MATH 161 SOLUTIONS: TEST I

The more you know, the less sure you are.

- Voltaire

1. *[University of Michigan]* For each of the following three sets of axes, exactly one of the following statements **(A)-(E)** is true. You may use a letter more than once. In the space provided next to each figure, enter the letter of the true statement for that figure. For each graph, note that the *x* and *y* scales are not the same.

(A) h is the derivative of f, and f is the derivative of g.

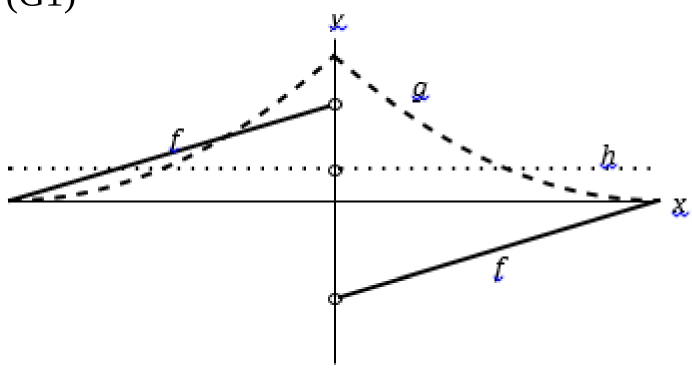
(B) g is the derivative of f, and f is the derivative of h.

(C) g is the derivative of h, and h is the derivative of f.

(D) h is the derivative of g, and g is the derivative of f.

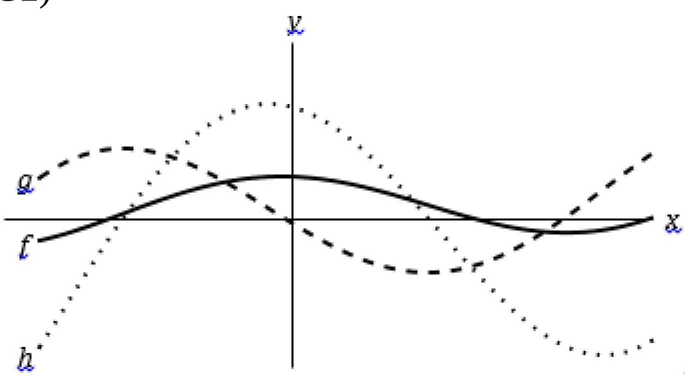
(E) none of (A) through (D) are possible

(G1)



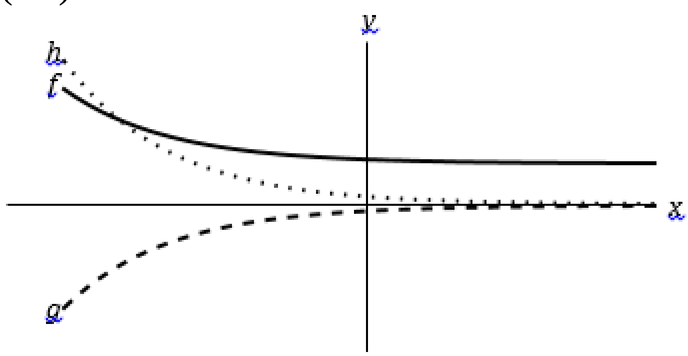
ANSWER: __A_____

(G2)



ANSWER: _E_______

(G3)



ANSWER: _D_____

2. Consider the rational function $f(x) = \dfrac{x(3x+1)^2(2x+3)^4}{(x-5)^7}$

(a) Find the *domain* of *f*.

Solution: f(x) is defined for all real numbers except 5

(b) Find the y–intercept of *f*.

Solution: The y-intercept of f is f(0) = 0.

(c) Find the zeroes of *f*.

Solution: f(x) = 0 when x = 0, -1/3, -3/2

(d) Find the vertical asymptote(s) of *f*.

Solution: A unique singularity occurs at x = 5; hence the only vertical asymptote is the line x = 5.

(e) Find the horizontal asymptote(s) of *f*.

Solution: For large |x|,

$$f(x) = \frac{x(3x+1)^2(2x+3)^4}{(x-5)^7} \approx \frac{x(3x)^2(2x)^4}{(x)^7} = \frac{x(9x^2)(16x)^4}{x^7} = 144$$

Thus the only horizontal asymptote is the line: y = 144

3. Albertine orders a large cup of coffee at Metropolis on Granville. Let F(t) be the temperature in *degrees Fahrenheit* of her coffee *t minutes* after the coffee is placed on her tray.

