MATH 161   PRACTICE TEST II

1.   (a) Let $y = (\arc\tan t)^7$.   Compute dy/dt.

 (b)   Let $g(x) = \cos(\ln x)$ Compute $g^{(100)}(x)$ and $g^{(101)}(x)$.

 (c)   Let $x = (\sinh(4t))^{1/2}$. Compute dx/dt.

 (d)   Let $z = (\ln(a + bx))^c$, where a, b, and c are constants.   Compute dz/dx.

 (e)   Let $G(x) = x^{5\cosh x}$.   Compute dG/dx.

2.   Sketch a continuous, differentiable graph with the following properties:

- local minima at 2 and 4
- global minimum at 2
- local and global maximum at 3
- no other extrema

3.   A warehouse orders and stores boxes. The cost of storing boxes is proportional to q, the quantity ordered. The cost of ordering boxes is proportional to $1/q$, because the warehouse gets a price cut for larger orders. The total cost of operating the warehouse is the sum of ordering costs and storage costs. What value of q gives the minimum cost?

4. Let $f(x) = x^4 - ax^2$.

(a) Find all possible critical points of f in terms of a.

(b) If $a < 0$, how many critical points does f have?

(c) If $a > 0$, find the x and y coordinates of all critical points of f.

(d) Find a value of a such that the two local minima of f occur at $x = \pm 2$.

5. Given $f(x) = x^6 - 3x^5$ on the interval [-1, 4].

(a) Find all critical points of f.

(b) Determine on which intervals f is increasing.

(c) Find and classify all local and global extrema of f.

(d) On which interval(s) is f concave up?   Find all the points of inflection.

(e) Sketch the graph of f using the above information.

6. Given the function $f(x) = x \ln(2x) - x$ on the closed interval $[1/(2e), e/2]$, find the global extrema, and points of inflection and use this information to sketch the graph.   Identify all local and global extrema as well as inflection points.   *Sketch.*

7. By using an appropriate tangent line approximation to a curve, estimate the value of $1/(0.9983)^{3/2}$.   Is your answer an *underestimate* or an *overestimate*?   *Sketch!*

8. For a science fair project, Albertine needs to build cylindrical cans with volume 300 cubic centimeters. The material for the side of a can costs 0.03 cents per cm^2, and the material for the bottom and top of the can costs 0.06 cents per cm^2. What is the cost of the least expensive can that she can build?

9. (a) Given the parameterized curve
$$x = t - \cos t$$
$$y = 1 + \sin t$$
where $-\pi \le t \le \pi$. Find an equation of the tangent line to this curve at the point corresponding to $t = \pi/4$.

(b) Find a parametric equation and a parameter interval for the motion of a particle that starts at $(5 + a, -3)$ and traces the circle
$$(x - 5)^2 + (y + 3)^2 = a^2$$
four times clockwise. (Assume that $a > 0$.)

(c) Find a parameterization for the line joining the points $P = (4, 5)$ and $Q = (-3, 11)$.

10. (a) Compute dy/dx given the curve implicitly defined by the equation:
$$x + 2y = x \sinh y + y \cosh x + 2015$$

(b) Find equations of the tangent and normal lines to the curve
$(y - x)^2 = 2x + 4$ at the point $P = (6, 2)$.

11. Water is flowing at the rate of 20 m^3/min from a shallow concrete conical reservoir (vertex down) of base radius 48 m and height 6 m. (Recall that the volume of a cone of radius r and height h equals $(1/3)\pi r^2 h$.)

(a) How fast (cm/min) is the water level falling when the water is 3 m deep?

(b) How fast (cm/min) is the radius of the water's surface changing then?

12. [University of Michigan] A model for the amount of an antihistamine in the bloodstream after a patient takes a dose of the drug gives the amount, a, as a function of time, t, to be
$a(t) = A(e^{-t} - e^{-kt})$. In this equation, A is a measure of the dose of antihistamine given to the patient, and k is a transfer rate between the gastrointestinal tract and

the bloodstream. A and k are positive constants, and for pharmaceuticals such as antihistamine, k > 1.

a. Find the location t = T_m of the non-zero critical point of a(t).

b. Explain why t = T_m is a global maximum of a(t) by referring to the expression for a(t) or a'(t).

c. The function a(t) has a single inflection point. Find the location t = T_I of this inflection point. You do not need to prove that this is an inflection point.

d. Using your expression for T_m from (a), find the rate at which T_m changes as k changes.

