Math 115 — Final Exam

Dec 12, 2014

Name: EXAM SOLUTIONS

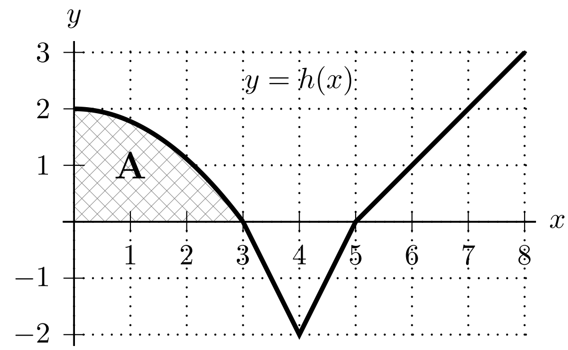
Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
8. Include units in your answer where that is appropriate.
9. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones.
10. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	13	
2	8	
3	12	
4	10	
5	10	
6	7	
7	10	
8	11	
9	8	
10	11	
Total	100	

1. [13 points]

The graph of a function $h(x)$ is shown on the right. The area of the shaded region A is 4, and $h(x)$ is piecewise linear for $3 \le x \le 6$.



Compute each of the following. If there is not enough information to compute a value exactly, write NOT ENOUGH INFO.

a. [2 points] Find $\int_0^3 (h(x)+2)\,dx$.

Solution:
$$\int_0^3 (h(x)+2)\,dx = \int_0^3 h(x)\,dx + \int_0^3 2\,dx = 4 + (3)(2) = 10\,.$$

Answer: $\int_0^3 (h(x)+2)\,dx =$ 10

b. [2 points] Find the average value of $h(x)$ on the interval $[0,4]$.

Solution: The average value is given by $\frac{1}{4-0}\int_0^4 h(x)\,dx = \frac{1}{4}(4-1) = \frac{3}{4}$.

Answer: $\frac{3}{4}$

c. [3 points] Let $J(x) = \sin(\pi h(x))$. Find $J'(3.5)$.

Solution: Since $J'(x) = \cos(\pi h(x))\pi h'(x)$, we have that
$$J'(3.5) = \cos(\pi h(3.5))\pi h'(3.5) = \cos(-\pi)\pi(-2) = 2\pi\,.$$

Answer: $J'(3.5) =$ 2π

d. [3 points] Let $H(x)$ be an antiderivative of $h(x)$ with $H(4) = 5$. Find an equation for the tangent line to the graph of $H(x)$ at $x = 4$.

Solution: Since $H'(4) = h(4) = -2$ and $H(4) = 5$, the tangent line to the graph of $H(x)$ at $x = 4$ is given by $y - 5 = -2(x-4)$.

Answer: $y = 5 - 2(x-4)$ **or** $y = 13 - 2x$

e. [3 points] Let $g(x) = e^x$. Find $\int_6^7 \big(g(x)h'(x) + g'(x)h(x)\big)\,dx$.

Solution: Note that $\frac{d}{dx}(g(x)h(x)) = g(x)h'(x) + g'(x)h(x)$. By the Fundamental Theorem of Calculus, the above integral is therefore equal to $g(7)h(7) - g(6)h(6) = 2e^7 - 1e^6$.

Answer: $\int_6^7 \big(g(x)h'(x) + g'(x)h(x)\big)\,dx =$ $2e^7 - e^6$

2. [8 points] A car is traveling on a long straight road. The driver suddenly realizes that there is a stop sign exactly 40 feet in front of the car and immediately hits the brakes. The car's velocity decreases for the next two seconds as the car slows to a stop.
Let $v(t)$ be the velocity of the car, in feet per second, t seconds after the driver hits the brakes. Some values of the function v are shown in the table below.

t	0	0.5	1	1.5	2
$v(t)$	40	32	23	12	0

a. [2 points] Estimate the car's acceleration 0.25 seconds after the driver hits the brakes. *Remember to show your work and include units.*

Solution: The acceleration at $t = 0.25$ can be approximated by the difference quotient

$$\frac{v(0.5) - v(0)}{0.5 - 0} = \frac{32 - 40}{0.5 - 0} = -16 \text{ ft/s}^2 .$$

Answer: -16 **ft/s^2**

b. [3 points] Based on the information in the table above, does the car first stop before, after, or at the stop sign? Or, is there not enough information to make this determination? Briefly explain your reasoning.