(a) Explain the meaning of the statement: F(9) = 167. (Use complete sentences. Avoid any mathematical terms!)

Solution: Nine minutes after being served her coffee, Albertine finds that the coffee is 167°F.

(b) Explain the meaning of the statement: $F^{-1}(99) = 17.5$

Solution: When the temperature of the coffee is 99°F, 17.5 minutes have elapsed since the coffee mug was placed on her tray.

(c) Give the *practical* interpretation of the statement: $F'(9) = -1.10$. (Use complete sentences. Do not use the words "derivative" or "rate" or any other mathematical term in your explanation.)

Solution: Nine minutes after being served her coffee, Albertine notices that the temperature of the coffee is decreasing by about 0.11°F every tenth of a minute during the next minute or two.

(d) What are the *units* of $F'(9)$?

Solution: °F/ minute (since $\Delta F/\Delta t$ represents change in temp/ change in time.)

(e) Using the information given in parts (a) and (c), estimate the temperature of Albertine's coffee seven minutes after she has been handed the coffee.

Solution: Since after nine minutes the temperature is 167°F, we would estimate that two minutes earlier the temperature was 167 + 2(1.1) = ***169.2 °F****.*

(e) **EXTRA CREDIT**

Explain the meaning of the statement:

$$(F^{-1})'(99) = -1$$

Solution: When the temperature of the coffee is 99°F, the time required for the temperature of the coffee to fall to 98 °F is one additional minute.

4. The following function is *not* continuous at the points x = 5/2 and x = 4/3.

$$g(x)=\frac{2x^5-19x^4+35x^3}{6x^2-23x+20}$$

(a) *[6 pts]* Does *g* possess a *continuous extension* at x = 5/2? If so, how should g(5/2) be defined? If not, explain!

Solution:

We begin by factoring:

$$g(x)=\frac{2x^5-19x^4+35x^3}{6x^2-23x+20}=\frac{x^3(2x-5)(x-7)}{(3x-4)(2x-5)}$$

If x ≠ 5/2, then cancellation yields:

$$g(x)=\frac{x^3(x-7)}{3x-4}$$

So we see that x = 5/2 is a removable discontinuity.

To create a continuous extension to g at x = 5/2, we define g(5/2) as follow:

$$g(5/2)=\frac{(5/2)^3(5/2-7)}{3(5/2)-4}=-\frac{1125}{56}\cong -20.09$$

(b) *[6 pts]* Does *g* possess a *continuous extension* at x = 4/3? If so, how should g(4/3) be defined? If not, explain!

Solution:

We begin by noting that, when x ≠ 5/2,

$$g(x)=\frac{x^3(x-7)}{3x-4}$$

Now, as x → 4/3, the numerator tends toward $(4/3)^3$ (4/3 – 7) ≅ -13.43, but the denominator tends toward 0. Consequently, the limit of g(x) as x → 4/3 does not exist. (This is an infinite discontinuity.)

5. *[University of Michigan]* Suppose $f(x) = \left(x + \frac{1}{2}\right)e^x$

a. Using the limit definition of the derivative, write an explicit expression for $f'(2)$. Your expression should not contain the letter "f". *Do not try to evaluate your expression.*

Solution: $$f'(2) = \lim_{h\to 0} \frac{(2+h+\frac{1}{2})e^{2+h} - \left(2+\frac{1}{2}\right)e^2}{h}$$

b. Albertine discovers that the derivative of $f(x)$ *is* $f'(x) = \left(x + \frac{3}{2}\right)e^x$. Using this formula for $f'(x)$, write an equation for the tangent line to the graph of $f(x)$ at $x = 2$.

Solution: The slope is $m = (2 + \frac{3}{2})\, e^2 = \frac{7}{2}\, e^2$ *and the y-coordinate when* $x = 2$ *is* $f(2) = \left(2 + \frac{1}{2}\right)e^2 = \frac{5}{2}e^2$. *Using the point-slope form for the tangent line, we obtain:*

$$y - \frac{5e^2}{2} = \frac{7}{2}e^2(x-2)$$

c. *[3 points]* Write an equation for the tangent line to the graph of $f(x)$ at $x = a$ where a is an unknown constant.