13. Let

$$y = \sqrt[3]{(x+11)(2x-1)(3x-44)(x^2+99)} .$$

Compute dy/dx.

14. [University of Michigan] A hoophouse is an unheated greenhouse used to grow certain types of vegetables during the harsh midwest winter. A typical hoophouse has a semi-cylindrical roof with a semi-circular wall on each end (see figure below). The growing area of the hoophouse is the rectangle of length ℓ and width w (each measured in feet) which is covered by the hoophouse. The cost of the semi-circular walls is $0.50 per square foot and the cost of the roof, which varies with the side length ℓ, is $1 + 0.001ℓ per square foot.
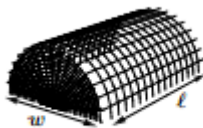

a. Write an equation for the cost of a hoophouse in terms of ℓ and w. (Hint: The surface area of a cylinder of height ℓ and radius r, not including the circles on each end, is A = $2\pi r\ell$.)

b.   Find the dimensions of the least expensive hoophouse with 8000 square feet of growing area.

15.   If two resistors of R_1 and R_2 ohms are connected in parallel in an electric circuit to make an R-ohm resistor, the value of R can be found from the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

If R_1 is decreasing at the rate of 1 ohm/sec and R_2 is increasing at the rate of 0.5 ohm/sec, at what rate is R changing when $R_1 = 75$ ohms and $R_2 = 50$ ohms?
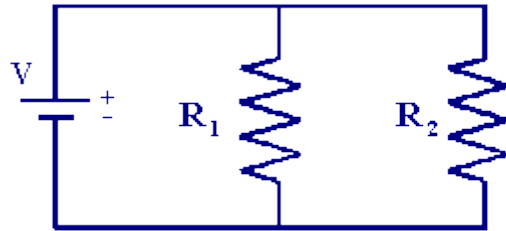
16. [University of Michigan]   Given below is the graph of a function h(t). Suppose j(t) is the local linearization of h(t) at t = 7/ 8 .

a. Given that h ' (7 /8) = 2/ 3 , find an expression for j(t).

b. Use your answer from (a) to approximate h(1).

c.   Is the approximation from (b) an over- or under-estimate? Explain.

d.   Using j(t) to estimate values of h(t), will the estimate be more accurate at t = 1 or at t = 3 /4 ? Explain.
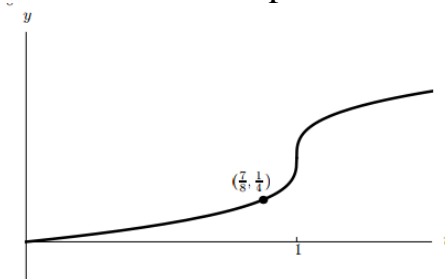
17.   A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east.   When the cruiser is 0.6 miles north of the intersection and the car is 0.8 miles to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph.   If the cruiser is moving at 60 mph at the instant of measurement, determine the speed of the car.

18.   (a)   By differentiating the equation $x^2 - y^2 = 1$ implicitly, show that
$$dy/dx = x/y.$$

(b)   By differentiating both sides of the equation $dy/dx = x/y$ implicitly, show that $d^2y/dx^2 = -1/y^3$.

19.   Derive a formula for the derivative of the *inverse* of sinh x.

20.   Find the *minimum value* of the expression $x^3 + y$ if $x + y = 4$ and $x \geq 0$. Justify your answer!

21.   *[If this is covered before the test]*

(a)   Carefully state Rolle's Theorem.

(b)   Carefully state the Mean Value Theorem.

