Math 115 — First Midterm

October 9, 2012

Name: EXAM SOLUTIONS

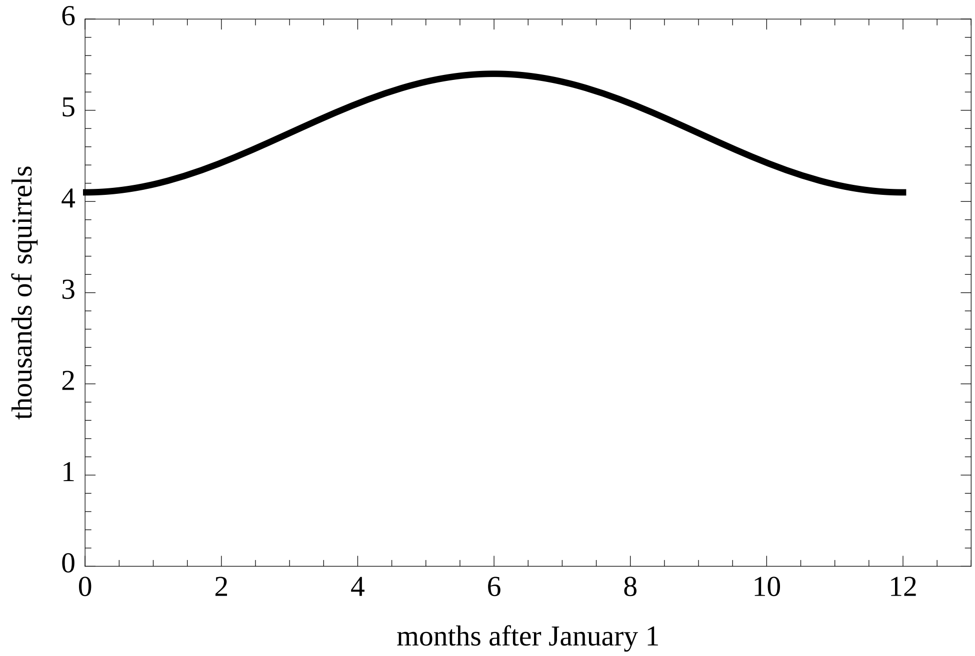
Instructor: ______________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 10 pages including this cover. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	10	
2	14	
3	12	
4	12	
5	12	
6	13	
7	13	
8	14	
Total	100	

1. [10 points] The population of squirrels in Ann Arbor oscillates sinusoidally between a low of 4.1 thousand on January 1 and a high of 5.4 thousand on July 1. Let $P(t)$ be the population, in thousands, of squirrels in Ann Arbor t months since January 1.

a. [5 points] On the axes below, graph the function P, showing at least one full period. Remember to label your axes and make sure important features of the graph are clear.



b. [5 points] Use your graph to find a formula for $P(t)$.

Solution: From the problem, we know that P is at its minimum when $t = 0$, so $-\cos$ is the natural choice of function to use. We need the appropriate values of A, B and C in $P(t) = -A\cos(Bt)+C$. We are given that the period is 12 months, so $B = 2\pi/12 = \pi/6$. A is the amplitude, which is half the distance between the maximum and minimum values, i.e. $A = (5.4 - 4.1)/2 = 0.65$. The value for C is the vertical shift, which is the average of the maximum and minimum. That is, $C = (5.4 + 4.1)/2 = 4.75$.

$$P(t) = -0.65\cos\left(\frac{\pi t}{6}\right) + 4.75$$

2. [14 points] Suppose p represents the price of a reuben sandwich at a certain restaurant on State St. $R(p)$ represents the number of reubens the restaurant will sell in a day if they charge \$$p$ per reuben.

a. [3 points] What does $R(5.5)$ represent in the context of this situation?

Solution: $R(5.5)$ is the number of reubens the restaurant will sell in a day if they charge 5.50 dollars per reuben.

b. [3 points] Assuming R is invertible, what does $R^{-1}(305)$ represent?

Solution: $R^{-1}(305)$ is the price (in dollars) per reuben when 305 are sold in a day.

c. [3 points] The owner of the restaurant also has a Church St location. It doesn't get quite as much business, and the owner finds that the State St store sells 35% more reubens than the Church St store sells at the same price. Let $C(p)$ be the number of reubens the Church St location sells in a day at a price of \$$p$ each. Write a formula for $C(p)$ in terms of $R(p)$.