Answer: (Circle one choice.)

Before the sign   After the sign   At the sign $\boxed{\text{Not enough info}}$

Reasoning:

Solution: The distance (in feet) travelled by the car before it stops is $\int_0^2 v(t)\,dt$. Since $v(t)$ is decreasing for $0 \le t \le 2$, the left-hand sum gives an overestimate for $\int_0^2 v(t)\,dt$, while the right-hand sum gives an overestimate.

Since

$$\text{Left-hand sum} = (0.5)(40) + (0.5)(32) + (0.5)(23) + (0.5)(12) = 53.5$$

and

$$\text{Right-hand sum} = (0.5)(32) + (0.5)(23) + (0.5)(12) + (0.5)(0) = 33.5,$$

we cannot determine whether or not $\int_0^2 v(t)\,dt$ is greater than or less than 40. (The estimates 53.5 and 33.5 are the best overestimate and underestimate, respectively, that we can make of the actual distance travelled based on the data we have.)

c. [3 points] How often would speedometer readings need to be taken so that the resulting left-hand Riemann sum approximates the actual distance traveled between $t = 0$ and $t = 2$ seconds to within 1 foot?

Solution: Since $v(t)$ is decreasing, the difference between the left-hand sum and the actual value is at most the difference between the left- and right-hand sums.
When readings are taken every Δt seconds, the difference between the two sums is $(\Delta t)|v(2) - v(0)|$. We choose Δt so that $1 \ge \Delta t|v(2) - v(0))| = \Delta t(40)$. Thus, $\Delta t \le 1/40$.

Answer: Readings would need to be taken once every $1/40$ **or** 0.025 seconds.

3. [12 points] Oren plans to grow kale on his community garden plot, and he has determined that he can grow up to 160 bunches of kale on his plot. Oren can sell the first 100 bunches at the market and any remaining bunches to wholesalers. The revenue in dollars that Oren will take in from selling b bunches of kale is given by

$$R(b) = \begin{cases} 6b & \text{for } 0 \le b \le 100 \\ 4b + 200 & \text{for } 100 < b \le 160. \end{cases}$$

a. [2 points] Use the formula above to answer each of the following questions.

i. What is the price (in dollars) that Oren will charge for each bunch of kale he sells at the market?

Answer: \$6

ii. What is the price (in dollars) that Oren will charge for each bunch of kale he sells to wholesalers?

Answer: \$4

For $0 \le b \le 160$, it will cost Oren $C(b) = 20 + 3b + 24\sqrt{b}$ dollars to grow b bunches of kale.

b. [1 point] What is the fixed cost (in dollars) of Oren's kale growing operation?

Answer: \$20

c. [4 points] At what production level(s) does Oren's marginal revenue equal his marginal cost?

Solution: Oren's marginal revenue is $R'(b) = 6$ for $0 < b < 100$ and $R'(b) = 4$ for $100 < b < 160$. His marginal cost is $C'(b) = 3 + 12/\sqrt{b}$.
Thus, $R'(b) = C'(b)$ for $b = 16$ and $b = 144$.

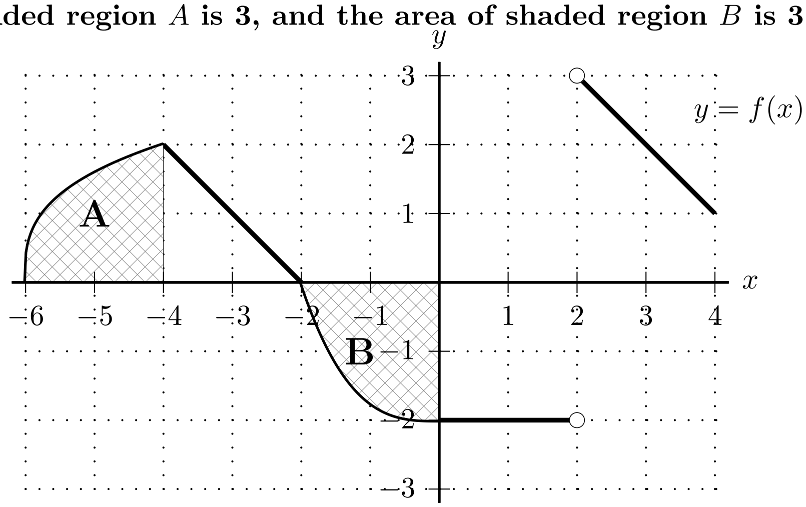
Answer: **at 16 bunches and 144 bunches**

d. [5 points] Assuming Oren can grow up to 160 bunches of kale, how many bunches of kale should he grow in order to maximize his profit, and what is the maximum possible profit? *You must use calculus to find and justify your answer. Be sure to provide enough evidence to justify your answer fully.*

