On my honor, as a student, I have neither given nor received unauthorized aid on this academic work. **Signed:** ______________________

Math 115 — Final Exam

Dec 12, 2014

Name: ______________________

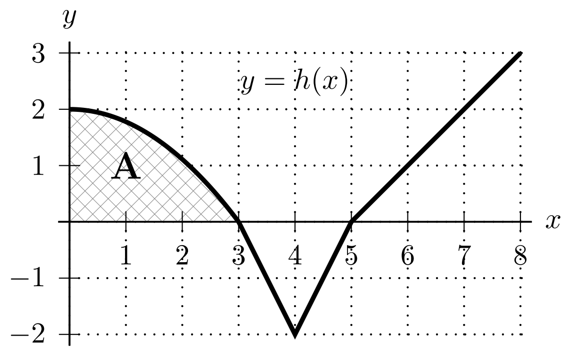
Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
8. Include units in your answer where that is appropriate.
9. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones.
10. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	13	
2	8	
3	12	
4	10	
5	10	
6	7	
7	10	
8	11	
9	8	
10	11	
Total	100	

1. [13 points]

The graph of a function $h(x)$ is shown on the right. The area of the shaded region A is 4, and $h(x)$ is piecewise linear for $3 \le x \le 6$.



Compute each of the following. If there is not enough information to compute a value exactly, write NOT ENOUGH INFO.

a. [2 points] Find $\displaystyle\int_0^3 (h(x)+2)\,dx$.

Answer: $\displaystyle\int_0^3 (h(x)+2)\,dx =$ ______________________

b. [2 points] Find the average value of $h(x)$ on the interval $[0,4]$.

Answer: ______________________

c. [3 points] Let $J(x) = \sin(\pi h(x))$. Find $J'(3.5)$.

Answer: $J'(3.5) =$ ______________________

d. [3 points] Let $H(x)$ be an antiderivative of $h(x)$ with $H(4) = 5$. Find an equation for the tangent line to the graph of $H(x)$ at $x = 4$.

Answer: ______________________

e. [3 points] Let $g(x) = e^x$. Find $\displaystyle\int_6^7 \big(g(x)h'(x) + g'(x)h(x)\big)\,dx$.

Answer: $\displaystyle\int_6^7 \big(g(x)h'(x) + g'(x)h(x)\big)\,dx =$ ______________________

2. [8 points] A car is traveling on a long straight road. The driver suddenly realizes that there is a stop sign exactly 40 feet in front of the car and immediately hits the brakes. The car's velocity decreases for the next two seconds as the car slows to a stop.
Let $v(t)$ be the velocity of the car, in feet per second, t seconds after the driver hits the brakes. Some values of the function v are shown in the table below.

t	0	0.5	1	1.5	2
$v(t)$	40	32	23	12	0

a. [2 points] Estimate the car's acceleration 0.25 seconds after the driver hits the brakes. *Remember to show your work and include units.*

Answer: ______________________

b. [3 points] Based on the information in the table above, does the car first stop before, after, or at the stop sign? Or, is there not enough information to make this determination? Briefly explain your reasoning.

Answer: (Circle one choice.)

Before the sign After the sign At the sign Not enough info

Reasoning:

c. [3 points] How often would speedometer readings need to be taken so that the resulting left-hand Riemann sum approximates the actual distance traveled between $t = 0$ and $t = 2$ seconds to within 1 foot?

Answer: Readings would need to be taken once every ______________________ seconds.

3. [12 points] Oren plans to grow kale on his community garden plot, and he has determined that he can grow up to 160 bunches of kale on his plot. Oren can sell the first 100 bunches at the market and any remaining bunches to wholesalers. The revenue in dollars that Oren will take in from selling b bunches of kale is given by

$$R(b) = \begin{cases} 6b & \text{for } 0 \le b \le 100 \\ 4b + 200 & \text{for } 100 < b \le 160. \end{cases}$$

a. [2 points] Use the formula above to answer each of the following questions.

i. What is the price (in dollars) that Oren will charge for each bunch of kale he sells at the market?