Solution: Employing virtually the same computation that we did in (b), we find:

$$y - \left(a + \frac{1}{2}\right)e^a = \left(a + \frac{3}{2}\right)e^a(x - a)$$

d. EXTRA CREDIT Using your answer from (c), find a value of a so that the tangent line to the graph of $f(x)$ at $x = a$ passes through the origin.

Solution: The point (0, 0) must satisfy the tangent line equation in (e). Hence

$$0 - \left(a + \frac{1}{2}\right)e^a = \left(a + \frac{3}{2}\right)e^a(0 - a)$$

Dividing each side by e^a *we obtain:*

$$-\left(a+\frac{1}{2}\right)=-a\left(a+\frac{3}{2}\right)$$

So -2a – 1 = -2a² – 3a

Simplifying: $2a^2 + a - 1 = 0$

Factoring: $(2a - 1)(a + 1) = 0$

Hence $a = ½$ *or* $a = -1$

6. Compute each of the following limits. *Justify your reasoning.*

$$(a)\quad \lim_{x\to 1} \frac{(x-2)^5}{x^{2015}+4x-6}$$

Solution: Using the laws of limits:

$$\lim_{x\to 1} \frac{(x-2)^5}{x^{2015}+4x-6} = \frac{\lim_{x\to 1}(x-2)^5}{\lim_{x\to 1}\left(x^{2015}+4x-6\right)} = \frac{(-1)^5}{1+4-6} = \frac{-1}{-1} = 1$$

$$(b)\quad \lim_{x\to 2}\left(1+\frac{8}{x}+\frac{x}{2}-5\sin(\pi x)+7\cos(\pi x)\right)$$

Solution: Using the laws of limits:

$$\lim_{x\to 2}\left(1+\frac{8}{x}+\frac{x}{2}-5\sin(\pi x)+7\cos(\pi x)\right)=$$

$$\lim_{x\to 2}1+\lim_{x\to 2}\frac{8}{x}+\lim_{x\to 2}\frac{x}{2}-\lim_{x\to 2}5\sin(\pi x)+\lim_{x\to 2}7\cos(\pi x)=$$

$$1+\frac{8}{2}+\frac{2}{2}-5\sin(2\pi)+7\cos(2\pi)=1+4+1-5(0)+7(1)=13$$

$$(c)\ \lim_{x\to\infty} \frac{x^4(3x-1)(5x^2-19x+1)^2}{(x-2)^7(4x-13)(x+43)}$$

Solution: *For large values of x:*

$$\frac{x^4(3x-1)(5x^2-19x+1)^2}{(x-2)^7(4x-13)(x+43)} \approx \frac{x^4(3x)(5x^2)^2}{(x)^7(4x)(x)} = \frac{75x^9}{4x^9} = \frac{75}{4}$$

$$(d)\ \lim_{x\to 0} \frac{\sin(5x)}{\tan(9x)}$$

Solution:

$$\frac{\sin(5x)}{\tan(9x)} = \frac{\sin(5x)}{\sin(9x)}\cos(9x) = \frac{5}{9}\left(\frac{\frac{\sin(5x)}{5x}}{\frac{\sin(9x)}{9x}}\right)\cos(9x) \to \frac{5}{9}\left(\frac{1}{1}\right)1 = \frac{5}{9}\ as\ x \to 0$$

7. (a) *[6 pts]* Find an equation of the *normal line* to the curve

$y = g(x) = \frac{x^2-1}{x^2+1}$ at x = 1.

Solution:

$$dy/dx = \frac{(x^2+1)d/dx(x^2-1)-(x^2-1)d/dx(x^2+1)}{(x^2+1)^2} =$$

$$\frac{(x^2+1)(2x)-(x^2-1)(2x)}{(x^2+1)^2}$$

Now, when x = 1, dy/dx = $\frac{(1^2+1)(2)-(1^2-1)(2)}{(1^2+1)^2} = 1$

Hence the slope of the normail line is m = -1.
Also, note that g(1) = 0.

Using the point-slope form of the line: $y - 0 = -1(x - 1)$

or **$y = 1 - x$**

(b) *[6 pts]* Find an equation of the *tangent line* to the curve $y = F(x) = e^x (x^2 + 4x + 1)$ at $x = 0$.

Solution:

$$dy/dx = e^x\, d/dx\left(x^2 + 4x + 1\right) + \left(x^2 + 4x + 1\right) d/dx\left(e^x\right) =$$
$$e^x (2x + 4) + \left(x^2 + 4x + 1\right)e^x$$

At $x = 0$, $dy/dx = e^0(4) + (1)\, e^0 = 5$.