Solution: Since Oren's profit function, $\pi(b) = R(b) - C(b)$, is continuous on $0 \le b \le 160$, it has a global maximum (by the Extreme Value Theorem) and the global maximum occurs at a critical point or an endpoint.
The critical points of $\pi(b)$ occur when $\pi'(b) = 0$ (at $b = 16$ and 144 (when MR=MC)), and when $\pi'(b)$ is undefined (at $b = 100$).
We check the value of $\pi(b)$ at the critical points and end points:
$\pi(0) = -20$, $\pi(16) - -68$, $\pi(100) = 40$, $\pi(144) = 36$, and $\pi(160) \approx 36.42$, and conclude that the maximum occurs at $b = 100$, with a resulting maximum profit of \$40.

Answer: bunches of kale: 100 and max profit: \$40

4. [10 points] A portion of the graph of $y = f(x)$ is shown below.
The area of shaded region A is 3, and the area of shaded region B is 3.



Let $F(x)$ be the continuous antiderivative of $f(x)$ with $F(0) = 1$ whose domain includes the interval $-6 \le x \le 4$.

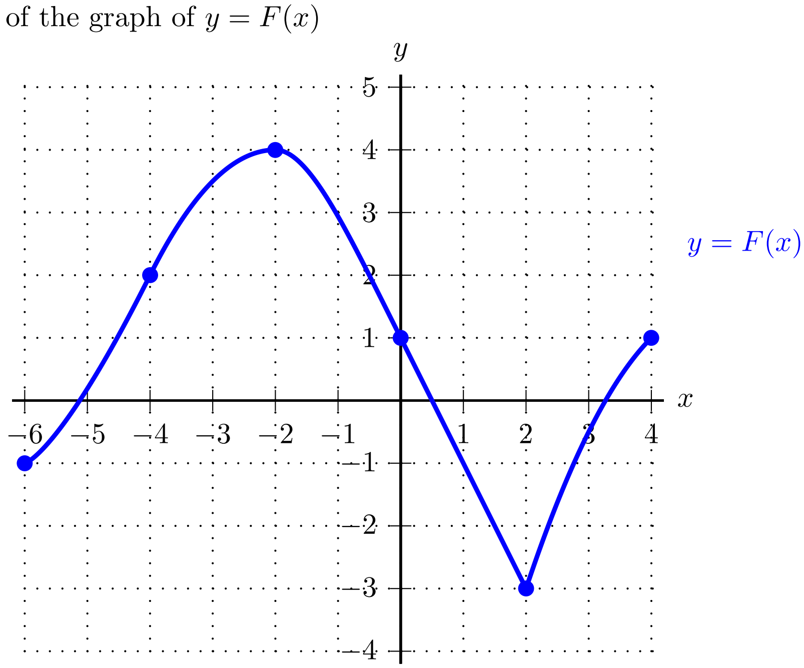
a. [3 points] For what value(s) of x with $-6 < x < 4$ does $F(x)$ have local extrema? If there are none of a particular type, write NONE. You do not need to justify your answers.

Answer: local max(es) at $x =$ -2

Answer: local min(s) at $x =$ 2

b. [7 points] Recall that $F(x)$ is the continuous antiderivative of $f(x)$ with $F(0) = 1$. On the axes below, draw the graph of $y = F(x)$ on the interval $-6 \le x \le 4$.
Be sure that you pay close attention to each of the following:

- the value of $F(x)$ at each of $x = -6, -4, -2, 0, 2, 4$
- where F is/is not differentiable
- where F is increasing/decreasing/constant
- the concavity of the graph of $y = F(x)$

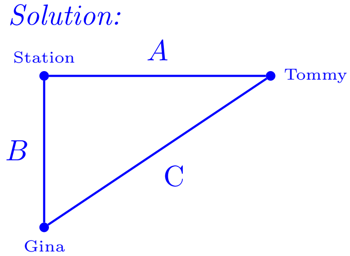


5. [10 points] Tommy and Gina were friends in high school but then went to college in different parts of the country. They thought they were going to see each other in Springfield over the December break, but their schedules didn't match up. In fact, it turns out that Tommy is leaving on the same day that Gina is arriving.