(c)   Define the function G on the interval [-1, 2] as follows:

$$G(x) = \begin{cases} 13 & \text{if } -1 \le x \le 0 \\ 13 + x^3 & \text{if } 0 \le x \le 2 \end{cases}$$

(d) Explain why G satisfies the hypotheses of the Mean Value
Theorem on the interval [-1, 2].   Sketch!

(e) Determine the value of c for the function G on the interval [-1, 2]
that is guaranteed by the Mean Value Theorem.

22. State the *Extreme Value Theorem*.   What happens if the closed interval
is replaced by an open interval?   Is continuity necessary?

23.   Using the method of *judicious guessing*, find an *antiderivative* for
each of the following functions. *Be certain to show your reasoning!*

(a) $\dfrac{(x+5)(2x-1)}{x^3}$

(b) $\dfrac{(\ln x)^{99}}{x}$

(c) $\dfrac{\cosh x}{\sqrt{5+3\sinh x}}$

(d) $\dfrac{\sec(4x)(\tan(4x)}{1+3\sec(4x)}$

24. The derivative of a continuous function g is given by

$$g'(x) = \frac{e^{-5x}(x+2)(x-3)^2(x^4+3)(x-9)}{(x-5)^{1/3}}$$

Determine all critical points of g, and classify each as a local max, local min, or neither. Carefully explain your reasoning for each classification.

25. Let y = (ln x)^x. Find dy/dx. (*Hint:* Use logarithmic differentiation.)

26. Using an appropriate tangent line approximation, estimate the value of each of the following quantities:

   (a) $(1.00013)^{1/9}$

   (b) $(0.99999)^5$

   (c) $e^{0.0007}$

   (d) $48.993^{1/2}$

27. Water is flowing at the rate of 50 m^3/min from a shallow concrete conical reservoir (vertex down) of base radius 45 and height 6 m.

   (a) How fast (in *cm/minute*) is the water level falling when the water is 5 meter deep?

   (b) How fast is the radius of the water's surface changing then? (Express your answer in *cm/min*.)

28.   Note that $f(x) = \sinh x$ is a strictly increasing function and thus has an inverse.   Using implicit differentiation, derive a formula for the derivative of its inverse, $\sinh^{-1} x$.

29.   Given $f(x) = x^4 - 4x^3 - 8x^2 + 1$ on the interval [-5, 5].

(a) Find all critical points of f.

(b) Determine on which intervals f is increasing.

(c) Using the information obtained above, sketch the graph of
$$y = f(x).$$

30.   Given $f(x) = x(x - 2)^4$ on the real line.

(a)   Find all critical points of f.

(b)   Determine on which intervals f is increasing.

(c)   Sketch the graph of f using the above information.

31.   Given $H(x) = x + 2 \sin x$ on the interval $[0, 4\pi]$.

(a)   Find all critical points of H.

(b)   Determine on which intervals H is increasing.

(c)   Sketch the graph of H using the above information.

32.   Given $G(x) = x^2 / (x^2 + 3)$ on the real line.

(a)   Find all critical points of G.

(b)   Determine on which intervals G is increasing.

(c)   Sketch the graph of G using the above information.

33. For each of the following functions, find an *anti-derivative*. Use the method of "judicious guessing" whenever possible.

(a) $8 \cos x$

(b) $x^3 + 3x + 1$

(c) $\sec^2 x$

(d) $\sin(5x)$

(e) $\sec(3x+1) \tan(3x+1)$

(f) $(3x + 11)^{1/2}$

(g) $\sinh(x+5)$

(h) $x\cosh(x^2)$

(i) $1/(3x - 4)$

34. A 13-ft ladder is leaning against a house when its base starts to slide away. When the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec.

(a) How fast is the top of the ladder sliding down the wall?

(b) At what rate is the area of the triangle formed by the ladder, wall, and ground changing?

(c) At what rate is the angle α between the ladder and the ground changing at that moment?

35. Find the equations of the tangent and normal lines to the implicitly defined curve $2xy + \pi \sin y = 2\pi$ at the point $Q = (1, \pi/2)$.