Answer: ____________________

ii. What is the price (in dollars) that Oren will charge for each bunch of kale he sells to wholesalers?

Answer: ____________________

For $0 \le b \le 160$, it will cost Oren $C(b) = 20 + 3b + 24\sqrt{b}$ dollars to grow b bunches of kale.

b. [1 point] What is the fixed cost (in dollars) of Oren's kale growing operation?

Answer: ____________________

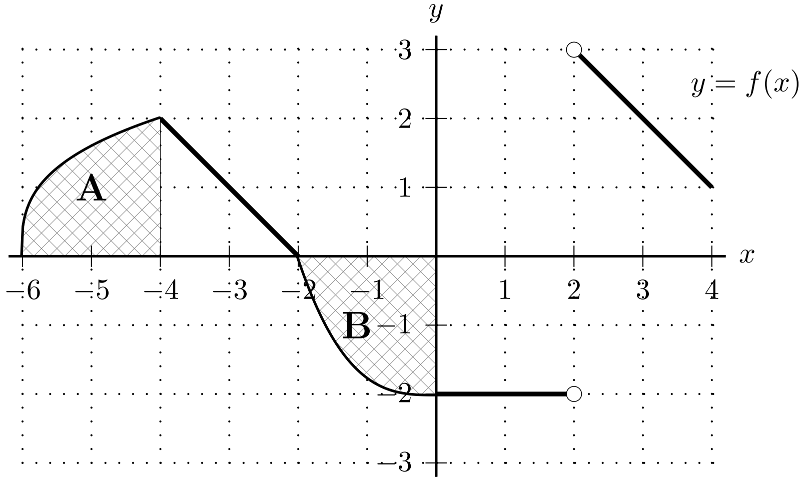
c. [4 points] At what production level(s) does Oren's marginal revenue equal his marginal cost?

Answer: ____________________

d. [5 points] Assuming Oren can grow up to 160 bunches of kale, how many bunches of kale should he grow in order to maximize his profit, and what is the maximum possible profit? *You must use calculus to find and justify your answer. Be sure to provide enough evidence to justify your answer fully.*

Answer: bunches of kale: _______________ and max profit: _______________

4. [10 points] A portion of the graph of $y = f(x)$ is shown below.
The area of shaded region A is 3, and the area of shaded region B is 3.



Let $F(x)$ be the <u>continuous</u> antiderivative of $f(x)$ with $F(0) = 1$ whose domain includes the interval $-6 \le x \le 4$.

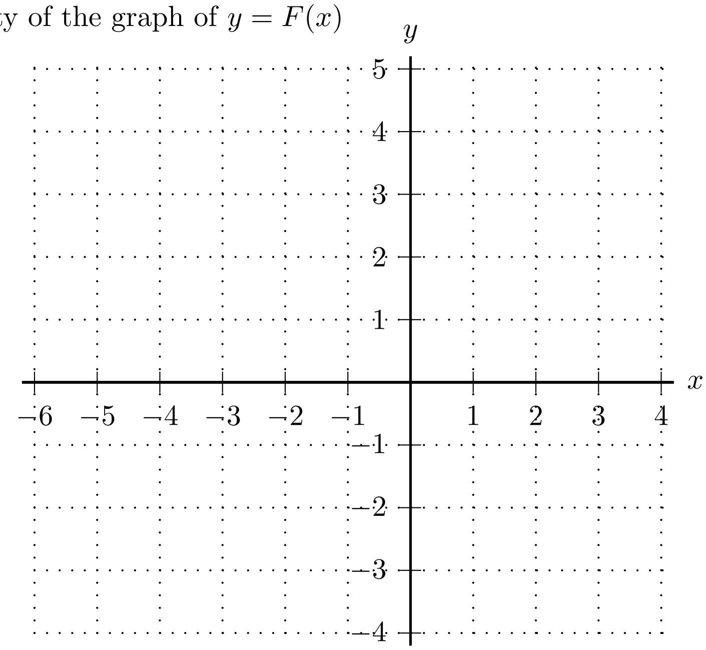
a. [3 points] For what value(s) of x with $-6 < x < 4$ does $F(x)$ have local extrema? If there are none of a particular type, write NONE. You do not need to justify your answers.