Shortly before Gina's train arrives in Springfield, she sends a text to Tommy to see where he is, and Tommy sends a text response to say that, sadly, his train has already left. At the moment Tommy sends his text, he is 20 miles due east of the center of the train station and moving east at 30 mph while Gina is 10 miles due south of the train station and moving north at 50 mph.

a. [2 points] What is the distance between Gina and Tommy at the time Tommy sends his text? *Remember to include units.*

Solution:



Let A, B, and C be the distances (in miles) as labeled in the drawing on the left. Then by the Pythagorean Theorem, $C^2 = A^2 + B^2$. When Tommy sends his text, $A = 20$ and $B = 10$ so we conclude that Gina and Tommy are $\sqrt{20^2 + 10^2} = \sqrt{500} = 10\sqrt{5} \approx 22.4$ miles apart.

Answer: $\sqrt{500} = 10\sqrt{5} \approx 22.4$ **miles**

b. [6 points] When Tommy sends his text, are he and Gina moving closer together or farther apart? How quickly? *You must show your work clearly to earn any credit. Remember to include units.*

Solution: With the notation from part (a), we have that

$$2C\frac{dC}{dt} = 2A\frac{dA}{dt} + 2B\frac{dB}{dt}.$$

When Tommy sends his text, we know that $\frac{dA}{dt} = 30$ and $\frac{dB}{dt} = -50$. Thus,

$$\frac{dC}{dt} = \frac{A\frac{dA}{dt} + B\frac{dB}{dt}}{C} = \frac{(20)(+30) + (10)(-50)}{\sqrt{500}} = 2\sqrt{5} \approx 4.47.$$

Since at this time the sign of $\frac{dC}{dt}$ is positive, Gina and Tommy are getting farther apart.

Answer: Tommy and Gina are getting (circle one) CLOSER TOGETHER $\boxed{\text{FARTHER APART}}$

at a rate of $2\sqrt{5} \approx 4.47$ **mph**

c. [2 points] Let $J(t)$ be the distance between Gina and Tommy t hours after Tommy sends his text. Use the local linearization of $J(t)$ at $t = 0$ to estimate the distance between Gina and Tommy 0.1 hours after Tommy sends his text. *Remember to show your work carefully.*

Solution: This local linearization is $L(t) = J(0) + J'(0)(t) = 10\sqrt{5} + 2t\sqrt{5}$. So the distance between Tommy and Gina 0.1 seconds after Tommy sends his text will be approximately $L(0.1) = 10\sqrt{5} + (0.1)(2\sqrt{5}) = 10.2\sqrt{5} \approx 22.8$ miles.

Answer: $10.2(\sqrt{5}) \approx 22.8$ **miles**

6. [7 points] Consider the family of functions given by

$$f(x) = \frac{ax}{e^{0.5(bx)^2}}$$

where a and b are constants with $a > 1$ and $b > 1$.
Note that the derivative and second derivative of $f(x)$ are

$$f'(x) = \frac{a(1 - b^2x^2)}{e^{0.5(bx)^2}} \qquad \text{and} \qquad f''(x) = \frac{ab^2x(b^2x^2 - 3)}{e^{0.5(bx)^2}}.$$

Find all global extrema of $f(x)$ on the interval $\left[\frac{1}{4b}, \infty\right)$. If there are none of a particular type, write NONE.

You must use calculus to find and justify your answers. Be sure to provide enough evidence to justify your answers fully.

