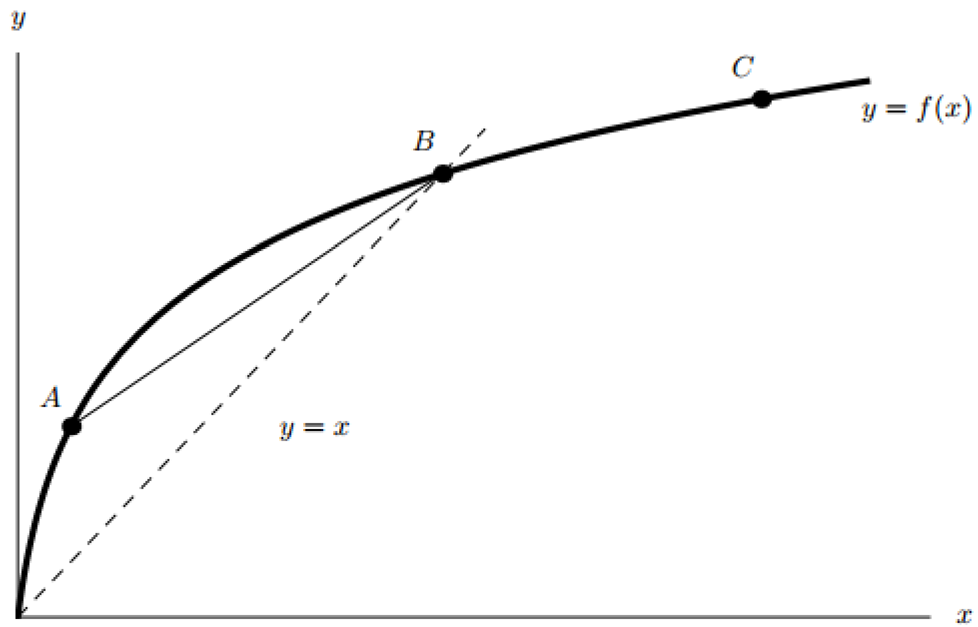
MATH 161 SOLUTIONS: QUIZ III

1. *[12 pts; University of Michigan]* For the graph of y = f(x) in the figure below, arrange the following numbers from smallest to largest:

A The slope of the curve at A.
B The slope of the curve at B.
C The slope of the curve at C.
AB The slope of the line *AB*.
0 The number 0.
1 The number 1.

Explain the positions of the number 0 and the number 1 in your ordering. Any unclear answers will not receive credit.



0 < **C** < **B** < **AB** < **1** < **A**

Solution: The number one and all other slopes are positive, so 0 must be the smallest number. The line $y = x$ has a slope of 1. The slope at C, the slope at B, and the slope of the line AB are each smaller than the slope of the line $y = x$ by looking at the picture. The slope at A is larger than the slope of $y = x$ also by the picture. Thus 1 is the second to largest number in the ordering.

2. (a) [10 pts] Using *only the definition of the derivative*, find the slope of the tangent line to the curve $G(x) = \sqrt{x+8}$ *at x = 1.* Show your work!

Solution:

$$\frac{G(1+h) - G(1)}{h} =$$

$$\frac{\sqrt{(1+h)+8} - \sqrt{9}}{h} =$$

$$\frac{\sqrt{9+h} - 3}{h} =$$

$$\left(\frac{\sqrt{9+h} - 3}{h}\right)\left(\frac{\sqrt{9+h} + 3}{\sqrt{9+h} + 3}\right) =$$