Answer: local max(es) at $x =$ ____________________

Answer: local min(s) at $x =$ ____________________

b. [7 points] Recall that $F(x)$ is the <u>continuous</u> antiderivative of $f(x)$ with $F(0) = 1$. On the axes below, draw the graph of $y = F(x)$ on the interval $-6 \le x \le 4$.
Be sure that you pay close attention to each of the following:

- the value of $F(x)$ at each of $x = -6, -4, -2, 0, 2, 4$
- where F is/is not differentiable
- where F is increasing/decreasing/constant
- the concavity of the graph of $y = F(x)$



5. [10 points] Tommy and Gina were friends in high school but then went to college in different parts of the country. They thought they were going to see each other in Springfield over the December break, but their schedules didn't match up. In fact, it turns out that Tommy is leaving on the same day that Gina is arriving.

Shortly before Gina's train arrives in Springfield, she sends a text to Tommy to see where he is, and Tommy sends a text response to say that, sadly, his train has already left. At the moment Tommy sends his text, he is 20 miles due east of the center of the train station and moving east at 30 mph while Gina is 10 miles due south of the train station and moving north at 50 mph.

a. [2 points] What is the distance between Gina and Tommy at the time Tommy sends his text? *Remember to include units.*

Answer: ____________________

b. [6 points] When Tommy sends his text, are he and Gina moving closer together or farther apart? How quickly? *You must show your work clearly to earn any credit. Remember to include units.*

Answer: Tommy and Gina are getting (circle one) CLOSER TOGETHER FARTHER APART

at a rate of ____________________

c. [2 points] Let $J(t)$ be the distance between Gina and Tommy t hours after Tommy sends his text. Use the local linearization of $J(t)$ at $t = 0$ to estimate the distance between Gina and Tommy 0.1 hours after Tommy sends his text. *Remember to show your work carefully.*

Answer: ____________________

6. [7 points] Consider the family of functions given by

$$f(x) = \frac{ax}{e^{0.5(bx)^2}}$$

where a and b are constants with $a > 1$ and $b > 1$.
Note that the derivative and second derivative of $f(x)$ are

$$f'(x) = \frac{a(1 - b^2x^2)}{e^{0.5(bx)^2}} \qquad \text{and} \qquad f''(x) = \frac{ab^2x(b^2x^2 - 3)}{e^{0.5(bx)^2}}.$$

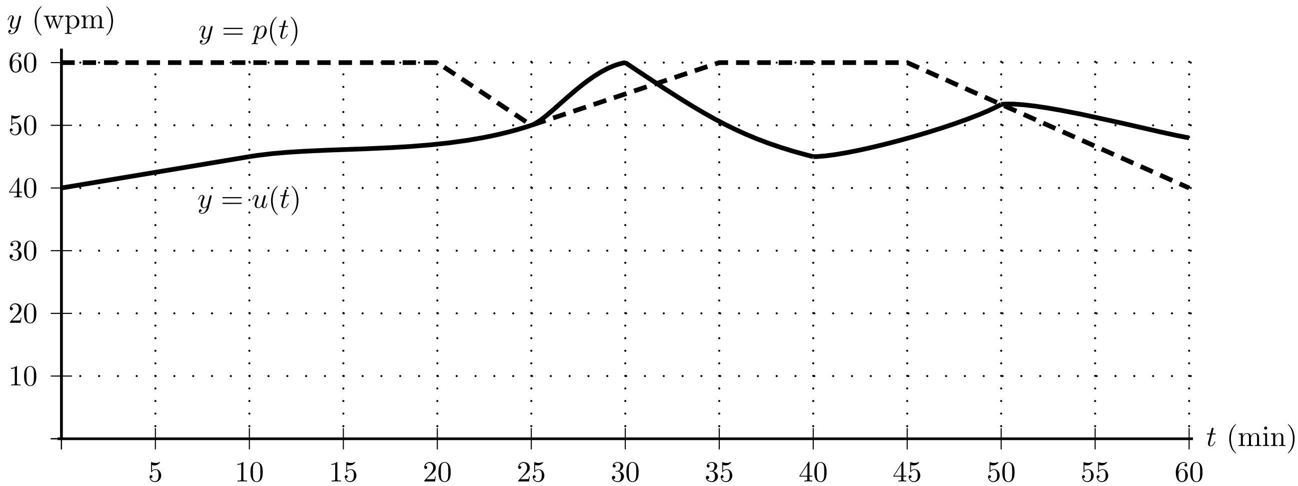
Find all global extrema of $f(x)$ on the interval $\left[\frac{1}{4b}, \infty\right)$. If there are none of a particular type, write NONE.
You must use calculus to find and justify your answers. Be sure to provide enough evidence to justify your answers fully.