Solution: In the domain $\left[\frac{1}{4b}, \infty\right)$ $f(x)$ has one critical point, which is at $x = \frac{1}{b}$. (Note that $f'(x) = 0$ for $x = \pm 1/b$ and $x = -1/b$ is not in our domain.)
Since

$$f''\left(\frac{1}{b}\right) = \frac{ab(-2)}{e^{0.5}} < 0$$

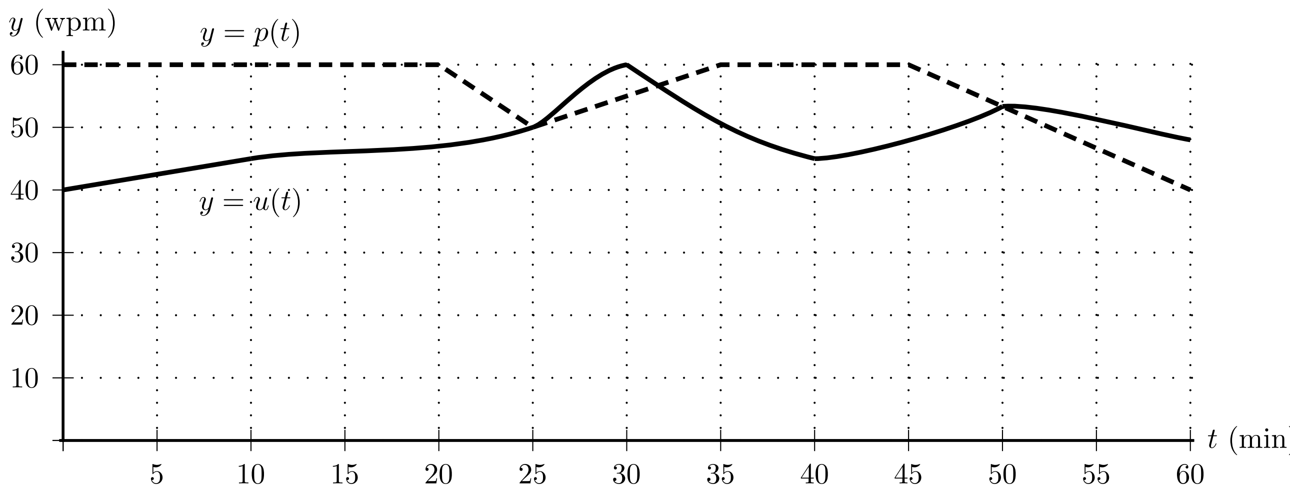
$x = 1/b$ is a local maximum of $f(x)$.
Since $f(x)$ has exactly one critical point in our domain and it is a local maximum, $x = 1/b$ is a global maximum. (Alternatively, one can note that sign of $f'(x)$ changes from positive to negative at $x = 1/b$ and does not change sign again in the interval (since no other critical points), so $f(x)$ is increasing from $\frac{1}{4b}$ to $\frac{1}{b}$ and always decreasing thereafter.)
The function $f(x)$ has no global minimum on the interval $[\frac{1}{4b}, \infty)$ since $\lim\limits_{x\to\infty} f(x) = 0$, but $f(x) > 0$ on $[\frac{1}{4b}, \infty)$. (Alternatively, one can note as above that $f(x)$ is increasing from $\frac{1}{4b}$ to $\frac{1}{b}$ and always decreasing thereafter so we compare $f(\frac{1}{4b})$ (which is strictly positive) to $\lim\limits_{x\to\infty} f(x) = 0$.)

Answer: global max(es) at $x =$ $1/b$

Answer: global min(s) at $x =$ NONE

7. [10 points] A history professor gives a 60 minute lecture, while one eager undergraduate student takes notes by typing what the professor says, word for word. Unfortunately, the student cannot always type as quickly as the professor is speaking.

Functions p and u are defined as follows. When t minutes have passed since the start of the lecture, the professor is speaking at a rate of $p(t)$ words per minute (wpm) while the undergraduate student is typing at a rate of $u(t)$ words per minute (wpm). Shown below are graphs of $y = p(t)$ (dashed) and $y = u(t)$ (solid).



a. [2 points] How many minutes after the start of the lecture is the student typing most quickly?

Solution: The student is typing most quickly when $u(t)$ is maximized, which is at $t = 30$.

Answer: 30 **minutes**

b. [3 points] Write a definite integral equal to the number of words the student types between the start of the lecture and the time the professor reaches the 600th **word** of the lecture. You do not need to evaluate the integral.