Solution: $R(p) = 1.35C(p)$, so $C(p) = \frac{R(p)}{1.35}$.

d. [5 points] The owner starts doing research on reuben sales at the State St location; he wants to know how the number of reubens sold is related to the price. He finds that every time he raises the price by \$1 per reuben, the number sold in a day decreases by 20%. Let the constant B represent the number of reubens sold in a day at the State St store if the the price of reubens is \$5 each. Write a formula for $R(p)$ involving the constant B. Assume the domain of R is $1 \leq p \leq 25$.

Solution: The growth factor is $1 - 0.2 = 0.8$, so $R(p) = R_0(0.8)^p$ and $R(5) = B = R_0(0.8)^5$.
So $R_0 = \dfrac{B}{(0.8)^5}$, and we get $R(p) = \dfrac{B}{(0.8)^5}(0.8)^p = B(0.8)^{p-5}$.

3. [12 points] For each of the following three sets of axes, exactly one of the following statements **(a)**-**(e)** is true. You may use a letter more than once. In the space provided next to each figure, enter the letter of the true statement for that figure. For each graph, note that the x and y scales are not the same.

(a) h is the derivative of f, and f is the derivative of g.

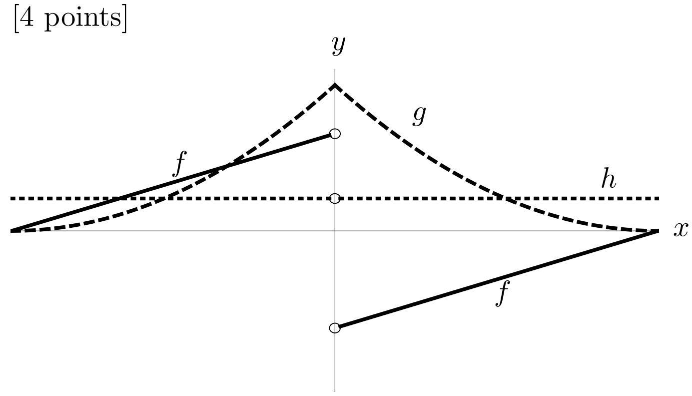
(b) g is the derivative of f, and f is the derivative of h.

(c) g is the derivative of h, and h is the derivative of f.

(d) h is the derivative of g, and g is the derivative of f.

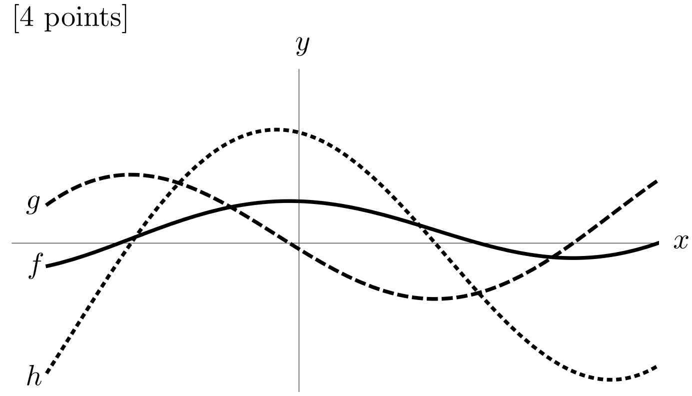
(e) None of **(a)**-**(d)** are possible.

a. [4 points]



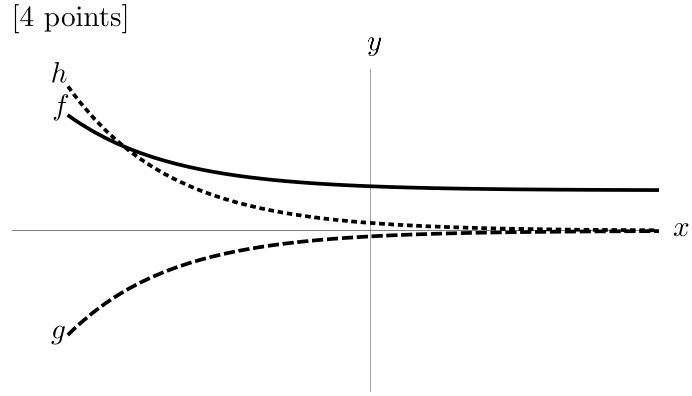
True statement **(a)**

b. [4 points]



True statement **(e)**

c. [4 points]



True statement **(d)**

4. [12 points] The Dow Jones Industrial Average (DJIA) is a stock market index which measures how the stocks of 30 large publicly-owned companies perform during a given period of time. On September 27, 2012 at 11:30am the DJIA was 13,420 and at 1:30pm, the DJIA was 13,520. Suppose $A = h(t)$ gives the value of the DJIA t hours after 9:00am on September 27, 2012 with $0 \le t \le 8$.

a. [4 points] Using the information given above approximate A using a linear function, $\ell(t)$. Write an expression for $\ell(t)$.

