PRACTICE PROBLEMS FOR TEST III

1. Find an anti-derivative of each of the following functions. Show your work!

   (a) $(1+3x)^{2.9}$

   (b) $\dfrac{\sinh x}{13+\cosh x}$

   (c) $(\sin x)^8 \cos^3 x$

   Hint: Express this as a function of sin x multiplied by cos x.

   (d) $\sqrt{\cos x}\ \sin^3 x$ *Hint:* Similar to (c).

2. Find the *indefinite integral* of each of the following functions. Show your work!

   (a) $\dfrac{1+4\ln x}{x}$

   (b) $x^3 e^{5+8x^4}$

   (c) $\dfrac{\arctan x}{1+x^2}$

3. Using de l'Hôpital's rule, compute the following limit:

$$\lim_{x \to 0} \frac{e^{2x} - 1 - 2x - 2x^2}{x^3}$$

4. Solve the following initial value problem:

$$\frac{dy}{dt} = t^3 \cos(t^4) + t + 4 \qquad \text{given that y = 3 when t = 0.}$$

5. We wish to approximate the value of

$$\int_1^5 (1 - 2^{-x})\, dx.$$

(a) Sketch the curve $g(x) = 1 - 2^{-x}$ over the interval [1, 5].

Using *two rectangles* of equal base length, compute each of the following estimates:

(b) left-hand endpoints

(c) right-hand endpoints

(d) midpoints

6. *Verify* the following anti-differentiation formula:

$$\int x \sin(2x)\, dx = -\frac{1}{2} x \cos(2x) + \frac{1}{4} \sin(2x) + C$$

7. Using only the area interpretation of the definite integral, compute each of the following. (Sketch!)

$$(a) \quad \int_{-2}^{1} |1 + 2x|\, dx$$

$$(b) \quad \int_{1}^{7} (1 + 3x)\, dx$$

$$(c)\quad \int_0^7 \sqrt{49-t^2}\, dt$$

8. Compute each of the following. *Simplify* your answers as much as possible.

$$(a)\quad \sum_{k=0}^{2} \frac{k}{k+1}$$

$$(b)\quad \sum_{i=1}^{2} \left(i^4 - 2i\right)$$

$$(c)\quad \sum_{j=1}^{5} \ln j$$

$$(d)\quad \sum_{m=1}^{2015} m$$

$$(e)\quad \sum_{j=1789}^{2015} \frac{1}{j} - \frac{1}{j+1}$$

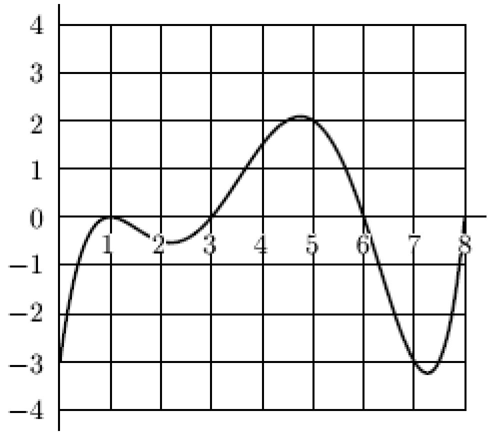
$$(f)\quad \sum_{k=1}^{2015} (-1)^k$$

9. Evaluate

$$\int_{-3}^{3} \left(x\sqrt{5+x^4} + \sqrt{9-x^2}\right) dx$$

by interpreting this definite integral in terms of areas. Sketch!

10. Consider the graph of y = f ′(x) drawn below. (*Note: This is not the graph of f.*)



(a) On which intervals, if any, is *f increasing?*

(b) At which values of *x* does *f* have a *critical point?*

(c) On which intervals, if any, is *f concave up?*

(d) Which values of *x*, if any, correspond to *inflection points* on the graph of *f*?

(e) Assume that f(0) = 0. Sketch a graph of *f*. (Your graph need only have the right general shape. You do not need to put units on the vertical axis.)

11. State the *two versions* of the Fundamental Theorem of Calculus.

12. Albertine has purchased a Chevy Volt that can accelerate from 0 ft/sec to 88 ft/sec in 5 seconds. The car's velocity is given below:

t (seconds)	0	1	2	3	4	5
v(t) (ft/second)	0	30	52	68	80	88

Using five rectangles, find upper and lower bounds (that is, over and under-estimates) for the distance traveled by Albertine's car in 5 seconds.

