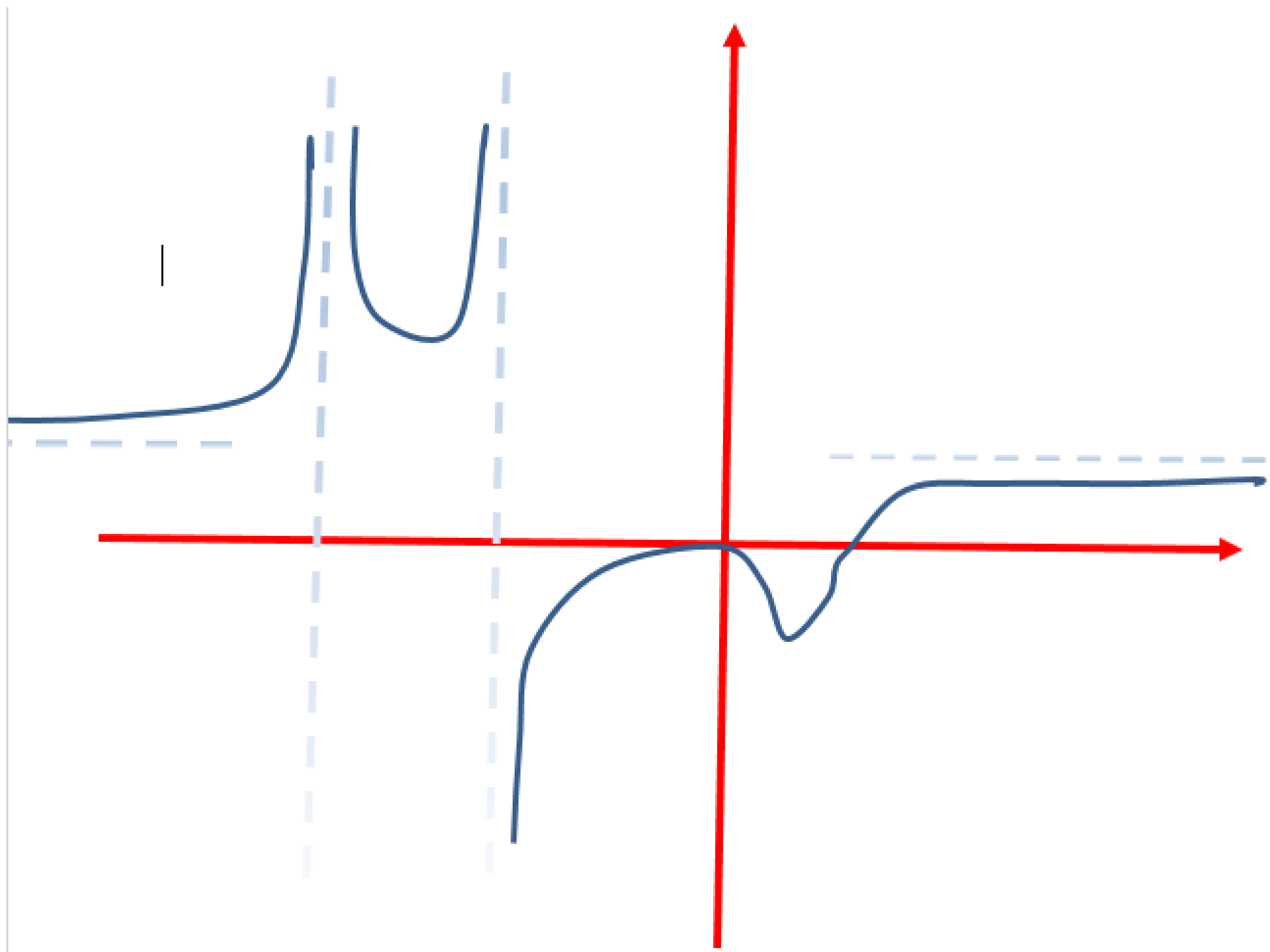
MATH 161 SOLUTIONS: QUIZ I

1. [10 pts] The graph of a rational function is shown below.

   Assume that

   Zeros: x = 0, x = 3
   Singularities: x = -2, x = -4
   Limiting behavior: y → 3 as |x| → ∞



Find an equation of a rational function that incorporates all of this information. (Note that this problem may have more than one correct answer.)

Solution:

Given the information about the zeros, we find that x and x + 3 must be factors of the numerator. Given the information about the singularities, x + 2 and x + 4 must be factors of the denominator.

Since the zero at x = 0 does not create a sign change, we find that x^2 or any even power of x, must a factor of the numerator. Since the singularity at x = -4 also results in no sign change, we find that $(x + 4)^2$ or any even power of x + 4, must be a factor of the denominator.