Answer: global max(es) at $x =$ ________________

Answer: global min(s) at $x =$ ________________

7. [10 points] A history professor gives a 60 minute lecture, while one eager undergraduate student takes notes by typing what the professor says, word for word. Unfortunately, the student cannot always type as quickly as the professor is speaking.

Functions p and u are defined as follows. When t minutes have passed since the start of the lecture, the professor is speaking at a rate of $p(t)$ words per minute (wpm) while the undergraduate student is typing at a rate of $u(t)$ words per minute (wpm). Shown below are graphs of $y = p(t)$ (dashed) and $y = u(t)$ (solid).



a. [2 points] How many minutes after the start of the lecture is the student typing most quickly?

Answer: ____________________

b. [3 points] Write a definite integral equal to the number of words the student types between the start of the lecture and the time the professor reaches the 600th **word** of the lecture. You do not need to evaluate the integral.

Answer: ____________________

c. [3 points] How many minutes after the start of the lecture is the student furthest behind in typing up the lecture? (In other words, after how many minutes is the difference between the total number of words the professor has spoken and the total number of words the student has typed the greatest?)

Answer: ____________________

d. [2 points] What is the average rate, in words per minute, at which the professor is speaking between $t = 40$ and $t = 60$?

Answer: ____________________

8. [11 points] Suppose k and p are positive constants. Consider the function

$$R(x) = p - \ln(x^2 + k)\,.$$

a. [5 points] Use the limit definition of the derivative to write down an explicit expression for $R'(3)$.
Your answer should not include the letter R.
Do not attempt to evaluate or simplify the limit.

Answer: $R'(3) =$

b. [4 points] Write out all the terms for the right-hand Riemann sum with three subdivisions of equal length which approximates the integral

$$\int_1^{13} R(x)\,dx\,.$$

Your answer should not include the letter R but may involve k and/or p.

c. [2 points] Is the right-hand Riemann sum with three subdivisions of equal length from part (b) an overestimate or an underestimate of $\int_1^{13} R(x)\,dx$, or is there not enough information to make this determination? Briefly explain your reasoning.

Answer: (Circle one choice.)

Overestimate Underestimate Not enough info

Reasoning:

9. [8 points] Consider the family of functions given by

$$I(t) = \frac{At^2}{B + t^2}$$

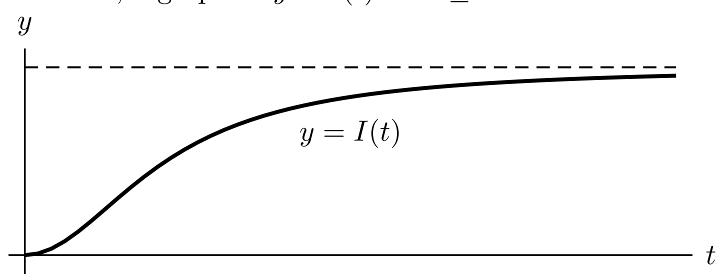
where A and B are positive constants. Note that the first and second derivatives of $I(t)$ are

$$I'(t) = \frac{2ABt}{(B + t^2)^2} \quad \text{and} \quad I''(t) = \frac{2AB(B - 3t^2)}{(B + t^2)^3}.$$

a. [2 points] Find $\lim_{t\to\infty} I(t)$. *Your answer may include the constants A and/or B.*