13. Using the FTC, evaluate each of the following. You *need not* simplify your answers.

$$(a)\ \int_0^{\frac{\pi}{4}} (\tan x)^4 \sec^2 x \ dx$$

$$(b)\ \int_0^1 \frac{e^{4x}}{5+3e^{4x}} \ dx$$

$$(c)\ \int_0^1 t \ \sqrt[4]{1+2t^2} \ dt$$

14. Using the method of judicious guessing evaluate:

$$\int \frac{1}{(\arcsin z)\ \sqrt{1-z^2}} dz$$

15. Using the FTC, compute g'(x) given that

$$g(x) = \int_0^x t^5 (5-4\ln t)^{13} \ dt$$

What is the value of g'(e) ?

16. Find an anti-derivative of each of the following functions. Show your work!

(a) tan x

(b) $\tan^2$ x

(c) $\sin^2$ x

Hint: Recall that $\sin^2 \theta = \dfrac{1-\cos 2\theta}{2}$

(d) $\cos^2$ x

Hint: Recall that $\cos^2 \theta = \dfrac{1+\cos 2\theta}{2}$

(e) $\sec^4$ x

Hint: Write $\sec^4$ x = ($\sec^2$ x)($\sec^2$ x) and then apply a basic identity.

17. Solve the initial value problem:

$$\frac{dy}{dx} = \frac{\pi}{4}\sec^2\left(\frac{\pi}{4}x\right) - \frac{2\ln x}{x}$$

given that y(1) = 13.

18. Evaluate each of the following indefinite integrals:

(a) $\int \frac{\cos x}{\sin x + 13}\, dx =$

(b) $\int x\sin(x^2+5)\, dx =$

(c) $\int x^2(11x^3+99)^{51}\, dx =$

19. Evaluate each of the following definite integrals using only the geometric interpretation of the definite integral. Explain your solution.

(a) $\int_0^2 |1-3x|\, dx$

(b) $\int_0^7 \sqrt{49-x^2}\, dx$

(c) $\int_{-7}^7 \frac{x^5}{1+x^2+\cos x}\, dx$

(d) $\int_{-5}^5 \left(3 + x(\cos x)e^{x^2}\right) dx$

20. Calculate each of the following sums. Simplify each answer.

(a) $\sum_{j=3}^{4} \sin\left(j\frac{\pi}{2}\right)$

(b) $\sum_{k=2}^{4} \frac{1}{k-1}$

(c) $\sum_{m=1}^{5} m^2$

21. Given that $\int_{-2}^{2} f(x)\,dx = 4 \;\; and \;\; \int_{2}^{5} f(x)\,dx = 3 \;\; and \;\; \int_{-2}^{5} g(x)\,dx = 2$

compute the value of $\int_{-2}^{5} \left(f(x)+2g(x)+5\right)\,dx.$

22. Find the *average value* of the function f(x) = 4 sin 2x over the interval [0, π/2].

23. Which values of *a* and *b* *minimize* the value of

$$\int_{a}^{b} (x^6 - 25x^4)\,dx$$

24. Find dy/dx given that

$$y = \int_{1}^{4x^2+1} \sin\sqrt{t}\;dt$$

25. Find the area bounded by the x-axis and the curve $y = x(x-3)^2$.

26. Find the area between the curves $y = 5 - x$ and $y = 7 - x^2$.

27. Find the area of the region in the first quadrant that is bounded above by $y = x^{1/2}$ and below by the x-axis and the line $y = x - 2$.

28. Compute the *average value* of the curve $y = \cos^2 4x$ over the interval $[0, \pi]$.

29. Let f(x) be a continuous function. Express

$$\lim_{n\to\infty} \frac{1}{n}\left(f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + f\left(\frac{3}{n}\right) + \ldots + f\left(\frac{n}{n}\right)\right)$$

as a definite integral.

30. Compute the value of the following limit

$$\lim_{n\to\infty} \frac{1^{15} + 2^{15} + 3^{15} + \ldots + n^{15}}{n^{16}}$$

by expressing this limit as a Riemann integral and then evaluating the integral.