So our first guess is:

$$y = \frac{x^2(x-3)}{(x+2)(x+4)^2}$$

Noting that the value of y as x → ∞ is 1, we have only to make one change:

$$y = \frac{3x^2(x-3)}{(x+2)(x+4)^2}$$

2. [5 pts each] Compute each of the following limits. (*Explain your reasoning*. You may use estimation techniques, tables, graphing calculators, etc.)

$$y = \frac{(x^3+11)^2(3x-91)^3}{(2x^2+5)^4(x+2015)}$$

Solution: Observe that:

$$\frac{(x^3+11)^2(3x-91)^3}{\left(2x^2+5\right)^4(x+2015)} \cong \frac{(x^3)^2(3x)^3}{(2x^2)^4 x} = \frac{27}{16}\left(\frac{x^9}{x^9}\right) \to \frac{27}{16} \text{ as } x \to \infty$$

$$(b) \quad \lim_{x\to 2} \frac{\frac{1}{x^2}-\frac{1}{4}}{x-2}$$

Solution:

$$\frac{\frac{1}{x^2}-\frac{1}{4}}{x-2} = \frac{\frac{4}{4x^2}-\frac{x^2}{4x^2}}{x-2} = \frac{4-x^2}{4x^2\left(x-2\right)} =$$

$$\frac{-(x-2)(2+x)}{4x^2(x-2)} = \frac{-(2+x)}{4x^2} \to -\frac{4}{16} = \frac{1}{4} \text{ as } x \to 2$$

$$(c) \quad \lim_{x\to 5} \frac{2x^2-9x-5}{x^2-8x+15}$$

Solution:

Observe that, as long as $x \neq 5$:

$$\frac{2x^2-9x-5}{x^2-8x+15} = \frac{(x-5)(2x+1)}{(x-5)(x-3)} = \frac{2x+1}{x-3} \to \frac{11}{2} \text{ as } x \to 5$$

(d) $\lim_{x\to 0} \dfrac{\sqrt{x+4}-2}{x}$

Solution: We begin by rationalizing the numerator of the algebraic expression,

$$\frac{\sqrt{x+4}-2}{x} = \left(\frac{\sqrt{x+4}-2}{x}\right)\left(\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}\right) = \frac{x}{(x)(\sqrt{x+4}+2)} =$$

$$\frac{1}{\sqrt{x+4}+2} \to \frac{1}{\sqrt{4}+2} = \frac{1}{4} \quad as \;\; x \to 0.$$

3. [10 pts] Does there exist a *continuous extension* to the curve

$$g(x) = \frac{3x^2-4x+1}{x^4-1}$$

at x = 1? If so, find it; if not explain! (Hint: Factor first.)

Solution: Let's begin by factoring, noting that the denominator is a difference of two squares.

$$g(x) = \frac{(x-1)(3x-1)}{(x^2+1)(x+1)(x-1)}$$

Now, as $x \to 1$, we can cancel the $x - 1$ factor occurring both in the numerator and the denominator. So, for $x \neq 1$

$$g(x) = \frac{3x-1}{(x^2+1)(x+1)}$$

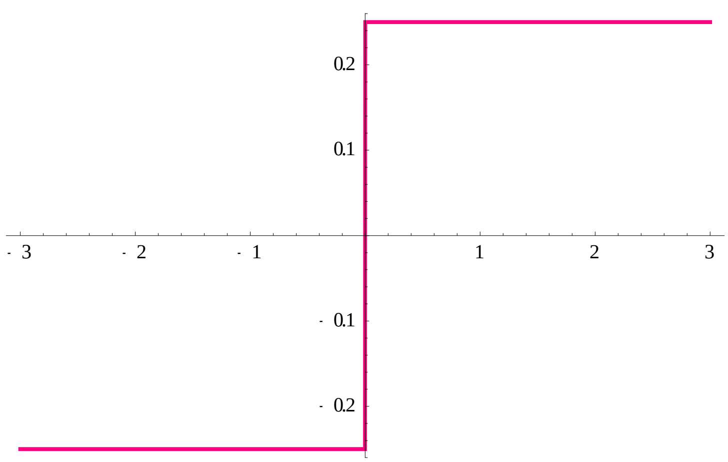
Now, as $x \to 1$, $g(x) \to 2/4 = ½$.

Thus x = 1 is a removable discontinuity, and we should assign the value of ½ to g(1).

4. [3 pts each] Identify the type of discontinuity that each of the following functions has at x = 0. (Choose from: removable, infinite, jump, or essential discontinuity.) You need not justify your answers.

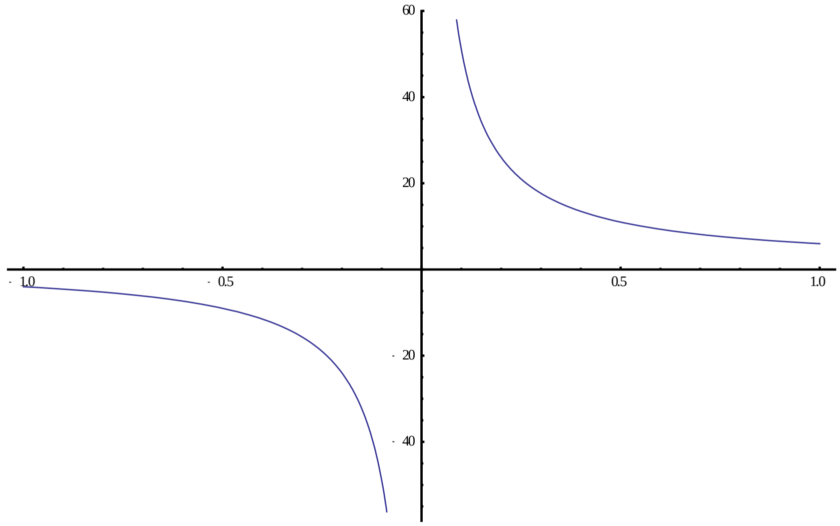
(a) $y = \dfrac{|x|}{4x}$

Solution: Since the limit of y as x→0 from the right is ¼, but the limit of y as x→0 from the left is -¼, the discontinuity at x = 0 is a jump discontinuity.