Solution: We are given $\ell(2.5) = 13420$ and $\ell(4.5) = 13520$. So the slope is $\dfrac{13,520 - 13,420}{4.5 - 2.5} = \dfrac{100}{2} = 50$. Now we can find our vertical intercept, b, by plugging in a point and solving: $13,420 = 50 \cdot 2.5 + b$ so $b = 13,295$.

$\ell(t) =$ $50t + 13,295$

b. [4 points] Your friend tells you that an exponential function would be more accurate in modeling A. If $g(t)$ is an exponential function which approximates A, what is the hourly growth rate of $g(t)$? What was the value of the DJIA at 2:30pm on September 27, 2012 according to this model?

Solution: 11:30 and 1:30 are two hours apart, so if a is the growth factor, $13420a^2 = 13520$, so $a \approx 1.0037$. So the growth rate is 0.0037, or 0.37%.
By 2:30, the DJIA will grow by one more growth factor, so it will be at $(13520)(1.0037) \approx 13570.3$

growth rate = 0.37% value of DJIA at 2:30pm = 13570.3

c. [4 points] In the end you realize the best model for A is a function of the form

$$p(t) = t^k + b$$

where k and b are constants. You also find out that at 9am on September 27, the DJIA was actually 13,402. Find values of k and b so that $p(t)$ approximates A.

Solution: We are given $p(0) = 13402$, so $b = 13402$.
We can use either of the other two points to get a value for k. Using the point at $t = 2.5$, we get

$$2.5^k + 13402 = 13420.$$

$$2.5^k = 18.$$

$$k \ln 2.5 = \ln 18.$$

This gives $k \approx 3.154$ (if we used the other point, we would get $k \approx 3.172$).

$k =$ 3.154 $b =$ 13,402

5. [12 points] A function g defined for all real numbers has the following properties:

(a) g is differentiable for $-1 \le x < 4$.

(b) $g'(x) \le 0$ for $-1 \le x < 4$.

(c) $g''(x) > 0$ for $2 < x < 4$.

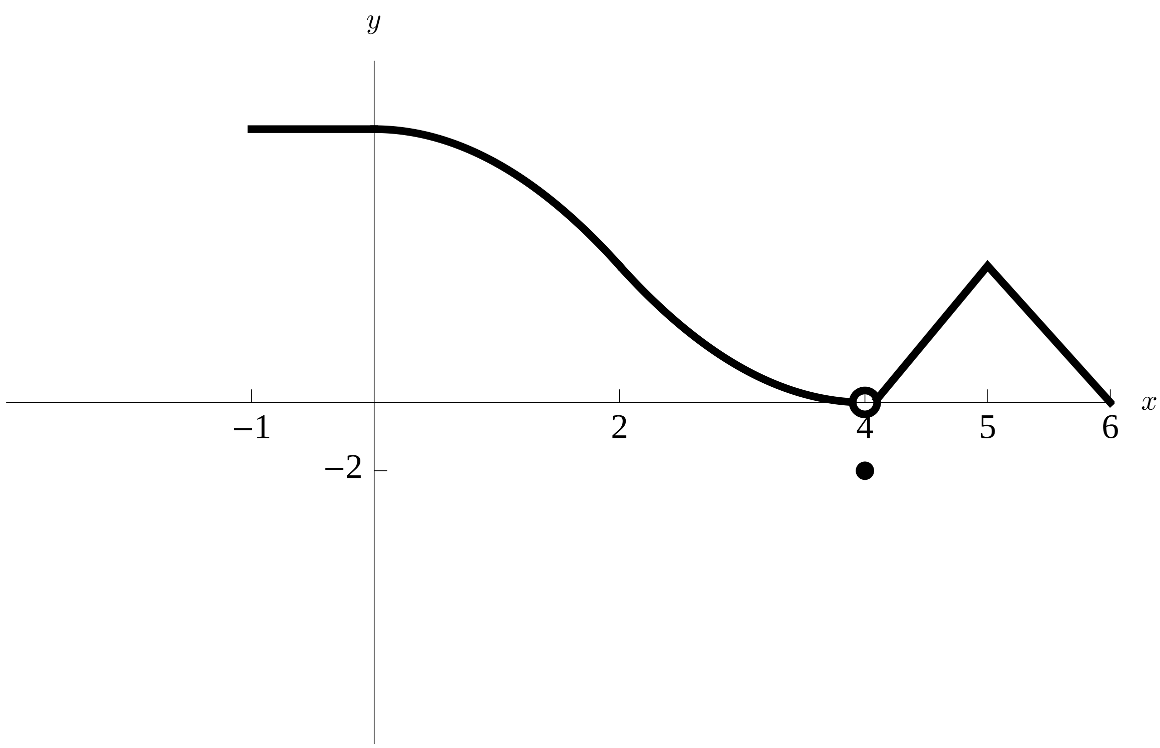
(d) $g(4) = -2$.