31. Evaluate each of the following.

(a) $\displaystyle \lim_{x\to 0} \frac{e^{x^3} - 1 - x^3}{x^6}$

(b) $\displaystyle \sum_{j=0}^{3} \frac{j}{j+3}$ *(and simplify)*

(c) $\displaystyle \sum_{k=0}^{4} \ln(k+1)$ *(and simplify)*

(d) $\displaystyle \int_0^9 4\sqrt{81 - x^2}\ dx$

32. Consider the following table of data:

x	-1.00	-0.25	0.50	1.25	2.00
F(x)	0.0000	2.6522	4.8755	6.8328	8.6790

Approximate the area below the graph of y = F(x) above the interval [-1, 2] using:

(a) left endpoints. *Sketch.*

(b) right endpoints. *Sketch.*

33. Suppose that Madame Verdurin wishes to find a solution to the equation

$$\ln x = x - 5.$$

(a) Sketch the graphs of the two curves (y = ln x and y = x – 5) on the same pair of axes. How many solutions would you expect to find?

(b) Define a function, G(x), whose roots will provide Verdurin with the answer to her quest.

(c) Assume that Verdurin's first guess is $x_0 = 6$. Verdurin uses Newton's method to compute x_1 and x_2. Which values should she obtain! Show your computation.

34. Solve the following initial value problem:

$$\frac{dy}{dt} = \frac{2}{1+4t^2} - \frac{3}{t+1}$$

given that y(0) = 11.

35. Verify the following integration formula:

$$\int \frac{x^2}{\sqrt{2x+3}}\,dx = \frac{1}{5}\sqrt{2x+3}\,\left(6-2x+x^2\right)+C$$

36. At time *t*, in seconds, the velocity, *v*, in miles per hour, of Albertine's new Prius is given by

$$v(t) = 5 + 0.8t^2 \text{ for } 0 \le t \le 8.$$

Use $\Delta t = 2$ to estimate the distance traveled during this time. Find the left endpoint and right endpoint sums, and the average of the two. *Sketch!*

37. Using the method of judicious guessing, find an *antiderivative* for each of the following functions. *Be certain to show your reasoning!*