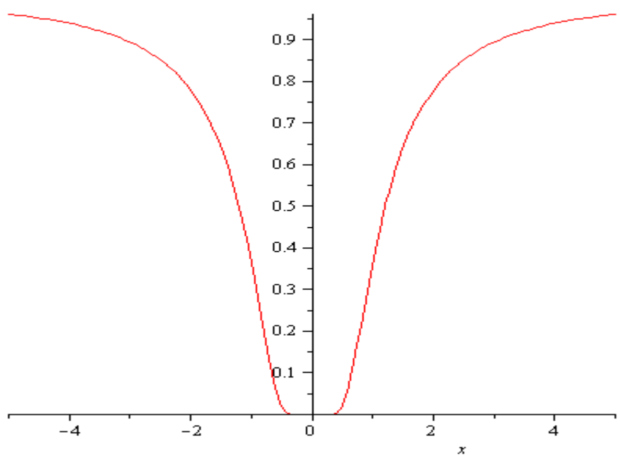
(b) $y = \dfrac{|x+5|}{x}$

Solution: Since the numerator tends toward 5 and the denominator tends toward 0, the limit does not exist. This is an infinite discontinuity



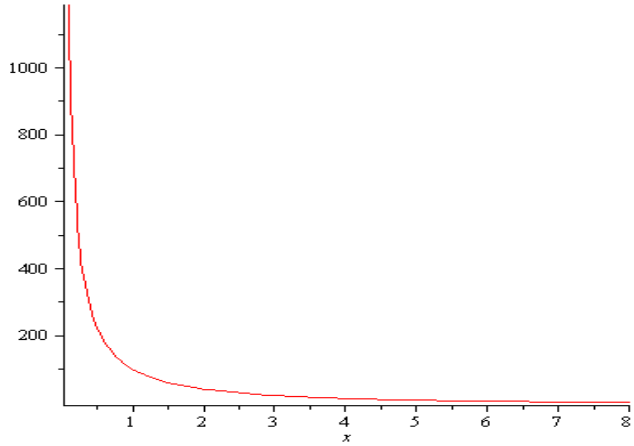
(c) $y = \dfrac{1}{e^{\frac{1}{x^2}}}$

Solution: Since $\lim_{x\to 0} e^{\frac{1}{x^2}} = \infty$ *it follows that* $y \to 0$ *as* $x \to 0$*. Hence the discontinuity is removable.*



(d) $y = \dfrac{(x-2015)^5}{x}$

Since $\lim\limits_{x \to 0+} \dfrac{(x-2015)^5}{x} = \infty$ *the discontinuity is infinite.*



Extra Credit:

(*University of Michigan* Calculus problem (first exam, 7 Oct 2014)

Consider the function h defined by

$$h(x) = \begin{cases} \dfrac{60(x^2 - x)}{(x^2+1)(3-x)} & \text{for } x < 2 \\ c & \text{for } x = 2 \\ 5e^{ax} - 1 & \text{for } x > 2 \end{cases}$$

where a and c are constants.

Find values of a and c so that both of the following conditions hold.

- $\lim\limits_{x \to 2} h(x)$ exists.
- $h(x)$ is not continuous at $x = 2$.

Note that this problem may have more than one correct answer. You only need to find one value of a and one value of c so that both conditions above hold. Remember to show your work clearly.

Solution: In order for $\lim_{x\to 2} h(x)$ to exist, it must be true that $\lim_{x\to 2^-} h(x) = \lim_{x\to 2^+} h(x)$. Now $\lim_{x\to 2^-} h(x) = \dfrac{60(2^2-2)}{(2^2+1)(3-2)} = 24$ and $\lim_{x\to 2^+} h(x) = 5e^{2a} - 1$. So it follows that $5e^{2a} - 1 = 24$. Solving for a, we have

$$\begin{aligned} 5e^{2a} - 1 &= 24 \\ e^{2a} &= 5 \\ 2a &= \ln(5) \\ a &= \ln(5)/2 \approx 0.804. \end{aligned}$$

When $a = \ln(5)/2$, $\lim_{x\to 2} h(x) = 5e^{(\ln(5)/2)*2} = 5e^{\ln(5)} - 1 = 24$. So, h is not continuous at $x = 2$ as long as $\lim_{x\to 2} h(x) \neq h(2)$. Since $h(2) = c$, we can choose c to be any number other than 24.

O dear Ophelia!
I am ill at these numbers:
I have not art to reckon my groans.

- HAMLET (Act II, Sc. 2)