Also note that $f(0) = e^0 = 1$.

So the equation of the tangent line at the point (0, 1) is given by:

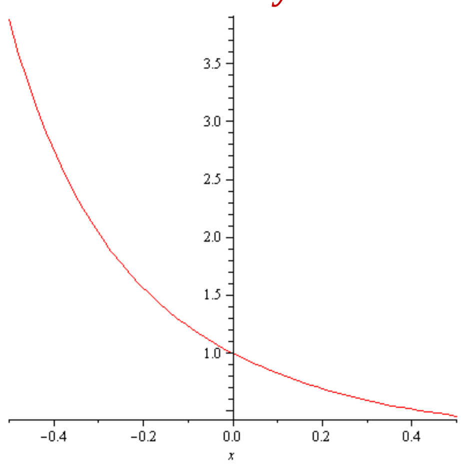
$y - 1 = 5(x - 0)$

or $y = 5x + 1$

8. Identify the *type of discontinuity* that each of the following functions has at $x = 0$. (Choose from: *removable*, *infinite*, *jump*, or *essential* discontinuity.) You need not justify your answers.

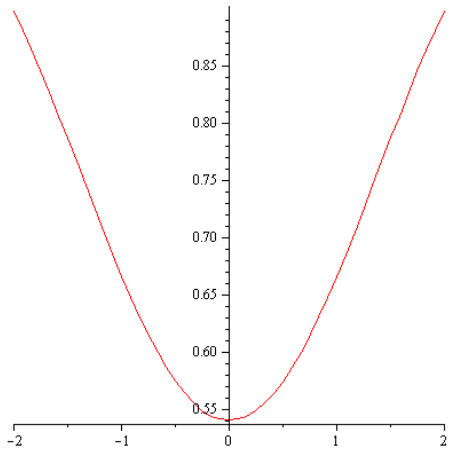
(a) $y = \dfrac{x^5 - x^4 + x^3 - x^2 + x}{x^2 + x}$

removable discontinuity *at $x = 0$*



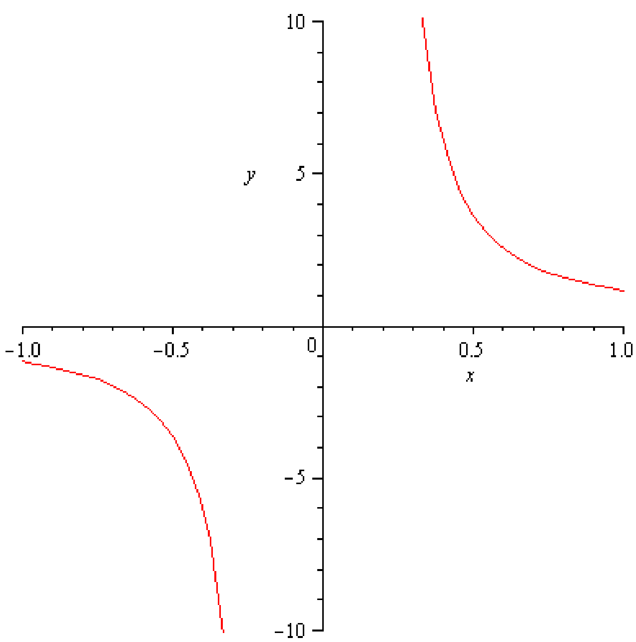
(b) $y = \cos\left(\frac{\sin x}{x}\right)$