(a) $\dfrac{(x+5)(2x-1)}{x^3}$

(b) $\dfrac{(\ln x)^{99}}{x}$

(c) $\dfrac{\cosh x}{\sqrt{5+3\sinh x}}$

(d) $\dfrac{\sec(4x)(\tan(4x)}{1+3\sec(4x)}$

38. Suppose that $\int_{-2}^{3}(f(x)-3x)dx = 0.$ Evaluate $\int_{-2}^{3}(f(x)-x^3)dx.$

39. Find the area of the region enclosed by $y = x(a^2 - x^2)^{1/2}$, where a is a positive constant and the y-axis.

40. Find the area of the region enclosed by $y = (|x|)^{1/2}$ and $5y = x + 6$.

41. Find the area of the region enclosed by $x = y^3 - y^2$ and $x = 2y$.

42. Find the area of the region enclosed by $y = 2 \sin x$ and $y = \sin (2x)$, $0 \le x \le \pi$.

43. *Find dy/dx given that* $y = \int_{\sec(3x)}^{x} \frac{1}{t^2 - 1}\, dt$

44. Evaluate the following limit:

$$\lim_{x \to 0} \frac{\sqrt{4+x} - 2}{x}$$

45. Evaluate the following limit

$$\lim_{x \to \infty} x^{3/x}$$

46. Evaluate the following limit

$$\lim_{x \to 0} \frac{\sqrt{1+2x} - 1 - x}{x^2}$$

47. Evaluate the following limit

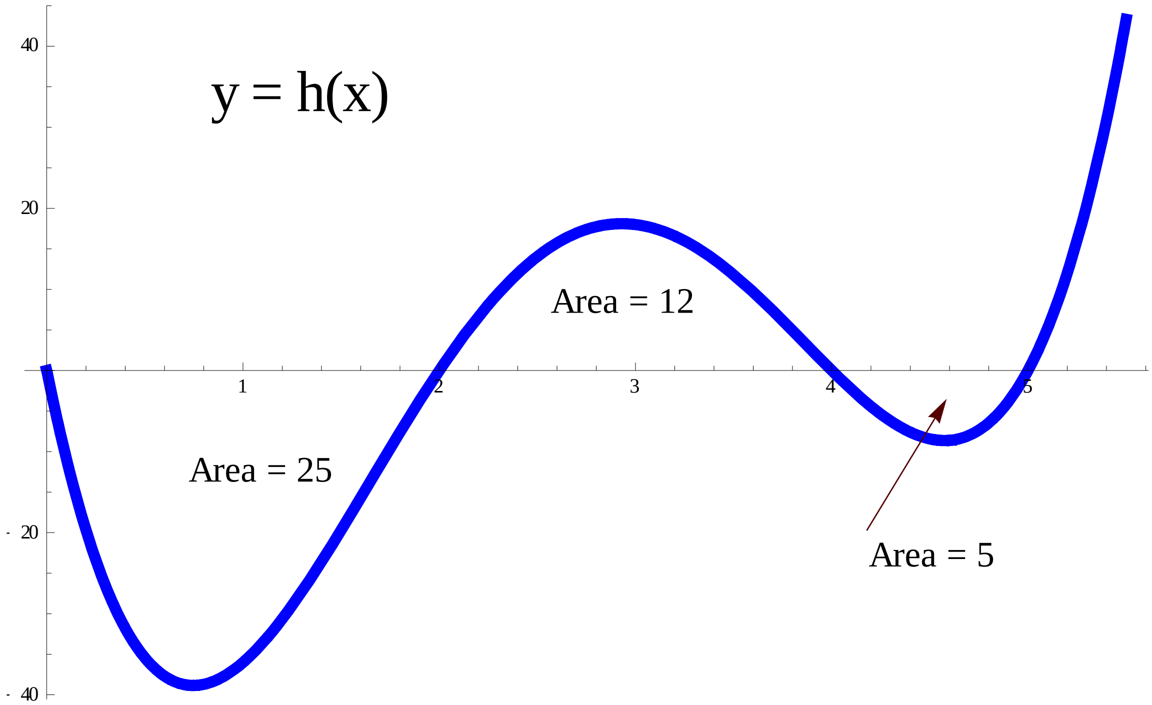
$$\lim_{x \to 0} \frac{\cos(4x) - 1 + 8x^2}{x^4}$$

48. Evaluate the following limit

$$\lim_{x \to 1} \frac{x^{1357} - 1}{x - 1}$$

49. Explain why the value of $\int_0^3 \cos^2(x^4)\, dx$ cannot equal 4.5.

50. *[U. Michigan]* Using the graph of y = h(x) shown below, compute each of the following quantities. If there is not enough information to compute the given quantity write “not enough information.” Justify each answer!



(A) $\int_{2}^{0} \left(h(x)+3\right)\, dx$

(B) $\int_{0}^{5} 3h(y)\, dy$

(C) $\int_{8}^{9} h(x-4)\, dx$

(D) The *average value* of h(x) on the interval [-2, 2], assuming that h(x) is an even function.

(E) H(2), where H is an anti-derivative of *h*.

(F) H(2) – H(0), where H is an anti-derivative of *h*.

51. (a) State **Rolle's Theorem**.

(b) Using Rolle's Theorem, prove that the function

$$g(x) = (x - 2) \ln (x + 1) + x \sin(4\pi x)$$

has *at least one* critical point between x = 0 and x = 2?   Explain!

52. (a) State the **Mean Value Theorem**.

(b) Show how the Mean Value Theorem applies to the function

f(x) = 4 + ln x on the interval $[1, e^3]$. Sketch! Find explicitly the *c* value.

53. Explain why any two anti-derivatives of a function F(x) must differ by a constant.

54. Find dy/dt given that

$$y = \int_1^{\sinh t} \cosh\sqrt{v}\, dv$$

55. (a) State **Rolle's Theorem**.

(b) Using Rolle's Theorem, prove that the function

$$g(x) = (x - 2) \ln (x + 1) + x \sin(4\pi x)$$

has *at least one* critical point between x = 0 and x = 2?   Explain!

56. *(U. Michigan)* A car, initially going 100 feet per second, brakes at a constant rate (constant negative acceleration), coming to a stop in 8 seconds. Let t be the time in seconds after the car started to brake.

(a) Sketch a graph of the velocity of the car from t = 0 to t = 8, being sure to include labels.

(b) Exactly how far does the car travel? Make it clear how you obtained your answer.

57. *(U. Michigan)* Suppose dg/dx > 0 on the interval [3, 5], g(3) = 12, and g(5) = 20.

We want to use a *Riemann sum* with equal-size subdivisions to approximate

$$\int_3^5 g(x)\,dx$$

If we want to guarantee that the error in our estimate is *no larger than ¼*, then what is the minimum number of subdivisions that we must use?

