- $\lim\limits_{x\to 5^+} f(x) = 2 + \lim\limits_{x\to 5^-} f(x)$.
- $\lim\limits_{x\to -\infty} f(x) = -4$.

Find the values of A, B, and C.
Show your work. You must give exact answers. Do not use decimal approximations.
For example, 0.333333333 *would* _not_ *be an acceptable answer if the answer were* $\frac{1}{3}$.

Solution: Because $f(x)$ is continuous at $x = 3$ (the first property), $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^+} f(x)$. So we have $3e^{3A} + B = C(3-3)^2 = 0$ and thus $3e^{3A} = -B$ (*).

Now, by the second property, we have $\lim\limits_{x\to 5^+} f(x) = 2 + \lim\limits_{x\to 5^-} f(x)$, so

$$\begin{aligned} \frac{130}{5} &= 2 + C(5-3)^2 \\ 26 &= 2 + 4C \\ 24 &= 4C \\ 6 &= C \end{aligned}$$

Thus $\boxed{C = 6}$.

Note that if $\lim\limits_{x\to -\infty} xe^{Ax}$ exists, then it is equal to 0 (and $A < 0$). By the third property, we therefore see that

$$-4 = \lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} (xe^{Ax} + B) = 0 + B = B.$$

So, $\boxed{B = -4}$, and using equation (*) above, we see that $3e^{3A} = -(-4)$ so $e^{3A} = \frac{4}{3}$ and $\boxed{A = \frac{1}{3}\ln(4/3)}$.

Answer: $A =$ $\frac{1}{3}\ln\left(\frac{4}{3}\right)$, $B =$ -4, and $C =$ 6

10. [9 points] Suppose data is collected at a U-M basketball game held at Crisler Center. Let $E(t)$ be the total amount of electricity, in megawatt-hours (MWh), that has been used by Crisler Center during the first t minutes of the basketball game, which starts at exactly 7:00 pm. Assume that E is invertible and that both E and E^{-1} are differentiable.

a. [3 points] Suppose b and c are positive constants. Use a complete sentence to give a practical interpretation of the equation

$$E(30 + b) = E(30) + c$$

in the context of this problem. Your sentence should involve the constants b and c but not "E". Be sure to include units.

Solution: In the b minutes after 7:30 pm, Crisler Center uses c MWh of electricity.

b. [3 points] Fill in the two answer blanks below to write a single mathematical equality involving the derivative of either E or E^{-1} which supports the following claim:

"During the basketball game, Crisler Center uses about 1.8 MWh of electricity during the first 3 seconds after 7:45 pm."

Answer: $E'(45)$ = 36

c. [3 points] Which of the sentences below best expresses the meaning of the equation

$$E^{-1}(20) = 1.5E^{-1}(12)$$

in the context of this problem? (Circle the one best choice.)

A. Crisler Center uses 50% more electricity during the first 20 minutes after the game starts than during the first 12 minutes after the game starts.

B. It takes half as long for Crisler Center to use the first 12 MWh of electricity during the game than for it to use the next 8 MWh.

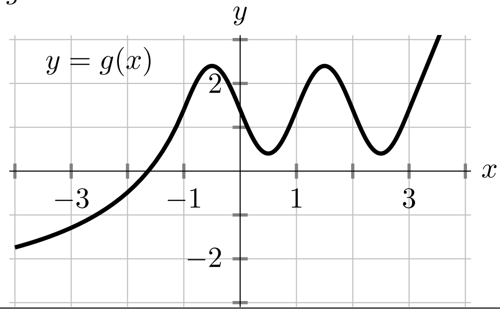
C. Crisler Center uses 50% as much electricity during the first 20 minutes after the game starts than during the first 12 minutes after the game starts.

D. It takes 50% longer for Crisler Center to have used a total of 20 MWh of electricity during the game than for it to use the first 12 MWh.