$$\frac{(9+h) - 9}{h(\sqrt{9+h} + 3)} =$$

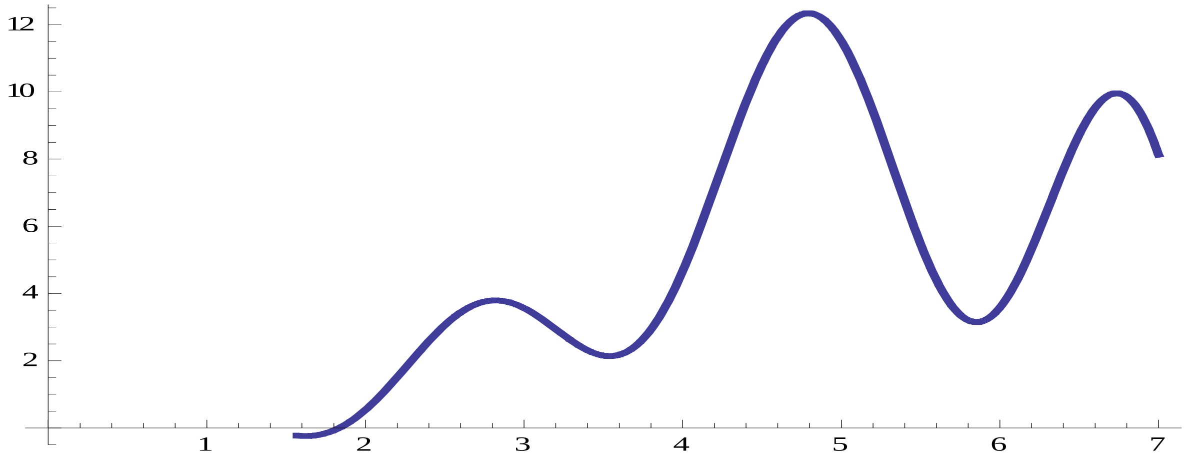
$$\frac{h}{h(\sqrt{9+h} + 3)} =$$

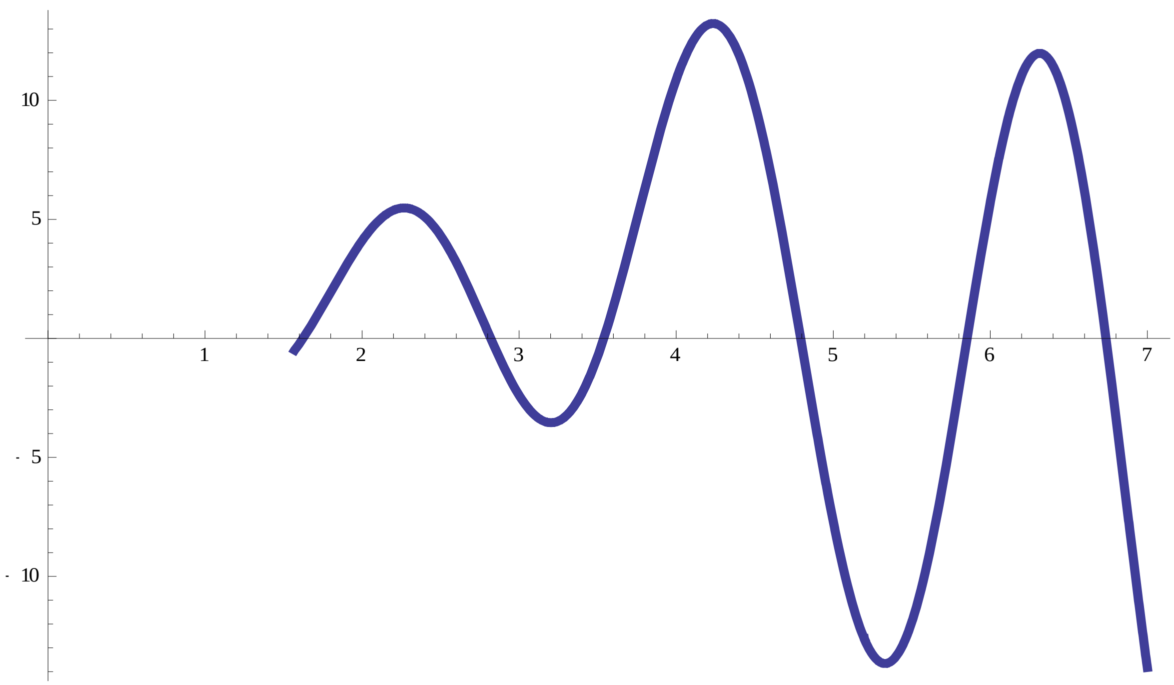
$$\frac{1}{\sqrt{9+h} + 3} \to \frac{1}{\sqrt{9+0} + 3} = \frac{1}{6}$$

(b) [3 pts] Using the result obtained in part (a), write an equation of the normal line to the curve y = G(x) at x = 1.

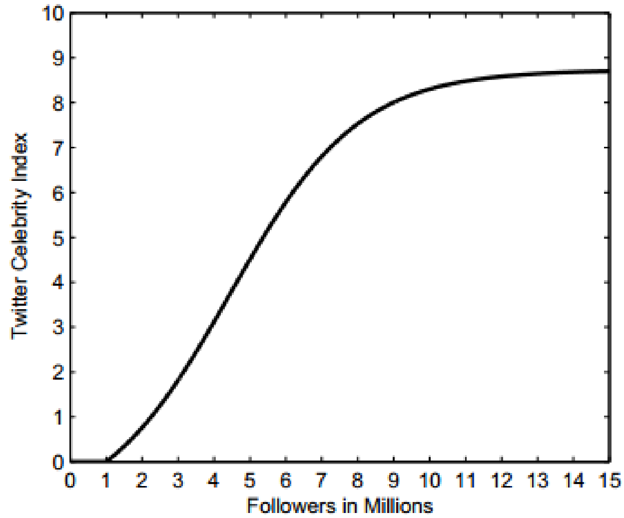
Solution: Since the point of tangency is (1, 3) and the slope of the tangent line is 1/6, the slope of the normal (perpendicular) line is -6. An equation of the normal line is y – 3 = -6(x – 1)

3. [8 pts] Below is the graph of $G(x) = x + x^{1.3} \cos(2x) \sin x$ defined on the interval [1.57, 7]. On the axes beneath the graph of *G*, sketch the graph of the derivative function, $y = G'(x)$.