36. Show that the normal line at *any* point of the circle $x^2 + y^2 = a^2$ passes through the origin.

37. A particle moves along the curve $y = x^{3/2}$ in the first quadrant in such a way that its distance from the origin increases at the rate of 11 units per second.   Find dx/dt when $x = 3$.

38. For each of the following functions, determine where the tangent line is horizontal.   (The x-coordinates of these critical points are sufficient.)

$$(a) \quad y = (3x-1)^8 (2x+1)^{13}$$

$$(b) \quad y = \frac{(4x+3)^{31}}{(x-2)^{15}}$$

$$(c) \quad y = (x+1)^4 e^{3x}$$

39. For each of the following curves, determine all local extrema and inflection points.   Identify intervals upon which the function is *concave up* or *concave down*.

   (a) $y = xe^{2x}$

   (b) $y = x^3/3 - x^2/2 - 2x + 1$

   (c) $y = (x^2 - 1)^{2/3}$

   (d) $y = x + \sin 2x$ on the interval $[-2\pi/3, 2\pi/3]$

   (e) $y = x(1 - x)^2$

   (f) $y = x^2(x^2 - 2)$

40. Sketch the graph of the function $F(t) = t^{1/3} (t^2 - 63)$.   Identify all zeroes

of F, as well as any and all local and global extrema.   Find any (and

all) inflection points. What is the equation of the tangent line to

$$y = F(t) \text{ at } t = 0?$$

41. *[if covered before test]*   (a)   State **Rolle's Theorem**.

(b)   Using Rolle's Theorem, prove that the function

$$g(x) = (x - 2) \ln (x + 1) + x \sin(4\pi x)$$

has at least one critical point between $x = 0$ and $x = 2$?   Explain!

42. *[if covered before test]*   (a)   State the **Mean Value Theorem**.

(b)   Show how the Mean Value Theorem applies to the function

$f(x) = 4 + \ln x$ on the interval $[1, e^3]$.   Sketch!   Find explicitly the

c value.

43.   The sum of a positive number and its reciprocal is to be as small as

possible.   What is the number?

44.   Find the point on the line $y = 2x + 5$ that is *closest* to the origin.

45.   Sketch the curve $y = 2x + \cos x$ on the interval $[0, 6\pi]$.   Find all

local/global extrema and points of inflection.

46.   Find the points on the hyperbola $x^2/4 - y^2/9 = 1$ that are *closest* to the

point (-1, 1).

47.   Given the graph of y = F′(x) below, sketch the graphs of y = F″(x) and
y = F(x).

$$y = F'(x)$$
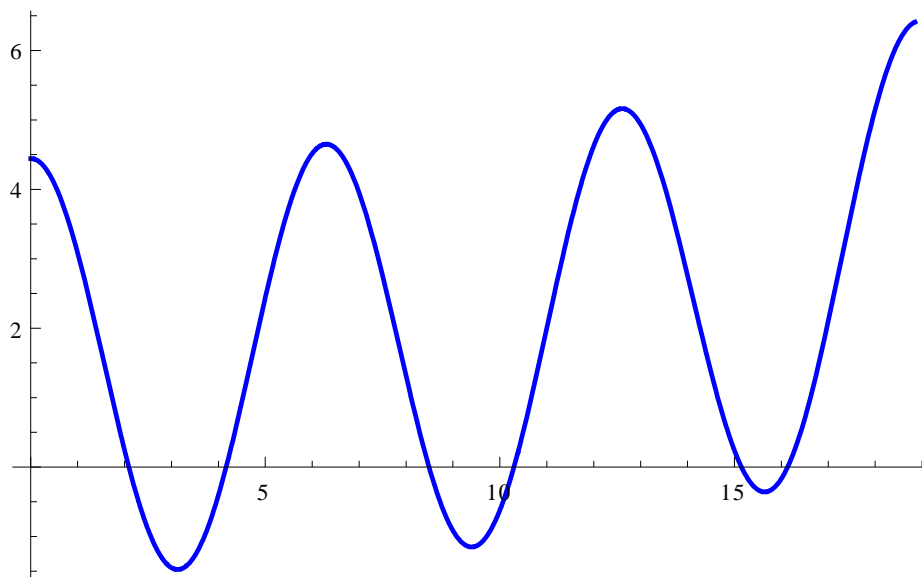
48.   For which value or values of the constant k will the curve

$$y = x^3 + kx^2 + 3x - 4$$

have *exactly one* horizontal tangent?