Solution: The professor reaches the 600th word of the lecture 10 minutes after the start of the lecture (since the professor is speaking at a constant rate of 60 wpm for the first 20 minutes). The total number of words the student has typed by then is $\int_0^{10} u(t)\,dt$.

Answer: $\displaystyle\int_0^{10} u(t)\,dt$

c. [3 points] How many minutes after the start of the lecture is the student furthest behind in typing up the lecture? (In other words, after how many minutes is the difference between the total number of words the professor has spoken and the total number of words the student has typed the greatest?)

Solution: This is equal to the time t when the difference between the total area under the graph of $p(t)$ and the total area under the graph of $u(t)$ is the greatest. Local maxima of this difference occur at $t = 25$ and $t = 50$. Since the area between the two graphs from $t = 25$ to $t = 32$ is smaller than the area between the two graphs from $t = 32$ to $t = 50$, the difference between the number of words spoken and typed is greatest when $t = 50$.

Answer: 50 **minutes**

d. [2 points] What is the average rate, in words per minute, at which the professor is speaking between $t = 40$ and $t = 60$?

Solution: This average rate is $\frac{1}{20}\int_{40}^{60} p(t)\,dt = \frac{1050}{20} = 52.5$ words per minute.

Answer: 52.5

8. [11 points] Suppose k and p are positive constants. Consider the function

$$R(x) = p - \ln(x^2 + k)\,.$$

a. [5 points] Use the limit definition of the derivative to write down an explicit expression for $R'(3)$.
Your answer should not include the letter R.
Do not attempt to evaluate or simplify the limit.

Answer: $R'(3) = \lim_{h \to 0} \dfrac{(p - \ln((3+h)^2 + k)) - (p - \ln(3^2 + k))}{h}$

b. [4 points] Write out all the terms for the right-hand Riemann sum with three subdivisions of equal length which approximates the integral

$$\int_1^{13} R(x)\,dx\,.$$

Your answer should not include the letter R but may involve k and/or p.

Solution: This right hand sum is given by

$$4R(5) + 4R(9) + 4R(13) = 4(p - \ln(25 + k)) + 4(p - \ln(81 + k)) + 4(p - \ln(169 + k))\,.$$

c. [2 points] Is the right-hand Riemann sum with three subdivisions of equal length from part (b) an overestimate or an underestimate of $\int_1^{13} R(x)\,dx$, or is there not enough information to make this determination? Briefly explain your reasoning.

Answer: (Circle one choice.)

Overestimate $\qquad$ $\boxed{\text{Underestimate}}$ $\qquad$ Not enough info

Reasoning:

Solution: Since $R'(x) = -\frac{2x}{x^2+k}$ is negative for $x > 0$, $R(x)$ is decreasing on the interval $1 \le x \le 13$. Thus, the above right-hand Riemann sum is an underestimate for the integral.

9. [8 points] Consider the family of functions given by

$$I(t) = \frac{At^2}{B + t^2}$$

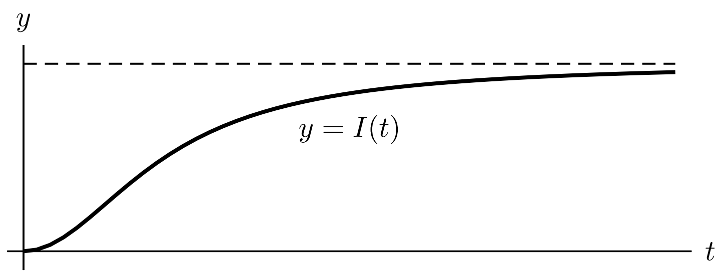
where A and B are positive constants. Note that the first and second derivatives of $I(t)$ are

$$I'(t) = \frac{2ABt}{(B+t^2)^2} \quad \text{and} \quad I''(t) = \frac{2AB(B-3t^2)}{(B+t^2)^3}.$$

a. [2 points] Find $\lim_{t\to\infty} I(t)$. *Your answer may include the constants A and/or B.*

Answer: $\lim_{t\to\infty} I(t) =$ A

A researcher studying the ice cover over Lake Michigan throughout the winter proposes that for appropriate values of A and B, the function $I(t)$ is a good approximation for the number of thousands of square miles of Lake Michigan covered by ice t days after the start of December. For such values of A and B, a graph of $y = I(t)$ for $t \geq 0$ is shown below.