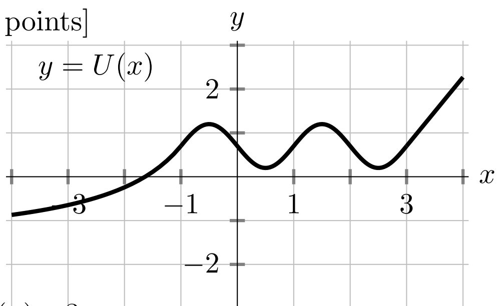
E. Crisler Center uses twice as much electricity during the first 20 minutes after the game starts than during the next 12 minutes.

F. It takes 50% less time for Crisler Center to have used a total of 12 MWh of electricity during the game than for it to use the first 20 MWh.

11. [11 points] A portion of the graph of a function g is shown below.
In each of parts **a.–d.** on this page, the corresponding portion of the graph of a function obtained from g by one or more transformations is shown, together with a list of possible formulas for that function. In each case, circle the one correct formula for the function shown.



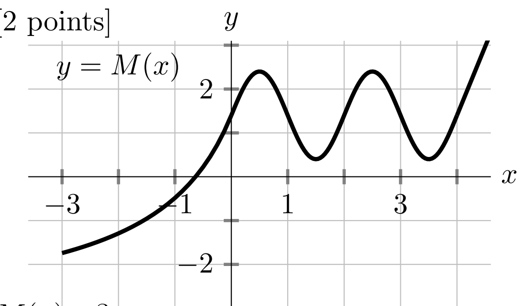
a. [2 points]



$U(x) = ?$
Circle the one correct choice below.

$g(x) - 1$	$g(0.5x)$	$\boxed{0.5g(x)}$
$g(x) + 1$	$g(2x)$	$2g(x)$
$g(x) - 1.5$	$g(x+1)$	$g(x-1)$

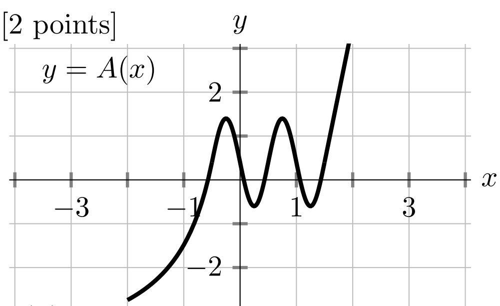
b. [2 points]



$M(x) = ?$
Circle the one correct choice below.

$g(x) - 1$	$g(0.5x)$	$0.5g(x)$
$g(x) + 1$	$g(2x)$	$2g(x)$
$g(x) - 1.5$	$g(x+1)$	$\boxed{g(x-1)}$

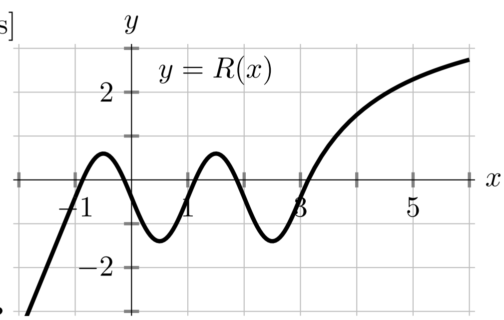
c. [2 points]



$A(x) = ?$
Circle the one correct choice below.

$g(2x) + 1$	$g(0.5x) + 1$	$g(x-2) - 1$
$\boxed{g(2x) - 1}$	$g(0.5x) - 1$	$2g(x-1)$
$2g(x+1)$	$0.5g(x+1)$	$0.5g(x-1)$

d. [2 points]



$R(x) = ?$
Circle the one correct choice below.

$g(-x-1) + 2$	$-g(x-1) - 2$	$-g(x+2) - 1$
$g(-x+1) - 2$	$-g(-x-2) - 1$	$-g(x-2) + 1$
$g(-x-2) + 1$	$\boxed{-g(-x+2) + 1}$	$-g(-x+1) + 2$

e. [3 points] A portion of the graph of the derivative of one of the five functions above is shown on the right. Which derivative is shown? Circle the one correct choice below.

$g'(x)$ $\boxed{U'(x)}$ $M'(x)$ $A'(x)$ $R'(x)$