Answer: $\lim_{t\to\infty} I(t) =$ ____________________

A researcher studying the ice cover over Lake Michigan throughout the winter proposes that for appropriate values of A and B, the function $I(t)$ is a good approximation for the number of thousands of square miles of Lake Michigan covered by ice t days after the start of December. For such values of A and B, a graph of $y = I(t)$ for $t \geq 0$ is shown below.



Based on observations, the researcher chooses values of the parameters A and B so that the following are true.

- $y = 21$ is a horizontal asymptote of the graph of $y = I(t)$.
- $I(t)$ is increasing the fastest when $t = 25$.

b. [6 points] Find the values of A and B for the researcher's model.
Remember to show your work carefully.

Answer: $A =$ ________________ and $B =$ ________________

10. [11 points] Suppose an online retailer uses robots to transport merchandise to the shipping area in its warehouse. Researchers are analyzing data from sales on November 28, 2014.

- Let $r(h)$ be the total number of kilometers the warehouse robots had traveled in the first h hours of November 28, 2014.
- Let $Q(h)$ be the total weight, in pounds, of the merchandise that had been transported to shipping by the warehouse robots in the first h hours of November 28, 2014.

Suppose that both $r(h)$ and $Q(h)$ are invertible and differentiable on the interval $0 < h < 24$. For each of the questions below, circle the <u>one</u> best answer. No points will be given for ambiguous or multiple answers.

a. [2 points] Which one of the following expressions is equal to the total number of pounds of merchandise the robots had transported to shipping on November 28 when the robots had traveled a total of 3 km that day?

i. $Q(r(3))$ ii. $r(Q(3))$ iii. $r^{-1}(Q(3))$ iv. $r(Q^{-1}(3))$ v. $Q(r^{-1}(3))$

b. [2 points] Let m be a positive constant. Which one of the following expressions is equal to the total number of kilometers the robots had traveled two hours after they had transported a total of m pounds of merchandise to shipping?

i. $r(m+2)$ ii. $r(Q^{-1}(m)+2)$ iii. $Q(2)+r(m)$ iv. $Q^{-1}(m+2)$ v. $Q'(m)+2$

c. [2 points] Which one of the following expressions is equal to the total number of pounds of merchandise transported by the warehouse robots between 1 am and 5 am?

i. $Q(5)$ ii. $Q'(5) - Q'(1)$ iii. $\int_1^5 Q(h)\,dh$ iv. $\int (Q(5) - Q(1))\,dh$ v. $\int_1^5 Q'(h)\,dh$

d. [2 points] Which one of the following expressions is equal to the average rate (in pounds per hour) at which merchandise was transported by the robots between 8 am and 10 am?

i. $\dfrac{Q'(10)+Q'(8)}{2}$ ii. $\dfrac{Q'(10)-Q'(8)}{2}$ iii. $\dfrac{Q(10)-Q(8)}{2}$ iv. $\int_8^{10} Q(h)\,dh$ v. $\int_8^{10} Q'(h)\,dh$

e. [3 points] Circle the one equation below that best supports the following statement:
On November 28, the warehouse robots had traveled a total of 29 kilometers about half an hour after they had traveled a total of 25 kilometers.

i. $r'(\frac{1}{2}) = -4$ ii. $r'(r^{-1}(25)) = 4$ iii. $r'(29) = 8$ iv. $(r^{-1})'(25) = \frac{1}{8}$ v. $(r^{-1})'(25) = 4$