Based on observations, the researcher chooses values of the parameters A and B so that the following are true.

- $y = 21$ is a horizontal asymptote of the graph of $y = I(t)$.
- $I(t)$ is increasing the fastest when $t = 25$.

b. [6 points] Find the values of A and B for the researcher's model.
Remember to show your work carefully.

Solution: From part (a) above, we know the graph of $y = I(t)$ has a horizontal asymptote at $y = A$. So $A = 21$.

$I(t)$ is increasing fastest when $I'(t)$ is maximized. For any value of t at which $I'(t)$ is maximized, t is a critical point of $I'(t)$, so $I''(t) = 0$ or $I''(t)$ is undefined. The function $I''(t)$ is defined for all t, and $I''(t) = 0$ if and only if $B - 3t^2 = 0$. So since $I'(t)$ is maximized when $t = 25$, we have $B - 3(25)^2 = 0$ so $B = 3(25)^2 = 1875$. (Alternatively, $B - 3t^2 = 0$ when $t = \pm\sqrt{B/3}$. The positive solution is t $t = \sqrt{B/3}$, so $\sqrt{B/3} = 25$ and $B = 1875$.)
Thus, if A, B are chosen so that $y = 21$ is a horizontal asymptote of the graph, $A = 21$. If A, B are chosen so that $I'(t)$ is maximized at $t = 25$, $25 = \sqrt{B/3}$. Thus, $B = 1875$.

Answer: $A =$ 21 and $B =$ 1875

10. [11 points] Suppose an online retailer uses robots to transport merchandise to the shipping area in its warehouse. Researchers are analyzing data from sales on November 28, 2014.

- Let $r(h)$ be the total number of kilometers the warehouse robots had traveled in the first h hours of November 28, 2014.
- Let $Q(h)$ be the total weight, in pounds, of the merchandise that had been transported to shipping by the warehouse robots in the first h hours of November 28, 2014.

Suppose that both $r(h)$ and $Q(h)$ are invertible and differentiable on the interval $0 < h < 24$. For each of the questions below, circle the one best answer. No points will be given for ambiguous or multiple answers.

a. [2 points] Which one of the following expressions is equal to the total number of pounds of merchandise the robots had transported to shipping on November 28 when the robots had traveled a total of 3 km that day?

i. $Q(r(3))$ ii. $r(Q(3))$ iii. $r^{-1}(Q(3))$ iv. $r(Q^{-1}(3))$ v. $\boxed{Q(r^{-1}(3))}$

b. [2 points] Let m be a positive constant. Which one of the following expressions is equal to the total number of kilometers the robots had traveled two hours after they had transported a total of m pounds of merchandise to shipping?

i. $r(m+2)$ ii. $\boxed{r(Q^{-1}(m)+2)}$ iii. $Q(2)+r(m)$ iv. $Q^{-1}(m+2)$ v. $Q'(m)+2$

c. [2 points] Which one of the following expressions is equal to the total number of pounds of merchandise transported by the warehouse robots between 1 am and 5 am?

i. $Q(5)$ ii. $Q'(5)-Q'(1)$ iii. $\int_1^5 Q(h)\,dh$ iv. $\int (Q(5)-Q(1))\,dh$ v. $\boxed{\int_1^5 Q'(h)\,dh}$

d. [2 points] Which one of the following expressions is equal to the average rate (in pounds per hour) at which merchandise was transported by the robots between 8 am and 10 am?

i. $\dfrac{Q'(10)+Q'(8)}{2}$ ii. $\dfrac{Q'(10)-Q'(8)}{2}$ iii. $\boxed{\dfrac{Q(10)-Q(8)}{2}}$ iv. $\int_8^{10} Q(h)\,dh$ v. $\int_8^{10} Q'(h)\,dh$

e. [3 points] Circle the one equation below that best supports the following statement:
On November 28, the warehouse robots had traveled a total of 29 kilometers about half an hour after they had traveled a total of 25 kilometers.

i. $r'(\frac{1}{2})=-4$ ii. $r'(r^{-1}(25))=4$ iii. $r'(29)=8$ iv. $\boxed{(r^{-1})'(25)=\frac{1}{8}}$ v. $(r^{-1})'(25)=4$