58. *(U. Michigan)* If

$$\int_{-1}^{4} \big(2f(x)-7\big)dx = -31$$

find the value of

$$\int_{-1}^{4} f(x)\ dx$$

59. State *three corollaries* to the Mean Value Theorem.

60. *(Apostol)* A function *f* is continuous everywhere and satisfies the equation

$$\int_1^x f(t)\ dt = -\frac{1}{2} + x^2 + x\sin 2x + \frac{1}{2}\cos 2x$$

for all x. Compute f(π/4) and df/dx at x = π /4.

61. *(Apostol)* Find a function *f* and a value of the constant *c* such that, for all real *x*,

$$\int_c^x f(t)\ dt = \cos x - \frac{1}{2}$$

62. Can the following limit be solved using L'Hopital's rule? *Explain.*

$$\lim_{x\to 1} \frac{3x^2+1}{2x-3}$$

63. Can the following limit be solved using only L'Hopital's rule? *Explain.*

$$\lim_{x\to 0}\frac{\sin x}{x^2+3x}$$

64. Find:

$$\lim_{n\to\infty}\frac{1}{n}\left(e^{1/n}+e^{2/n}+\ldots+e^{(n-1)/n}+e^{n/n}\right)$$

65. Can the following limit be solved using L'Hopital's rule? *Explain.*

$$\lim_{x\to 0}\frac{\cos 2x-1}{x^3+5x^2}$$

66. Can the following limit be solved using only L'Hopital's rule? *Explain.*

$$\lim_{x\to\infty}\frac{e^x}{x^{13}}$$

67. Can the following limit be solved using only L'Hopital's rule? Explain.

$$\lim_{x\to 0}\frac{\sqrt{x}}{\ln x}$$

68. *(MIT 18.01 final)*

Use L'Hopital's rule to compute the following limits:

$$(i)\quad \lim_{x\to 0}\frac{a^x-b^x}{x}\quad \textit{where } 0<a<b$$

$$(ii)\quad \lim_{x\to 1}\frac{4x^3-5x+1}{\ln x}$$

69. *(U. Michigan)* The rate at which a coal plant releases CO_2 into the atmosphere t days after 12:00 am on January 1, 2015 is given by the function $E(t)$ measured in tons per day. Suppose that $\int_0^{31} E(t)\,dt = 223$.

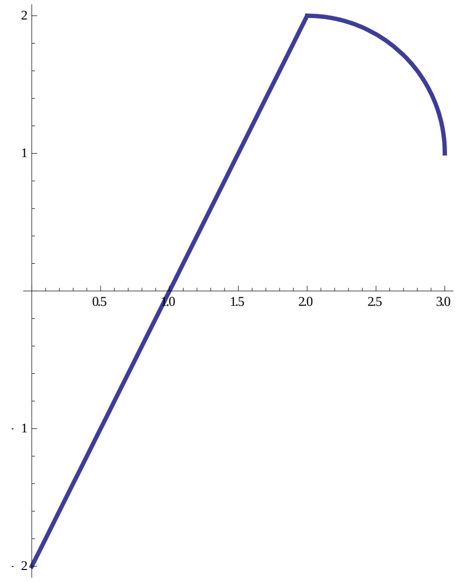
(a) Give a practical interpretation of $\int_{31}^{59} E(t)\,dt$.

(b) Give a practical interpretation of E(15) = 7.1.

(c) The plant is upgrading to “clean coal” technology which will cause its July 2015 CO_2 emissions to be one fourth of its January 2015 CO_2 emissions. How much CO_2 will the coal plant release into the atmosphere in July?

(d) Using a left-hand sum with four subdivisions, write an expression which approximates $\int_{31}^{59} E(t)\,dt$.

70. *(U. Michigan)* Shown below is a graph of a function f(t). The graph consists of a straight line between t = 0 and t = 2 and a quarter circle between t = 2 and t = 3.



Calculate the following using the graph and the properties of integrals.

(a) $3\int_0^3 (2+f(t))\,dt$

(b) The *average value* of *f* on the interval [1, 3].

71. *(U. Michigan)* A car, initially going 100 feet per second, brakes at a constant rate (constant negative acceleration), coming to a stop in 8 seconds. Let t be the time in seconds after the car started to brake.

(a) Sketch a graph of the velocity of the car from t = 0 to t = 8, being sure to include labels.

(b) Exactly how far does the car travel? Make it clear how you obtained your answer.

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$$\int_3^5 g(x)\,dx$$

If we want to guarantee that the error in our estimate is no larger than ¼, then what is the *minimum* number of subdivisions that we must use?

Learning without thought is labor lost; thought without learning is perilous.

- Confucius