removable discontinuity *at x = 0*



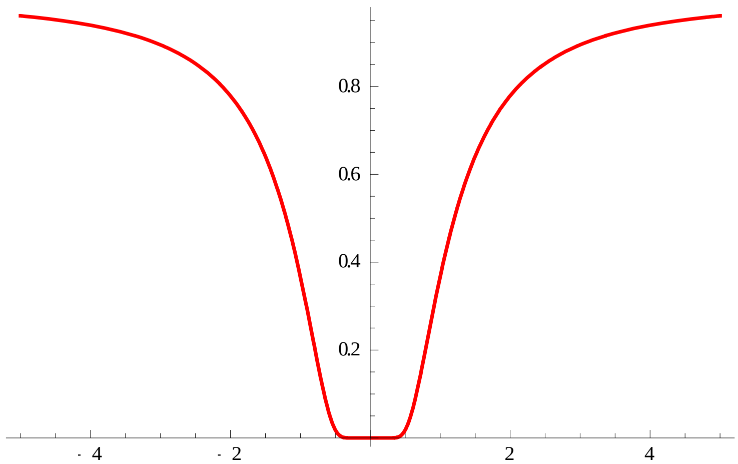
(c) $y = \sinh\left(\frac{1}{x}\right)$

infinite discontinuity *at x = 0*



(d) $y = e^{-1/x^2}$

removable discontinuity *at $x = 0$*



9. (a) Let $G(x) = a\, e^{bx}$, where *a* and *b* are non-zero *constants*.
Albertine, our friend, informs us that $(d/dx)\, e^{bx} = b\, e^{bx}$.
Find $G^{(2015)}(x)$.

Solution:

$G\,'(x) = ab\, e^{bx}$

$G\,'(x) = (d/dx)\, G\,'(x) = (d/dx)\,(ab\, e^{bx}) = ab^2\, e^{bx}$

$G'''(x) = (d/dx)\, G\,'(x) = (d/dx)\,(ab^2\, e^{bx}) = ab^3\, e^{bx}$

$G^{(4)}(x) = ab^4\, e^{bx}$

$G^{(5)}(x) = a\, b^5\, e^{bx}$

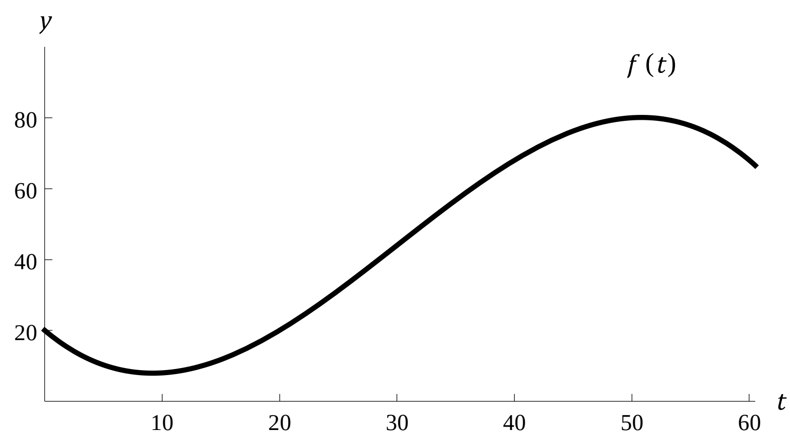
Based upon this pattern, we infer that $G^{(n)}(x) = ab^n\, e^{bx}$

Thus $G^{(2015)}(2015) = \mathbf{a\, b^{2015}\, e^{2015b}}$

(b) *[University of Michigan]* Your pet parakeet, Odette, is flying in a straight path that at first is toward you and then away from you for a minute. After t seconds, she is $f(t)$ feet away from you, where

$$f(t) = \frac{-t(t-20)(t-70)}{500} + 20, \qquad 0 \le t \le 60.$$

A graph of $y = f(t)$ is shown below.



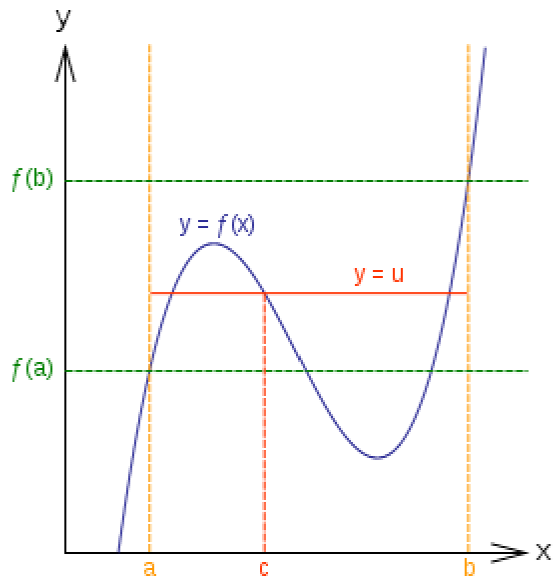
Without doing any calculations, determine which is greater: the *average velocity of Odette over the entire minute*, or her *instantaneous velocity at 30 seconds*. Explain, referring to the graph. (The absence of a clear explanation will result in no credit.)