(e) $\lim_{x \to 4} g(x) = 0$.

(f) g is continuous at $x = 5$ but not differentiable at $x = 5$.

(g) $g'(0) = 0$.

On the axes below, draw a possible sketch of $y = g(x)$ on the domain $-1 \le x \le 6$, including labels.



6. [13 points] Toby listens to music as he walks to class in the morning and notices an interesting phenomenon: the tempo of the music affects his walking speed and thus the time it takes him to get to class. Let $C(b)$ be the number of minutes it takes Toby to get to class when he is listening to music with a tempo of b beats per minute (bpm). You may assume Toby's house is 1.2 miles from his first class.

For parts (a)-(c), write a single mathematical equation using C, C^{-1}, and/or their derivatives that describes the given situation.

a. [3 points] The tempo of the music Toby is listening to when it takes him 32 minutes to get to class is 89 bpm.

Solution: $C^{-1}(32) = 89$ or $C(89) = 32$.

b. [3 points] If Toby gets to class in 30 minutes, and he wants to take 31 minutes to get there instead, he should decrease the tempo of his music by approximately 4 bpm.

Solution: $(C^{-1})'(30) = -4$ or $(C^{-1})'(31) = -4$.

c. [3 points] Toby's average velocity when he listens to music with a tempo of 115 bpm is 0.047 miles per minute.

Solution: $\dfrac{1.2}{C(115)} = 0.047$

For part (d) give a practical interpretation of the given mathematical equation.

d. [4 points] $C'(81) = -0.5$

Solution: If Toby increases the tempo of his music from 81 to 82 bpm, he will get to class approximately 30 seconds faster.

7. [13 points] f is a continuous, differentiable function defined for all real numbers. Some values of f and its derivative are given in the table below.

x	0	1	2	3	4	5	6	7
$f(x)$	-11.2	-4.0	-1.1	-0.5	-0.1	2.0	7.9	19.6
$f'(x)$	9.9	4.7	1.4	0.2	0.9	2.1	5.9	11.7

a. [4 points] Estimate the derivative of f at $x = 5, 6$, and 7, and fill in the remainder of the table.

Solution: Using left-hand estimates,
$f'(5) \approx \dfrac{2.0 - (-0.1)}{5 - 4} = 2.1$
$f'(6) \approx 7.9 - 2.0 = 5.9$
$f'(7) \approx 19.6 - 7.9 = 11.7.$

b. [2 points] Estimate $f''(1)$ using the data given.

Solution: We can use a left-hand estimate, a right-hand estimate, or find both and average them:
Left: $4.7 - 9.9 = -5.2$
Right: $1.4 - 4.7 = -3.3$
Average: $\dfrac{-5.2 - 3.3}{2} = -4.25$

c. [4 points] Assuming the concavity of f doesn't change on the interval $5 \le x \le 7$, is the graph of f concave up or concave down on that interval? Explain.

Solution: f is concave up on the interval $5 \le x \le 7$, because our estimates for the derivative are increasing on this interval.