4. *[University of Michigan problem]* The Twitter Celebrity Index (TCI) measures the celebrity of Twitter users; the function T(x) takes the number of followers (in millions) of a given user and returns a TCI value from 0 to 10. Below is a graph of this function.



Use the graph above to help you answer the following questions. *Use complete sentences!*

(a) (3 pts) Explain in practical terms what $T(13.72) = 8.67$ means.

When a Twitter user has 13.72 million followers, s/he has a Twitter Celebrity Index of 8.67.

(b) (3 pts) Explain in practical terms what $T^{-1}(4.25) = 4.88$ means.

When a Twitter user has Twitter Celebrity index of 4.25, s/he has 4.88 million followers.

(c) (3 pts) Explain in practical terms what $T'(10) = 0.2278$ means.

When a Twitter user has 10 million followers, adding 100,000 followers will increase her TCI by roughly 0.02278.

EXTRA CREDIT

The curves $y = x^2 + ax + b$ and $y = cx – x^2$ have a common tangent line at the point P = (1, 0). Find *a*, *b* and *c*.
Hint: Recall the following result: $(d/dx)(\alpha x^2 + \beta x + \delta) = 2\alpha x + \beta$

Solution:

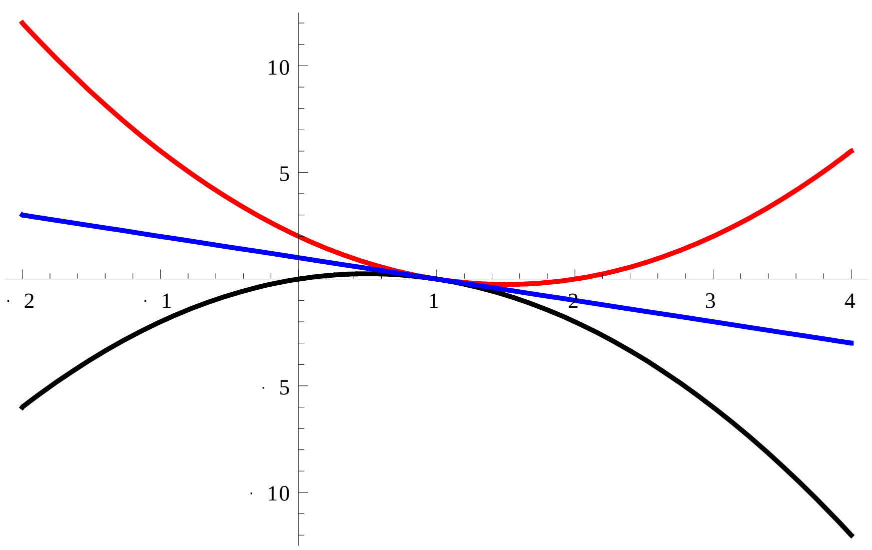
Since each parabola passes through (1, 0), we must have:

$$0 = 1 + a + b \text{ and } 0 = c - 1$$

Thus $a + b = -1$ *and* $\mathbf{c = 1}$.
Now the slope of the first parabola at $x = 1$ *is* $dy/dx = 2x + a$ *(at* $x = 1$*)* $= 2 + a$
And the slope of the second parabola at $x = 1$ *is* $dy/dx = c - 2x$ *(at* $x = 1$*)* $= c - 2$.
Since the two parabolas share a common tangent at (1, 0), we must have:

$$2 + a = c - 2.$$

Hence $a = c - 4 = 1 - 4 = -3$.
Finally, $b = -1 - a = -1 - (-3) = 2$.
Summarizing: $\mathbf{a = -3, b = 2, c = 1}$
Below is a sketch of the two parabolas and their common tangent line at (1, 0).



It's very good jam," said the Queen.
"Well, I don't want any today, at any rate."
"You couldn't have it if you did want it," the Queen said. "The rule is jam tomorrow and jam yesterday but never jam today."
"It must come sometimes to 'jam to-day,'" Alice objected.
"No it can't," said the Queen. "It's jam every other day; today isn't any other day, you know."
"I don't understand you," said Alice. "It's dreadfully confusing."

\- Lewis Carroll, **Through the Looking Glass**