49.   Using the method of *judicious guessing*, find an anti-derivative of
each of the following functions:

a) $\pi + x^3 + \dfrac{1}{\pi x^3}$

b) $\cosh x - 3 \sin 3x + \sec^2(4x)$

c) $\dfrac{x^3 + x^2 + x + 1}{x}$

d) $(x^3 - 3)(x^2 - 4)$

e) $x^4 + \sin x + e^{1789x}$

f) $x \sin(x^2) + 1$

g) $\sec^2(13x) + (\sec x)(\tan x)$

h) $9 \cosh 2014x$

i) $\dfrac{\arctan}{1 + x^2}$

j) $x^4(3 + 4x^5)^6$

k) $\dfrac{e^x}{\sqrt{1 - e^{2x}}}$

l) $\dfrac{1}{x(4 + 5\ln x)}$

m) $\dfrac{\arcsin}{\sqrt{1 - x^2}}$

50. Find the *indefinite integral* of each of the following functions. Show your work!

(a) $\dfrac{\sqrt{\ln x}}{x}$

(b) $\dfrac{1 + 4\ln x}{x}$

(c) $x^3 e^{1 + x^4}$

(d) $x^2(1 + x)(2 + x)$

51. A plasma TV screen of height 36 inches is mounted on a wall so that its lower edge is 12 inches above eye-level of an observer. How far from the wall should the observer stand so that the viewing angle θ subtended at her eye by the TV screen is as large as possible?

52. A grain silo has the shape of a right circular cylinder surmounted by a hemisphere. If the silo is to have a volume of 504π ft^3, determine the radius and height of the silo that requires the least amount of material to build.
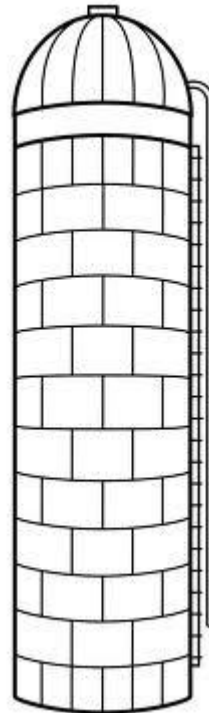
53. Find the global extrema of $y = \cos x - 3x$ on $[0, 2\pi]$.

54. Sketch the graph of the function $g(x) = x^2(x-1)^2$ on the interval $[0, 2]$. Locate any (and all) local and global extrema.   Discuss concavity.

55.   Find the local extrema of $y = x^3 - 3x^2 - 24x + 32$ employing the second derivative test.

56.   Sketch the graph of $y = e^{2/x}$.   Locate any local or global extrema. Discuss concavity.

57.   Suppose that the derivative of the function

$$y = f(x) \text{ is } y' = (x-1)^2(x-2).$$

Find all local extrema and points of inflection.

58.   Suppose that the derivative of the function $y = g(x)$ is

$$y' = x^2(x-2)^3(x+3).$$

Find all local extrema and points of inflection.

59.   Find the values of constants a, b, and c so that the graph of $y = (x^2 + a) / (bx + c)$ has a local minimum at $x = 3$ and a local maximum at $(-1, -2)$.

60.   Does $F(x) = x^3 + 2x + \tan x$ have any local extrema.   Why?

61. *[if covered before test]*   Explain why any two anti-derivatives of a function F(x) *must differ by a constant.*

62.   Suppose that a is a positive constant. Consider the function

$$f(x) = x^3/3 - 4a^2x.$$

Determine all local and global extrema of *f* on the interval [-3a, 5a].

63. Solve the *initial-value problem:*

$$\frac{dy}{dx} = \frac{(\ln x)^4}{x}; \quad y(e) = 2012$$

Practice, the master of all things.

- Augustus Octavius