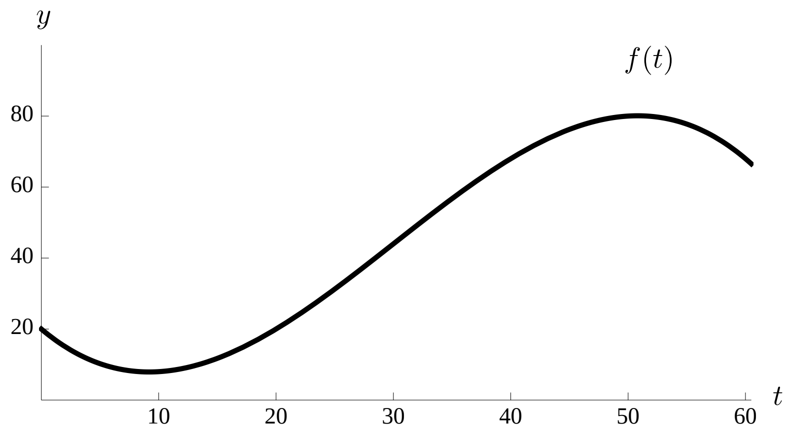
d. [3 points] Using your answer from part (c), is your approximation for $f'(7)$ an overestimate or an underestimate? Explain.

Solution: Our approximation for $f'(7)$ is an underestimate. Our estimate was the slope of a secant line on the left, which will be smaller than the slope of the tangent line since f is concave up.

8. [14 points] Your pet bird is flying in a straight path toward you and away from you for a minute. After t seconds, she is $f(t)$ feet away from you, where

$$f(t) = \frac{-t(t-20)(t-70)}{500} + 20, \qquad 0 \leq t \leq 60.$$

A graph of $y = f(t)$ is shown here.



a. [3 points] Without doing any calculations, determine which is greater: the average velocity of the bird over the entire minute, or her instantaneous velocity after 30 seconds. Explain, referring to the graph.

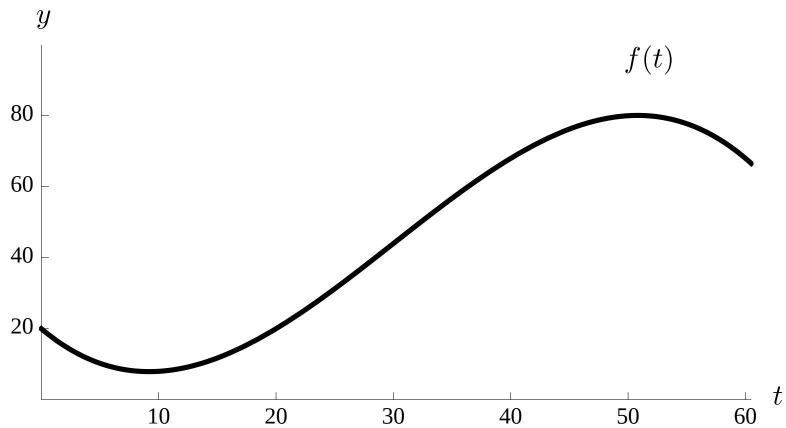
Solution: The slope of the secant line from $t = 0$ to $t = 60$ is the average velocity over the minute and slope of the tangent line at $t = 30$ is the instantaneous velocity after 30 seconds. If you draw these lines on the graph, you can see that the tangent line clearly has a larger slope. Thus, her instantaneous velocity after 30 seconds is greater than her average velocity over the entire minute.

b. [3 points] Calculate the exact value of the average velocity of the bird over the entire minute.

Solution: Average velocity $= \dfrac{f(60) - f(0)}{60 - 0} = \dfrac{68 - 20}{60} = 0.8$ ft/s

8. (continued) The formula for f and its graph are repeated below for your convenience.

$$f(t) = \frac{-t(t-20)(t-70)}{500} + 20, \qquad 0 \le t \le 60.$$



c. [4 points] Write an explicit expression for the velocity of the bird at time t using the limit definition of velocity. Final answers containing the letter f will receive no credit. Do not evaluate your expression.

Solution: $f'(t) = \lim_{h\to 0} \dfrac{\left(\frac{-(t+h)(t+h-20)(t+h-70)}{500} + 20\right) - \left(\frac{-t(t-20)(t-70)}{500} + 20\right)}{h}$

d. [4 points] After a minute, you scare the bird, and she flies away at 9 feet/sec. Write a formula for a continuous function $f(t)$ describing the distance between you and the bird for $0 \le t \le 180$.

Solution: After 60 seconds, the bird is $f(60) = 68$ ft away. So we want to find a formula for a line with slope 9 passing through $(60, 68)$. Plugging in and solving for the vertical intercept b, we get $68 = 9 \cdot 60 + b$. So $b = -472$. We can then write this as a piecewise function:

$$f(t) = \begin{cases} \dfrac{-t(t-20)(t-70)}{500} + 20 & 0 \le t \le 60 \\ 9t - 472 & 60 < t \le 180 \end{cases}$$