Solution:

The slope of the secant line from t = 0 to t = 60 is the average velocity over the minute and the slope of the tangent line at t = 30 is the instantaneous velocity after 30 seconds. If you draw these lines on the graph, you can see that the tangent line clearly has a larger slope. Thus, Odette's instantaneous velocity after 30 seconds is greater than her average velocity over the entire minute.

10. (a) State carefully the *Intermediate Value Theorem* for a function y = G(x). *Provide a sketch that illustrates the statement.*

Let y = f(x) be a continuous function on the interval [a, b]. Let u be any number between f(a) and f(b). Then there exists a number c in the interval [a, b] such that f(c) = u.



(b) Let G(x) = $x^5 - 5 - \ln x$. *Using the Intermediate Value Theorem*, carefully explain why the equation G(x) = 0 has *at least one* solution. (Your calculator is not needed here.)

Solution:

Note that G(1) = 1 – 5 – ln 1 = -4 < 0 and G(e) = $e^5 - 5 - \ln e =$

$e^5 - 6$ *>* $2^5 - 6 = 28 > 0$.

Also note that G is continuous on the interval [0, π]. Since G(0) < 0 < G(e), the IVT guarantees the existence of a root of the equation G(x) = 0 in the interval [0, e].

11. Suppose that *f* and *g* are differentiable functions satisfying:

$$f(3) = -2,\ g(3) = -4,\ f'(3) = 3, \text{ and } g'(3) = -1.$$

(a) Let H(x) = (f(x) + 2g(x) + 1)(f(x) – g(x) – 4). Compute H′(3)

Solution: Using the product rule:

H′(x) = (f(x) + 2g(x) + 1) d/dx(f(x) – g(x) – 4) + (f(x) – g(x) – 4) d/dx(f(x) + 2g(x) + 1)

= (f(x) + 2g(x) + 1) (f ′(x) – g ′(x)) + (f(x) – g(x) – 4)(f ′(x) + 2g ′(x))

Thus H′(3) = (f(3) + 2g(3) + 1) (f ′(3) – g ′(3)) + (f(3) – g(3) – 4)(f ′(3) + 2g ′(3))

= (-2 – 8 + 1) (3 – (-1)) + (-2 – (-4) – 4)(3 + 2(-1)) = (-9)(4) + (-2)(1) = -38

(b) Let $M(x) = \dfrac{2f(x) + 3g(x)}{2 - 3g(x)}$. Compute M ′(3)

Solution: Using the quotient rule:

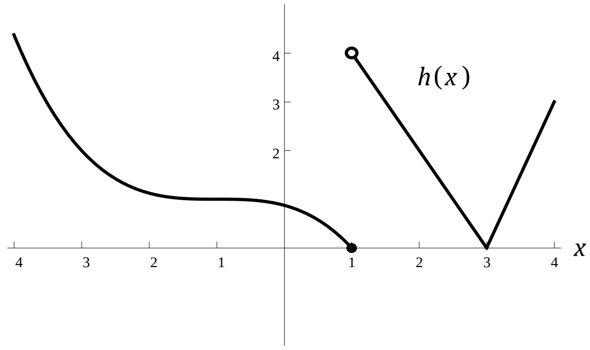
$$M(x) = \frac{(2 - 3g(x))d/dx(2f(x) + 3g(x)) - (2f(x) + 3g(x))d/dx(2 - 3g(x))}{(2 - 3g(x))^2} =$$

$$\frac{(2 - 3g(x))(2f'(x) + 3g'(x)) - (2f(x) + 3g(x))(-3g'(x))}{(2 - 3g(x))^2}$$

$At\ x = 3,\ M'(3) =$

$$\frac{(2 - 3(-4))(2(3) + 3(-1)) - (2(-2) + 3(-4))(-3(-1))}{(2 - 3(-4))^2} = \frac{45}{98}$$

12. *[University of Michigan]* The graph of a function $h(x)$ is given below.



a. *[2 point]* List all x-values with $-4 < x < 4$ where $h(x)$ is not continuous. If there are none, write NONE.

Solution: h(x) is not continuous at x = 1. Notice it is a jump discontinuity.

b. *[2 point]* List all x-values with $-4 < x < 4$ where $h(x)$ is not differentiable. If there are none, write NONE.

Solution: h(x) is not differentiable at two points: x = 1 and x = 3.

c. On the axes provided, carefully draw a graph of $y = h'(x)$. Be sure to label important points or features on your graph.